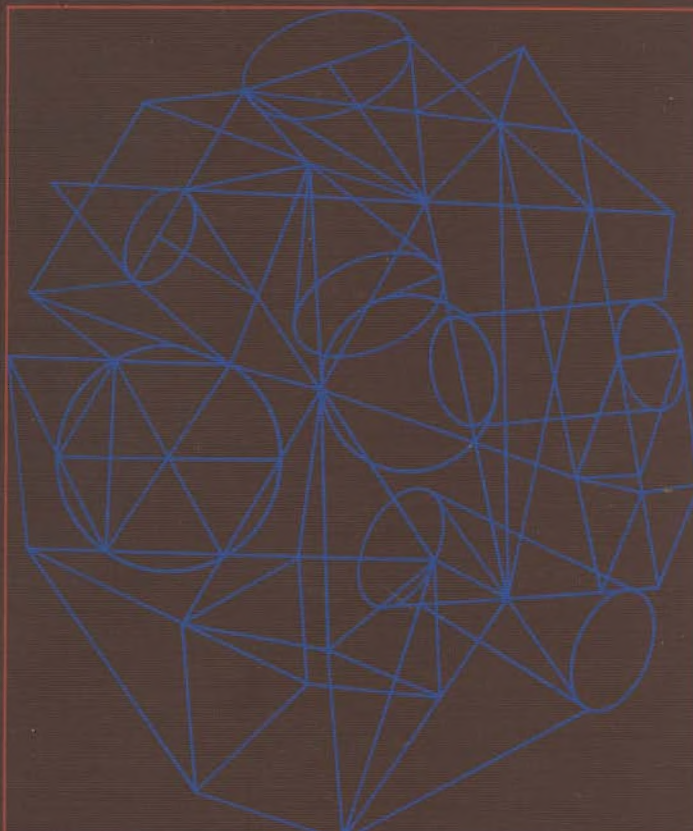
GEOMETRY

LAWRENCE A. RINGENBERG / RICHARD S. PRESSER

UNDER THE EDITORIAL DIRECTION OF ROY DUBISCH & ISABELLE P. RUCKER



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Preface

Geometry is one of a series of five mathematics textbooks for junior and senior high schools. It is designed for a one-year course for students with a background of informal geometry and elementary algebra such as that included in *Mathematics I*, *Mathematics II*, and *Algebra* of this series.

In its formal development of Euclidean geometry, this textbook features an integrated treatment of plane and solid geometry with an early introduction of coordinates. Coordinates on a line, in a plane, and in space relate numbers and points. They are used to gain knowledge of geometrical figures and to simplify the development of formal geometry.

Students use their knowledge of elementary algebra throughout the book to help them learn geometry. In doing this they maintain and strengthen their competence in algebra.

The postulates in this book form the basis of a rigorous, yet plausible, development of a first course in formal Euclidean geometry. In some instances, statements that are proved in more advanced treatments are accepted as postulates here. This has been done to decrease the length of the development and to make the development appropriate for high school students.

Geometry is an important subject because it is practical and useful and at the same time abstract and theoretical. There are two main objectives in this geometry textbook. One is to help students learn a body of important facts about geometrical figures. These facts, interpreted physically, are facts about the space in which we

live. These facts are important for intelligent citizenship and for success in many careers. The other main objective is to help students attain a degree of mathematical maturity. In the elementary and junior high schools, students are encouraged to learn by observing and manipulating physical objects. There is considerable emphasis on intuitive and inductive reasoning. The power and beauty of mathematics, however, is due primarily to its abstractness. The generality of its theorems makes possible a variety of applications. Understanding mathematics "in the abstract" is tantamount to understanding the deductive method in mathematics. In studying this book students will develop their capacity to reason deductively and hence their ability to read and write proofs.

Contents

Chapter 1

Sam Falk /Monkmeyer

Points, Lines, and Planes

1.1 INFORMAL GEOMETRY

Geometry began informally more than 2000 years ago in Babylon and Egypt. The meaning of geometry is, literally, "earth measurement"; hence it is not surprising that early knowledge of the subject was concerned largely with measurements of lengths, areas, and volumes. Such knowledge was a necessity, especially to the ancient Egyptians who almost annually were forced to restore to their river-side farms the boundary markers which were washed away by heavy flooding of the River Nile.

Then, as well as now, men learned by experience. The ancient Egyptians developed many rules-of-thumb for surveying fields and roads and for making calculations related to building dwellings and pyramids. These rules were based on many observations, intuition, and reasoning. As long ago as 500 B.C. men knew how to find the areas of rectangles, triangles, and trapezoids, but they had made little progress in formal geometry, that is, in developing geometry as a system that explains why the rules worked.

As a school subject today geometry is both informal and formal. In the elementary schools geometry is largely informal. It is physical geometry. Students work with physical objects or with pictures that represent physical objects. General statements or rules are based on intuitive reasoning and inductive reasoning. Intuitive reasoning is what might be called common sense, or, as some might say, reasoning in a hurry. Inductive reasoning is reasoning based on numerous examples. We discuss an example of each.

Intuitive Reasoning

Let us take for granted that we know what is meant by a triangle. Figure 1-1 shows a right triangle $\triangle ABC$ with right angle at C.

Figure 1-1

Suppose that the lengths of the sides $\overline{AC}$ and $\overline{BC}$ are b inches and a inches, respectively. What is the area of the triangle? We say "area of the triangle," but we really mean the area of the figure or region that consists of the triangle and its interior. In earlier mathematics classes we have learned that the area is $\frac{1}{2}ab$ square inches. Suppose someone asks why. One answer might be: "It is intuitively obvious. You can see it by looking at a picture of a rectangle with sides of lengths a and b and with one diagonal drawn." (See Figure 1-2.) If the person who asked why responds with "Now I see why," he is responding to an intuitive feeling of the "rightness" of things rather than to a logical argument or to inductive reasoning. This is an example of intuitive reasoning.

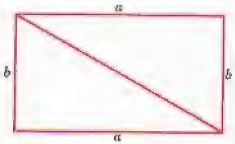


Figure 1-2

Inductive Reasoning

Let us take for granted that we know what is meant by an angle and its measure. We record the measures of angles as numbers, omitting the degree symbol. This is consistent with our formal point of view regarding measure developed later. Suppose that each student in an elementary geometry class is given a set of plastic triangles numbered 1, 2, 3, 4, as suggested by Figure 1-3, and a protractor. The instructions

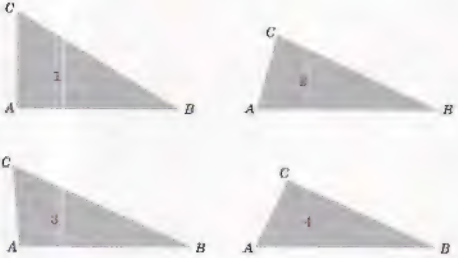


Figure 1-3

direct the students to find and record the measures of the angles of the triangles and then to find the sum of the measures for each triangle as in the following table. The object of the lesson is to make it plausible that the sum of the measures of the angles of a triangle is 180. If the students reason that this is a correct conclusion on the basis of the measurements they have made, this is an example of inductive reasoning.

	$m\angle A$	$m\angle B$	$m\angle C$	Sum
Triangle 1	90	32	58	180
2	75	26	78	179
3	95	25	62	182
4	65	25	90	180

1.2 THE IDEA OF FORMAL GEOMETRY

Suppose that as a result of inductive reasoning we have made a general statement about all triangles, or about all rectangles, or about all circles. We think that the statement is true, but how can we know for certain? Is it possible to know something for certain about all triangles? It would seem necessary to test every triangle, which is clearly impossible. Therefore a different approach is required, the formal approach.

The formal approach involves deductive reasoning, that is, logical arguments by which general statements are obtained from previously accepted statements. Two examples follow.

Deductive Reasoning

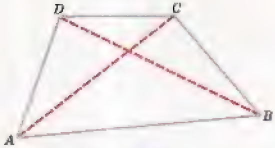
Example Suppose that you are given a set of plastic convex quadrilaterals numbered 1, 2, 3, . . . , 10 as suggested by Figure 1-4. Suppose that there are really ten of them, although only four of them are shown.

Figure 1-4

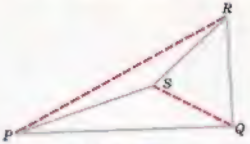
Suppose also that you know that the sum of the measures of the angles of every triangle is 180. Perhaps you know this through the example of inductive reasoning in Section 1.1. At any rate you know or are told this. This knowledge about all triangles is considered as "given" in the situation of this example. The instructions are to find the sum of the measures of the angles of each quadrilateral without making any measurements using a protractor.

A quadrilateral is *convex* only if it has the property that if either pair of its opposite vertices is joined by a segment, called a diagonal of the quadrilateral, then all of that diagonal except its endpoints lies inside the quadrilateral. Figure 1-5a shows a convex quadrilateral. In Figure 1-5b the segment $\overline{PR}$, except for the points P and R, lies outside the quadrilateral. Hence $PQRS$ is not a convex quadrilateral.

A picture of a quadrilateral and one of its diagonals provides a clue for finding the sum of the measures of the angles. (See Figure 1-6.) Quadrilateral $ABCD$ may be considered representative of any convex quadrilateral. From this figure we see that the sum of the measures of



(a) Convex



(b) Not convex

Figure 1-5

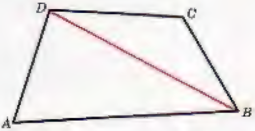


Figure 1-6

the angles of a quadrilateral is the sum of the measures of the angles of two triangles. Therefore the sum of the measures of the angles of a quadrilateral is $2 \cdot 180$ or 360. Without using a protractor we decide that the sum of the measures of the angles of each quadrilateral is 360. We arrive at this conclusion by deductive reasoning.

Let us examine this example carefully so that we may understand more clearly the nature of deductive reasoning. We took some things for granted or as given. From these we deduced the answer, or *conclusion*. The given statements are called *hypotheses*, or collectively, the *hypothesis*. Thus the essence of deductive reasoning is to obtain a conclusion from a hypothesis. The hypothesis may or may not be true. If it is true in a given situation and if the reasoning involved in reaching the conclusion is correct, then the conclusion is true in that situation.

Let us look at this idea expressed in symbols. Suppose that we are given the hypothesis, denoted by H, and we want to establish the conclusion, denoted by C. What we want to prove is not "C" but "If H, then C." Many statements in mathematics are in this "If H, then C" form. How can such a statement be useful? It is useful in any situation where the "H" is true. Because if we know "H" is true and if we know "If H, then C" is true, we also know that "C" is true. In the problem about quadrilaterals, the hypothesis and the conclusion may be identified as follows:

H: $ABCD$ is a convex quadrilateral.

C: The sum of the measures of the angles of $ABCD$ is 360.

If $ABCD$ is a quadrilateral like the one in Figure 1-5a, then "H" is true, and since "If H, then C" is true, it is correct to conclude that "C" is true. If $ABCD$ is a quadrilateral like the one in Figure 1-5b, then "H" is false, and although "If H, then C" is true, it is incorrect to conclude that "C" is true.

Example This example in which deductive reasoning is used to establish a general statement concerns a property of the natural numbers, 1, 2, 3, ... Undoubtedly, you know an even number is a number that is 2 times an integer. Thus x is an even number if there is an integer y such that $x = 2y$. Some even numbers are 2, 16, 168, and 2466. If you add any two of these numbers, you get an even number. Is it then true that the sum of any two even numbers is an even number? Of course it is. Let us prove it, however, by using deductive reasoning.

H: x and y are even numbers.
C: $x + y$ is an even number.

Our task is to prove: If H, then C.

Proof: Since x is even, there is an integer u such that $x = 2u$. Since y is even, there is an integer v such that $y = 2v$. Then

$$x + y = 2u + 2v$$

and, using the distributive property, we get

$$x + y = 2(u + v).$$

But $u + v$ is an integer. Therefore, since $x + y$ is 2 times an integer, $x + y$ is an even number.

Notice that we did not prove that x is even or that y is even. We proved that *if* x and y are even numbers, *then* $x + y$ is an even number.

As we said, formal geometry involves deductive reasoning. An important feature of a formal geometry is its structure or arrangement. The geometry of this book is elementary Euclidean geometry, carefully arranged so that we can see how the various parts fit together and how some things depend on other things. Formal geometry might be thought of as geometry for the lazy person. In formal geometry we prefer a general statement that tells something significant about *all* triangles rather than a hundred statements about a hundred triangles.

Our starting point in formal geometry is a set of statements about some of the simplest objects of geometry. We do not try to tell what these objects are by definitions. Definitions would involve other words that we would need to define in turn, and so we accept some concepts

as basic and undefined. We might decide, for example, to start with triangles because almost everyone has an idea of what a triangle is. But do we actually know what a triangle is? A triangle is made up of three segments. The notion of segment is more basic than the notion of triangle. Every segment is a set of points. The idea of a point is more fundamental than the idea of a segment. Every segment is a part of some line. Perhaps a line is simpler to think about than a segment.

In formal geometry we usually consider points, lines, and planes as the basic building blocks. We do not define these words. How then can we be sure that we know anything at all about them? On the basis of our experience with physical objects we identify the most basic properties that points, lines, and planes have in relation to one another. We formulate these as statements that we accept without proof. We call these statements **postulates.**

The foundation for formal geometry, then, is a set of statements, the postulates, which we accept without proof. The postulates are statements about the basic objects in geometry. We agree that what we know about these objects is what the postulates say, and nothing more, at least at the start. What else can we possibly know about these objects? We can know what we formally assume in the definitions and what we deduce by logical reasoning.

As an example suppose that we start with the following six postulates and one definition. (This is not for keeps—just for this example!) Our notations are self-explanatory.

POSTULATE A Every line is a set of points and contains at least two distinct points.

POSTULATE B Every segment is a subset of a line and contains (besides other points perhaps) exactly two distinct points called its endpoints.

POSTULATE C If A and B are any two distinct points, there is exactly one line $\overleftrightarrow{AB}$ containing A and B, and exactly one segment $\overline{AB}$ with A and B as endpoints.

POSTULATE D Every plane is a set of points and contains at least three points that do not all lie on one line, that is, three noncollinear points.

POSTULATE E If A, B, C are three noncollinear points, then there is exactly one plane containing A, B, and C.

POSTULATE F If a plane contains two distinct points A and B, then it contains the segment $\overline{AB}$.

Definition If A, B, C are any three noncollinear points, then the union of the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ is a triangle; we denote it by $\triangle ABC$.

In this example, what do we know for sure? We know what is said in these seven statements, the six postulates and the definition. What else do we know? We know anything else that we deduce by logical reasoning from them. We shall deduce one statement and call it a theorem.

THEOREM Every plane contains at least one triangle.

Proof: Let α (read "alpha") be any plane. Then α contains three noncollinear points (Postulate D); call them A, B, C. Then there is exactly one segment $\overline{AB}$ with endpoints A and B, exactly one segment $\overline{BC}$ with endpoints B and C, and exactly one segment $\overline{CA}$ with endpoints C and A (Postulate C). Then there is a triangle $\triangle ABC$ (by our definition). Each of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ lies in α (Postulate F). Therefore $\triangle ABC$, which is their union, lies in α and the proof is complete.

In proving a theorem it is usually a good idea to include one or more figures to suggest the given situation. In this situation we start with a plane α as suggested in Figure 1-7. Next we reason deductively to get three noncollinear points A, B, C in α as suggested in Figure 1-8. Finally, we reason deductively to show that there is a triangle $\triangle ABC$ and that it lies in α as suggested in Figure 1-9.

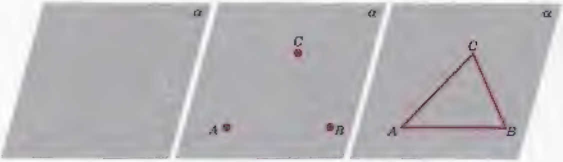


Figure 1-7 Figure 1-8 Figure 1-9

Our theorem may seem trivial to you. Our goal, however, was simply to show an easy example of a theorem obtained by deductive reasoning from a set of postulates.

EXERCISES 1.2

In Exercises 1–6, a situation is given and a question is asked. Obtain an answer in each case using intuitive reasoning.

1. Given the two numbers

$$x = 21.3 + 27.4 \qquad \text{and} \qquad y = 28.5 + 16.2 + 5.9,$$

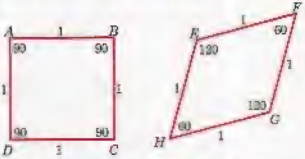
which is larger, x or y?

2. Given the two numbers

$$p = (1.6)(754) \qquad \text{and} \qquad q = (1.6)(896),$$

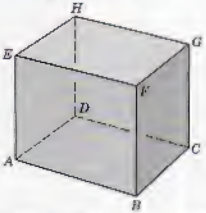
is pq greater than, equal to, or less than $(2)(765)(896)$?

3. The following figure shows a square and a parallelogram with side lengths and angle measures as labeled. Do these figures have equal areas? If not, which one has the larger area?



4. The figure below shows two parallel lines. Is there a plane containing these two lines?

5. The figure shows a rectangular box with the vertices labeled. Is there a plane containing the four points E, G, C, A?



6. Given the situation of Exercise 5, is there a plane containing the four points A, B, C, H?

In Exercises 7–12, a hypothesis H and a conclusion C are given. State whether you think the conclusion follows logically from the hypothesis. Be prepared to defend your answer.

7. H: x is an odd number.
   C: x^2 is an odd number.
8. H: x is a multiple of 3.
   C: x^2 is a multiple of 6.
9. H: x is a multiple of 3.
   C: x^2 is a multiple of 9.
10. H: $\triangle ABC$ is a right triangle with sides of lengths 3 in., 4 in., and 5 in.
    C: The area of $\triangle ABC$ is 6 sq. in.
11. H: p and q are two distinct intersecting planes.
    C: The intersection of p and q is a line.
12. H: S is a sphere and p is a plane that intersects S.
    C: The intersection of S and p is a circle or a point.

1.3 THE IDEAS OF POINT, LINE, AND PLANE

Every day in the world around us we observe objects of different sizes and shapes. We notice that some of these objects have corners, edges, and sides, and that some of their parts are "straight," some are "flat," and some are "round." Touching and seeing certain objects help us to classify them according to their characteristics.

In arithmetic the idea of a number is a mathematical idea that grew out of a need to classify certain sets according to how many objects they contained. But no one has ever seen or touched a number. In geometry the ideas of point, line, and plane are mathematical ideas that grew out of a need to classify certain sets of figures and to measure their boundaries or the regions bounded by the figures. But no one has ever seen or touched a point, a line, or a plane. Just as in arithmetic you studied numbers and the operations on them, in geometry you will study points, lines, and planes and how they relate to one another.

In the same way that deductive reasoning must be based on certain assumptions (postulates) that we accept without proof so must our definitions be based on certain terms that we make no attempt to define. This is necessary in order to avoid "circular" definitions, that is, a chain of definitions which eventually comes back to the first word being defined. If a person does not know the meaning of any of the words in the chain, the definition is of no value to him. For example, in defining the word "dimension" one dictionary uses the word "magnitude." It defines "magnitude" in terms of "size." When looking up

"size" in this same dictionary, it gives "dimension or magnitude." Thus, if someone does not know the meaning of any of the terms magnitude, size, or dimension, the dictionary is not very useful. Somewhere in the cycle it is necessary to know the meaning of a word based on experience.

The basic undefined terms in our geometry are *point*, *line*, and *plane*. Our postulates give these terms the meaning that we wish them to have. You undoubtedly already have an intuitive feeling for the concept of a point, a line, or a plane. We can think in a vague fashion of a point as having "position" but no "size"; of a line as being "straight," having "direction," but no "width"; and a plane as being "flat" but having no "thickness." When we "mark a point" or "draw a line" on our paper or on the chalkboard, we are merely drawing a picture of what we think a point or a line should be. These pictures or figures help us to see and discover some of the relationships that exist among points, lines, and planes and help us to keep these relationships straight in our minds. However, any deductions that we reach must be justified strictly on the basis of our postulates, definitions, and theorems and not on what appears to be true from a figure. As we progress in our study of geometry, we will use figures more and more freely. If our figures are drawn carefully enough, they generally will not give us false information or lead us to false conclusions.

Indeed, trying to deduce all the theorems and work all the problems in this text without drawing any figures would be a tedious and difficult task. It would be somewhat like a carpenter attempting to construct a house from memory, without the aid of any blueprints or floor plans. The resulting structure might be quite different from the house he planned to build.

We use figures frequently to help explain what our postulates, definitions, and theorems say. You are encouraged to do the same. When it is practical, you should restate a theorem or a problem in terms of a figure that shows the relationships that are given in the theorem. If you are careful not to include in the figure any special properties that are not given in the theorem, then the figure should be a valuable aid in understanding the theorem and also in proving it.

EXERCISES 1.3

1. Name three physical objects that convey the idea of a point.
2. Name three physical objects that convey the idea of a line.
3. Name three physical objects that convey the idea of a plane.

Exercises 4–9 refer to the cube shown in Figure 1-10. Answer each question in Exercises 4–7 with the word point, line, plane, or space.

4. Each corner (vertex) of the cube suggests a $\boxed{?}$.
5. Each side (face) of the cube suggests a $\boxed{?}$.
6. Each edge of the cube (such as $\overline{AB}$) suggests a $\boxed{?}$.
7. The interior of the cube is a subset of $\boxed{?}$.
8. How many vertices does the cube have? How many faces? How many edges?
9. Let V represent the number of vertices, E the number of edges, and F the number of faces. Compute $V - E + F$ for the cube.

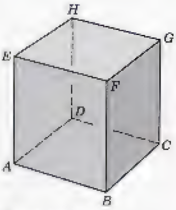
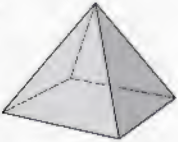


Figure 1-10

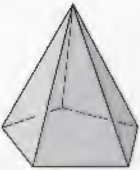
In Exercises 10–17, copy and complete the table on the following page for the pyramids and prisms shown below.

10.

11.



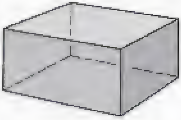
12.

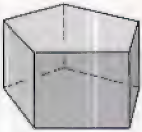


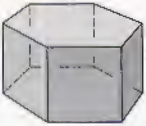
13.

14.

15.



16. 

17. 

	V	E	F	$V - E + F$
10.	—	—	—	—
11.	—	—	—	—
12.	—	—	—	—
13.	—	—	—	—
14.	—	—	—	—
15.	—	—	—	—
16.	—	—	—	—
17.	—	—	—	—

18. Did you get the same number for $V - E + F$ in Exercises 9–17? If you have counted correctly, $V - E + F$ equals 2 in each case. Figures like those in Exercises 10–17 are called *polyhedrons*. On the basis of your answers for Exercises 9–17, do you think that $V - E + F = 2$ for every polyhedron? Compute V, E, F, and $V - E + F$ for the star-shaped polyhedron shown in Figure 1-11. (You can construct this polyhedron by gluing together 16 triangles and 2 squares.)

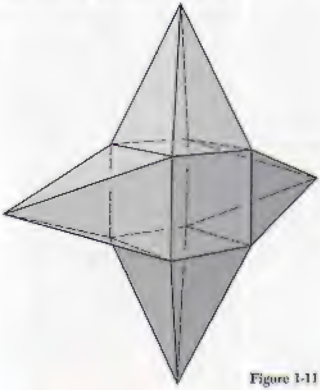


Figure 1-11

1.4 SETS

In our development of formal geometry we shall often speak of sets of points, or sets of lines, or sets of planes, as well as sets of numbers. Indeed, we shall agree later in this chapter that every line is a set of points and that every plane is a set of points. Although you are familiar with the language of sets, we shall review it briefly.

Recall that a set is simply a collection of objects. The objects may be numbers, people, points, or whatever. The objects of a set are called the **members** or the **elements** of a set. The symbol $\in$ indicates that an object is an element of a given set. Thus, to indicate that the number 2 is an element of the set $\{1, 2, 3, 4\}$, we write

$$2 \in \{1, 2, 3, 4\}.$$

We read the expression $2 \in \{1, 2, 3, 4\}$ as "2 is an element of the set $\{1, 2, 3, 4\}$" or simply as "2 is in $\{1, 2, 3, 4\}$." Similarly, if point A is an element of the set of points in line l, we write

$$A \in l$$

and read this as "point A is on line l."

It is often convenient to use set-builder notation to indicate the members of a particular set. For example,

$$\{x : x \text{ is an integer and } -2 < x < 4\}$$

is just another way of naming the set $\{-1, 0, 1, 2, 3\}$. We read the expression $\{x : x \text{ is an integer and } -2 < x < 4\}$ as "the set of all numbers x such that x is an integer and x is greater than -2 and less than 4." If we wished to include -2 and 4 in this set, we would write $-2 \leq x \leq 4$ rather than $-2 < x < 4$.

Example 1 Let R represent the set of all real numbers. Suppose that we wish to picture (graph) the set

$$C = \{x : x \in R \text{ and } -3 < x \leq 3\}$$

on a number line. Another way of writing the expression $-3 < x \leq 3$ is $x > -3$ *and* $x \leq 3$.

The number line in Figure 1-12 pictures the set

$$A = \{x : x \in R \text{ and } x > -3\}.$$

Figure 1-12

Note that the point corresponding to the number -3 has been circled to indicate that -3 is not an element of this set.

The number line in Figure 1-13 pictures the set

$$B = \{x : x \in R \text{ and } x \leq 3\}.$$

Why is the point corresponding to the number 3 circled and shaded in the figure?

Figure 1-13

It is clear from the two number lines in Figures 1-12 and 1-13 that the numbers that are common to *both* sets (those that belong to both *A and B*) are the real numbers between -3 and 3 and including 3. This is shown on a single number line in Figure 1-14.

Figure 1-14

We call the set shown in Figure 1-14 the intersection of the two sets in Figure 1-13. The intersection symbol $\cap$ is used in forming a symbol to denote the intersection of two sets. Thus to indicate the intersection of the two sets

$$A = \{x : x \in R \text{ and } x > -3\}$$

and

$$B = \{x : x \in R \text{ and } x \leq 3\}$$

we write $A \cap B$ and we see that this intersection is the set

$$C = \{x : x \in R \text{ and } -3 < x \leq 3\}.$$

Therefore $C = A \cap B$.

In set language the connective "and" is related to intersection. Thus the elements that belong to sets A and B in Example 1 are those elements that are common to the two sets; in other words, the intersection of the two sets is the set C.

It may happen that the intersection of two sets is empty, meaning that the two sets have no elements in common. The symbol $\varnothing$ indicates the empty or null set. For example, since the set E of even integers and the set O of odd integers have no integers in common we write

$$E \cap O = \varnothing.$$

In formal geometry, when we say that one set intersects another set, we mean that *the two sets have at least one element in common.*

In this case, the intersection of the two sets cannot be the empty set. For example, consider the three sets

$$D = \{2, 3, 5, 7\},$$
$$F = \{0, 4, 6, 9\},$$
$$G = \{3, 5, 8, 10\}.$$

Sets D and F *do not intersect* since they have no elements in common, and we write $D \cap F = \varnothing$. However, sets D and G do intersect and we write $D \cap G = \{3, 5\}$.

In listing the elements of a set, as for D, F, and G, the order in which the elements appear is not important. Thus, if $D = \{2, 3, 5, 7\}$, then we may also write $D = \{2, 5, 7, 3\}$. The intersection of two sets is independent of order, too. Thus, if A and B are any two sets, then

$$A \cap B = B \cap A.$$

Example 2 Let set $D = \{x : x \in R \text{ and } x < -3 \text{ or } x > 3\}$. Let us picture the sets

$$A = \{x : x \in R \text{ and } x < -3\}$$

and

$$B = \{x : x \in R \text{ and } x > 3\}$$

separately on two parallel number lines. (*Note:* A and B are not the same sets as in Example 1.)

The number line (a) in Figure 1-15 shows the graph of the set

$$A = \{x : x \in R \text{ and } x < -3\}$$

and the number line (b) shows the graph of the set

$$B = \{x : x \in R \text{ and } x > 3\}.$$

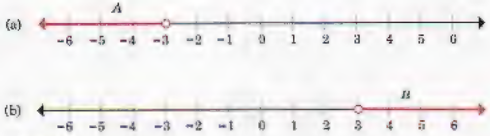


Figure 1-15

Those numbers that belong to set A *or* set B are the numbers that lie to the left of -3 *or* to the right of 3. Figure 1-16 on page 17 shows this on a single number line.

Figure 1-16

We call the set shown in Figure 1-16 the union of the two sets in Figure 1-15. The union symbol ∪ is used in forming a symbol to denote the union of two sets. Thus to indicate the union of two sets

$$A = \{x : x \in R \text{ and } x < -3\}$$

and

$$B = \{x : x \in R \text{ and } x > 3\},$$

we write $A \cup B$ and see that this union is the set

$$D = \{x : x \in R \text{ and } x < -3 \text{ or } x > 3\}.$$

Therefore $D = A \cup B$.

Example 2 illustrates how the connective "or" is related to unions in the language of sets. Thus the elements that belong to sets A or B are those elements that are either in set A or in set B or in both. The set of all elements that belong to A or B is the union of the two sets A and B which, we have seen, is the set D.

The union of two sets is independent of their order. Hence, if A and B are any sets, then $A \cup B = B \cup A$.

The ideas of union and intersection are extended easily to apply to any number of sets. For example, if A, B, and C are sets, then $A \cap B \cap C$ is the set of all elements each of which is in all three of the sets A, B, and C. Describe the intersection of the two sets A and B of Example 2.

Example 3 Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$. The union of A and B is the set of numbers in A or B, and we write

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}.$$

It should be noted that if a number is in set A, or in set B, or in both set A and set B, then this number is in $A \cup B$. Thus 1 and 2 belong in $A \cup B$ because 1 and 2 are in set A. The numbers 6 and 7 belong in $A \cup B$ because 6 and 7 are in set B. Finally, the numbers 3, 4, and 5 belong in $A \cup B$ because 3, 4, and 5 are in both set A and set B. The intersection of A and B is the set whose elements are common to both sets A and B, and we write

$$A \cap B = \{3, 4, 5\}.$$

Recall that a set A is a subset of set B if every element of A is also an element of B. (Write $A \subset B$ for "A is a subset of B.") Thus, if $A = \{a, b\}$, then each of the sets $\{a\}$, $\{b\}$, $\{a, b\}$, and $\varnothing$ is a subset of A.

EXERCISES 1.4

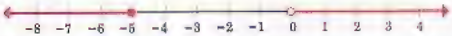
1. Let $A = \{x : x \in R \text{ and } x < 1\}$ and $B = \{x : x \in R \text{ and } x \geq -1\}$.
   (a) Graph the set A, the set B, and the set $A \cap B$ on separate number lines.
   (b) Use set-builder notation to indicate the intersection of sets A and B.
   (c) Use set-builder notation to indicate the union of sets A and B.

Exercises 2–8 refer to the sets

$$A = \{-3, -1, 1, 3\},\ B = \{-2, 0, 2, 4\}, \text{ and}$$
$$C = \{-2, -1, 0, 1, 2\}.$$

2. Find $A \cap B$.
3. Find $A \cap C$.
4. Find $B \cap C$.
5. Find $A \cup B$.
6. Find $A \cup C$.
7. Find $A \cup B \cup C$.
8. Find $(A \cup B) \cap C$.

9. The figure below shows the graph of a set of numbers on a real number line. Use set-builder notation to describe this set.



10. The figure below shows the graph of a set of numbers on a real number line. Use set-builder notation to describe this set.

1.5 CONJUNCTIONS AND DISJUNCTIONS

Similar to the relationships of two sets and their union and intersection are the relationships of two statements and their conjunction and disjunction. In mathematics it is very important to understand clearly the relationship of truth in given statements to truth in statements formed from the given ones. In this section we consider the relation of two statements to the statements formed from them by connecting them first by "and" and then by "or."

Suppose that p is a statement and that q is a statement. (By a *statement* we mean a sentence that is either true or false.) We call the statement "p and q" the conjunction of the statement p and the statement q. For example, if p is the statement

$$4 = 4$$

and q is the statement

$$4 = 5,$$

the conjunction of p and q is the statement

$$4 = 4 \qquad \text{and} \qquad 4 = 5.$$

We agree to call a conjunction of two statements true if and only if the statements are both true.

> If p is true and q is true, then p and q is true; if p and q is true, then p is true and q is true.

Thus the conjunction $4 = 4$ and $4 = 5$ is false because the statement $4 = 5$ is false. On the other hand, the conjunction $4 = 4$ and $5 = 5$ is true. Of course, if both of the statements p and q are false, then the conjunction p and q is false.

Let us see how the conjunction of two statements is related to the intersection of two sets. In Example 1 of Section 1.4, we considered the intersection of two sets

$$A = \{x : x \in R \text{ and } x > -3\} \qquad \text{and} \qquad B = \{x : x \in R \text{ and } x \leq 3\}.$$

We saw that the intersection of these two sets is the set of all numbers that belong to both A and B. In other words, if x is a number, then it is true that x is an element of A and of B ($x \in A \cap B$) if and only if it is true that x is an element of A and it is true that x is an element of B. For example, $2 \in A$ and $2 \in B$; hence $2 \in A \cap B$. On the other hand, $5 \in A$ and $5 \notin B$ (5 is not an element of B); hence $5 \notin A \cap B$.

Thus if the statements $x \in A$ and $x \in B$ are true, then the conjunction of these statements ($x \in A \cap B$) is also true. But if either of the statements $x \in A$ and $x \in B$ is false, then $x \in A \cap B$ is false.

Now consider the statement p or q where p and q are statements. We call the statement p or q the disjunction of the two statements p and q. For example, if p is the statement

$$4 = 4$$

and q is the statement

$$4 = 5,$$

the disjunction of p and q is the statement $4 = 4$ or $4 = 5$.

We agree to call the disjunction of two statements true if and only if either, or both, of the two statements is true.

► If p is true and q is true, then p or q is true; if p is true and q is false, then p or q is true; if p is false and q is true, then p or q is true; if p or q is true, then either p is true and q is false, or p is false and q is true, or p and q are both true.

Thus the disjunction $4 = 4$ or $4 = 5$ is considered to be true because $4 = 4$ is true. Similarly, the disjunction $4 = 4$ or $5 = 5$ is true since the statements $4 = 4$ and $5 = 5$ are both true. On the other hand, the disjunction $4 = 5$ or $7 < 5$ is false since both of the statements $4 = 5$ and $7 < 5$ are false.

Let us see how the disjunction of two statements is related to the union of two sets. In Example 2 of Section 1.4 we considered the union of the two sets

$$A = \{x : x \in R \text{ and } x < -3\} \qquad \text{and} \qquad B = \{x : x \in R \text{ and } x > 3\}.$$

We saw that the union of these two sets is the set of numbers belonging to either A or B. In other words, if x is a number, then it is true that x is an element of A *or* B ($x \in A \cup B$) if and only if at least one of the statements x *is an element of* A and x *is an element of* B is true. For example, $5 \notin A$, but $5 \in B$; hence $5 \in A \cup B$. On the other hand, $2 \notin A$ and $2 \notin B$; hence $2 \notin A \cup B$.

Thus, if at least one of the statements $x \in A$ and $x \in B$ is true, then the disjunction of these two statements ($x \in A \cup B$) is also true. But if both of the statements $x \in A$ and $x \in B$ are false, then the statement $x \in A \cup B$ is false.

EXERCISES 1.5

■ Exercises 1–6 refer to Figure 1-17. Imagine that each of the four lines a, b, c, d in the figure is a set of points.

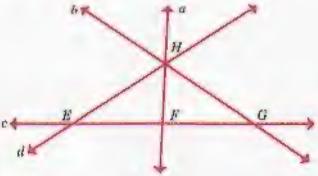


Figure 1-17

1. The set $\{E\}$ is the intersection of which two lines?
2. What is the intersection of b and c?
3. What is the intersection of the three sets a, b, and d?
4. Does $a \cap b = a \cap d$?
5. What is $a \cap c$?
6. What is the intersection of the four sets a, b, c, and d?

The following table shows the possible truth values (T for true, F for false) that can be assigned to the statements p and q. In Exercises 7–10, copy and complete the table for the statements p and q and p or q.

	p	q	p and q	p or q
7.	T	T	T	?
8.	T	F	F	?
9.	F	T	?	T
10.	F	F	?	?

11. If $x \in A$ and $x \notin B$, is x an element of $A \cap B$? Of $A \cup B$?

In Exercises 12–15, P and Q represent sets. Copy and complete the table using T for true or F for false.

	$x \in P$	$x \in Q$	$x \in P \cap Q$	$x \in P \cup Q$
12.	T	T	T	?
13.	T	F	?	?
14.	F	T	?	?
15.	F	F	?	F

16. Two sets are said to be *equal* if they contain exactly the same elements. If $A = \{x : x \text{ is an integer and } -1 < x < 5\}$ and $B = \{0, 1, 2, 3, 4\}$, does $A = B$? Why or why not? If $C = \{2, 3, 0, 1, 4\}$, does $B = C$?
17. In Exercise 16 is A a subset of B? Is B a subset of A? Is B a subset of C? Is C a subset of B?
18. If $A \subset B$ and $B \subset A$ are both true, what can you conclude?
19. If A and B are sets and $A = B$, does $B = A$?
20. If A and B are sets and $A \subset B$, is $B \subset A$?

21. (a) List all the subsets of the set $\{1, 2, 3\}$. (Remember that the empty set $\varnothing$ is a subset of every set.)
    (b) How many distinct subsets does a set consisting of three elements have? Four elements? n elements?
22. If $A = \varnothing$ and $B = \{3, 4, 5, 6\}$, what is $A \cap B$? $A \cup B$?
23. (a) Given that x is a number such that $x^2 = 16$, find one possible value for x. Is there another possible value for x? What is the solution set of the equation $x^2 = 16$?
    (b) Write the solution set of the equation $x^2 + 7 = 56$.

In Exercises 24–29, write the solution set of the equation.

24. $3x - 5 = 13$
25. $5(2x - 9) = 10$
26. $3(2x - 7) = x + 9$
27. $2(x + 3) = 5x - 8$
28. $x^2 - 5 = 31$
29. $x^2 - 2x - 15 = 0$

30. If $F = \{(x, y) : x \in R,\ y \in R, \text{ and } y = 2x - 3\}$, we find that when $x = 2$, $y = 2 \cdot 2 - 3 = 1$. Thus the ordered pair $(2, 1)$ is an element of the set F. To check this, note that "$2 \in R$, $1 \in R$, and $1 = 2 \cdot 2 - 3$" is a true statement. This is the statement you get if you replace x by 2 and y by 1 in the sentence which follows the colon in the set-builder symbol.
    (a) What value of y is paired with x when $x = \frac{1}{2}$?
    (b) Find five more ordered pairs of numbers that belong to the set F.
31. Describe the solution set of the equation $2(x - 2) = 2x - 4$.
32. Describe the solution set of the equation $2(x - 2) = 2x - 5$.

1.6 THE INCIDENCE RELATIONSHIPS OF POINTS AND LINES

We are now ready to begin the formal development of geometry. We do not state all the postulates of our formal geometry at once, but rather introduce them throughout the text as needed. In this chapter, we state the first eight of our postulates. These postulates are concerned with points on lines, in planes, and in space; lines through points, in planes, in space; and planes through points, through lines, and in space. Figure 1-18 shows a picture of a line l and a point P. As the figure suggests, point P is on line l and line l passes through point P. Since l is a set of points and P is an element of this set, it is correct to say that P is "in" l or that P is a point "of" l. However, we usually say that

Figure 1-18

P is "on" l and that l "passes through" P. Another way of saying this is to say that the point and line are incident. It is for this reason that our first postulates, which we state in this section and in Section 1.7, are called *Incidence Postulates.*

For our first postulates we draw on our experience with physical representations of points and lines. When you use your ruler or straightedge to draw a line through two points, such as P and Q in Figure 1-19, you are, of course, actually working with physical representations of points and lines and not the abstract points and lines of our geometry.

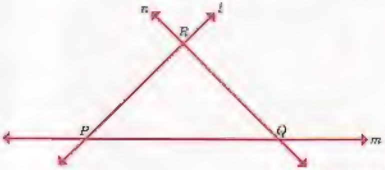


Figure 1-19

We decide what we are going to accept without proof about geometric points and lines by looking at the real world. In fact, we need only look at Figure 1-19 to arrive at our first three postulates. First, however, we state two definitions.

Definition 1.1 Space is the set of all points.

Definition 1.2 The points of a set are collinear if and only if there is a line which contains all of them. The points of a set are noncollinear if and only if there is no line which contains all of them.

Plane Postulates of Incidence

POSTULATE 1 *(The Three-Point Postulate)* Space contains at least three noncollinear points.

POSTULATE 2 *(The Line-Point Postulate)* Every line is a set of points and contains at least two distinct points.

POSTULATE 3 *(The Point-Line Postulate)* For every two distinct points, there is one and only one line that contains both points.

Postulate 3 is sometimes shortened to read "two points *determine* a line." When we use the word "determine" here, we use it in the sense of Postulate 3, which states that, given two distinct points, there is exactly one line which contains them. Similarly, when we say that "three noncollinear points determine three lines," we mean that there are exactly three lines each containing two of the three points.

You may feel that these postulates do not tell us very much about points and lines. Your experience has led you to believe that there are a great many points on a line and in space, and you may wonder why we do not say so in the postulates. We feel that it is more instructive to proceed as we have done. Shortly we will have introduced enough postulates that we will be able to prove that there are infinitely many points on a line and in space. However, there is not much we can prove from only the three postulates. Figure 1-19 should suggest to you at least two statements concerning lines that are not stated in the postulates. Before formulating these statements, we will introduce some symbols for points, lines, and planes.

Notation. Capital (upper case) letters are used to denote points and small (lower case) letters to denote lines. For example, in Figure 1-19 P, Q, and R denote points and l, m, and n denote lines. We may also name a line by naming two points that determine the line. Thus either of the symbols $\overleftrightarrow{PQ}$ or $\overleftrightarrow{QP}$ could be used to name the line that is determined by points P and Q. It may also be convenient to name several different lines by using the same letter with distinguishing subscripts such as $l_1, l_2, l_3, \ldots$.

In the figures we represent a line by the symbol $\longleftrightarrow$ where the arrowheads are used to indicate that the line does not stop where our picture stops but continues indefinitely in both directions.

We often use the Greek letters $\alpha, \beta, \gamma, \ldots$ (alpha, beta, gamma, . . .) to name a plane. Sometimes it is convenient to name a plane by naming three noncollinear points that the plane contains, such as "plane PQR" in Figure 1-19.

Let us return now to our discussion of the relationships between points and lines. Figure 1-19 suggests, but our postulates do not tell us, that there are at least three distinct lines in space. Let us now attempt to deduce this statement from the three postulates we already have.

The Three-Point Postulate tells us that there are at least three points in space that do not all lie on one line. For convenience we name these points P, Q, and R. It follows from the Point-Line Postulate that

points P and Q determine exactly one line $\overleftrightarrow{PQ}$, that points Q and R determine exactly one line $\overleftrightarrow{QR}$, and that points P and R determine exactly one line $\overleftrightarrow{PR}$. No two of these lines are the same. For if they were, points P, Q, and R would be collinear, and this would contradict the fact that P, Q, R do *not* all lie on one line. This completes the proof of the following theorem. (Usually the proof of a theorem follows the statement of the theorem. For Theorem 1.1, however, the proof precedes the statement of the theorem.)

THEOREM 1.1 There are at least three distinct lines in space.

We observe from Figure 1-19 that no two distinct lines intersect in more than one point. We proceed now to deduce this statement from our postulates.

If l_1 and l_2 are any two distinct lines that intersect, then, by definition of intersect, we know they have at least one point in common. Call this point P. Now, either they have a second point in common or they do not. Let us suppose that they do have a second point Q in common as suggested in Figure 1-20.

Figure 1-20

The Point-Line Postulate tells us that for every two distinct points there is one and only one line that contains them. Thus $l_1 = l_2$ (that is, they are the same line). But this contradicts our hypothesis that l_1 and l_2 are distinct lines. Since our supposition that l_1 and l_2 intersect in a second point leads us to a contradiction, we must conclude that l_1 and l_2 intersect in no more than one point. We have proved the following theorem.

THEOREM 1.2 If two distinct lines intersect, then they intersect in exactly one point.

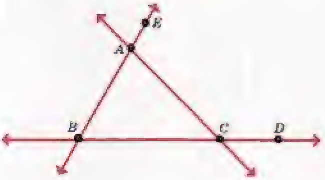
Does Figure 1-20 confuse you because the marks that represent lines do not look as you think straight lines should look? Actually the word *straight* is not part of our formal geometry. We want a formal

geometry of points, lines, and planes that is consistent with our experiences in informal geometry. In particular, we want lines in our formal geometry to have the property of straightness. It is true that we cannot draw a convincing picture of two distinct lines with two distinct points in common. This is evidence at the informal geometry level consistent with Theorem 1.2.

EXERCISES 1.6

1. Which postulate asserts that there are at least two distinct points on every line?
2. Restate the Line-Point Postulate and the Point-Line Postulate in your own words.
3. Restate the definition of "intersect" as the word applies to two sets of points.
4. If points A, B, C, and D are distinct, and if line l contains points A, B, and C and line m contains points A, C, and D, what can you conclude from this?
5. In proving Theorem 1.1 we started with three noncollinear points and showed that there are three distinct lines each containing two of the three noncollinear points. Suppose we start with four points, no three of them collinear. How many distinct lines are there each containing two of the four points?
6. Suppose we start with five distinct points, no three of them collinear. How many distinct lines are there each containing two of the five points?
7. Suppose we start with six distinct points, no three of them collinear. How many distinct lines are there each containing two of the six points?
8. Consider your answers to Exercises 5, 6, and 7. Can you predict how many lines there are for seven distinct points each containing two of the seven points if no three of the points are collinear? If so, how many?
9. Let m and n be different lines. Let A be a point such that $A \in m$ and $A \in n$. Let B be a point such that $B \in m$ and $B \in n$. What must be true of points A and B? What postulate or theorem supports your answer?
10. Let A and B be different points. Let line l_1 contain A and B and let line l_2 contain A and B. What can you conclude about l_1 and l_2? What postulate or theorem supports your conclusion?

11. Consider the set consisting of the five points, and no others, named in the following figure. If the points appear to be collinear, assume that they are.



(a) Identify, by listing the members, two subsets each containing three collinear points.
(b) Identify, by listing the members, four subsets each containing four noncollinear points of which three points are collinear.
(c) List the members of one subset containing four noncollinear points of which no three are collinear.
(d) Name three lines which are not drawn in the figure, but each of which contains two points named in the sketch.

12. From which postulate does it follow that no line contains all points of space?

CHALLENGE PROBLEM. Start with three distinct objects such as three boys named Jerry, Jim, and John. Create a structure as follows: Each boy is a point and there is no other point. Thus space = {Jerry, Jim, John}. There are three distinct lines a, b, and c (and no others) as follows: a = {Jerry, Jim}, b = {Jim, John}, c = {Jerry, John}. Exercises 13–30 are questions about the Jerry-Jim-John structure. Exercises 31–34 are general questions.

13. Are the points, Jerry, Jim, and John, collinear?
14. Does space contain at least three noncollinear points?
15. Does the structure satisfy the Three-Point Postulate?
16. Is a a set of points?
17. Does a contain at least two distinct points?
18. Is b a set of points?
19. Does b contain at least two distinct points?
20. Is c a set of points?
21. Does c contain at least two distinct points?
22. Does the structure satisfy the Line-Point Postulate?
23. Is {Jerry, Jim} a set of two distinct points?

24. Is {Jerry, John} a set of two distinct points?
25. Is {Jim, John} a set of two distinct points?
26. Are there other sets of two distinct points?
27. Is there one and only one line that contains Jim and John?
28. Is there one and only one line that contains John and Jerry?
29. Is there one and only one line that contains Jerry and Jim?
30. Does the structure satisfy the Point-Line Postulate?
31. Can we prove, using only the plane postulates of incidence, that space contains at least four distinct points?
32. What do the Plane Postulates of Incidence tell us about the nature of an individual point?
33. What do the first three postulates tell us about the straightness of a line?
34. What do the first three postulates tell us about segments?

1.7 THE INCIDENCE RELATIONSHIPS OF POINTS, LINES, AND PLANES

We are now ready to list our remaining Incidence Postulates. We begin by stating the following useful definition.

Definition 1.3 The points of a set are coplanar if and only if there is a plane which contains all of them. The points of a set are noncoplanar if and only if there is no plane which contains all of them.

Space Postulates of Incidence

POSTULATE 4 (*The Four-Point Postulate*) Space contains at least four noncoplanar points.

POSTULATE 5 (*The Plane-Point Postulate*) Every plane is a set of points and contains at least three noncollinear points.

POSTULATE 6 (*The Point-Plane Postulate*) For every set of three noncollinear points there is one and only one plane that contains them.

POSTULATE 7 (*The Flat-Plane Postulate*) If two distinct points of a line belong to a plane, then every point of the line belongs to that plane.

POSTULATE 8 (*The Plane-Intersection Postulate*) If two distinct planes intersect, then their intersection is a line.

Note that our postulates do not include a statement about the number of planes in space. Theorem 1.3 states that there are at least two distinct planes in space. In the Exercises that follow you will be asked to prove that there are at least four distinct planes in space.

From Postulate 1 we know that there are at least three noncollinear points in space. Call these points A, B, and C. By Postulate 6 there is exactly one plane α that contains these points. From Postulate 4 there is a fourth point D that is not in plane α. Points A, B, and D are not collinear, for if they were, point D would lie in plane α. Why? By Postulate 6 again there is exactly one plane β that contains points A, B, and D. Planes α and β are not the same plane, for if they were, points A, B, C, and D would be coplanar, which is contrary to the way they were chosen. We have proved the following theorem.

THEOREM 1.3 Space contains at least two distinct planes.

It is often helpful to draw diagrams or pictures showing the relationships between points, lines, and planes when making a deduction such as the one for Theorem 1.3. Figure 1-21 shows points A, B, and C in plane α and points A, B, and D in plane β. (Although we often use a quadrilateral to represent a plane, be careful not to think of the sides of the quadrilateral as the "edges" or "ends" of the plane.)

Figure 1-21 also helps us to "see" that when two planes intersect, their intersection is a line as we have stated in Postulate 8. What is the intersection of a line and a plane not containing the line? When we look at Figure 1-22, it appears that the intersection is a single point.

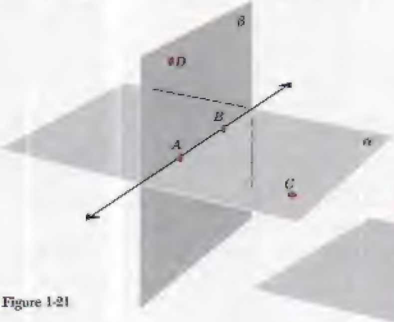


Figure 1-21

Figure 1-22

THEOREM 1.4 If a line intersects a plane that does not contain the line, then the intersection is a single point.

We can prove Theorem 1.4 by the same technique used in proving Theorem 1.2. Read again the proof of Theorem 1.2 which appears before the statement of the theorem. Now try to answer the questions in the argument (proof) that follows.

By hypothesis and the definition of intersect, line m and plane α (Figure 1-22) have at least one point P in common. How do we know that plane α does not contain every point of line m? Suppose line m and plane α have a second point Q in common. (Here we are examining one of only two possibilities: either the line and plane have a second point in common or they do not. Of course, we are trying to prove that they do not.) But if a plane contains two distinct points of a line, then it contains the entire line. (What postulate are we using here?) What does this last conclusion, namely, that the plane contains the entire line, contradict? Since we have agreed to accept our postulates as true statements about points, lines, and planes, what must we conclude? Does this prove the theorem?

It follows from Postulate 6 (the Point-Plane Postulate) that three noncollinear points determine exactly one plane. That is, there is exactly one plane which contains any set of three noncollinear points.

We conclude this section with two additional theorems regarding the existence of planes.

THEOREM 1.5 If m is a line and P is a point not on m, then there is exactly one plane that contains m and P.

Note that we must prove two things.

1. There is at least one plane that contains m and P.
2. There is no more than one plane that contains m and P.

These two statements illustrate the ideas of *existence* and *uniqueness*. When we prove existence, we show that there is *at least one* plane containing m and P. When we prove uniqueness, we show that there is *at most* one such plane. If we can prove both existence and uniqueness, then we know that there is *exactly one* such plane.

In the following proof name the postulate or theorem that justifies the statement preceding the question "Why?".

Proof of 1. **Existence.** There is a plane α that contains line m and point P (Figure 1-23).

There are at least two distinct points on line m. Why? Call these points A and B. The points A and B do not lie on any line except m. Why? By hypothesis, point P does not lie on line m. Hence points A, B, and P are noncollinear. There is a plane α that contains points A, B, and P. Why? If plane α contains points A and B, then it contains line m. Why? Hence there is a plane that contains line m and point P.

Figure 1-23

Proof of 2. ***Uniqueness.*** There is no more than one plane that contains line m and point P. Suppose that there is a second plane β which contains m and P. Then β contains the noncollinear points A, B, and P. Thus we have two distinct planes, α and β, each containing three noncollinear points. What postulate does this contradict? Does this prove that there is no more than one plane containing m and P?

THEOREM 1.6 If two distinct lines intersect, then there is exactly one plane that contains them.

Let lines m and n intersect in the point P (Figure 1-24).

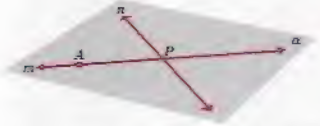


Figure 1-24

We must prove the following:

1. ***Existence.*** There is at least one plane that contains m and n.
2. ***Uniqueness.*** There is no more than one plane that contains m and n.

In the following proofs name the postulate or theorem that justifies the statement preceding the question "Why?"

Proof of 1: Lines m and n have exactly one point in common. Why? There is a point A on m that is different from P. Why? A is not on n by Theorem 1.2. (Two distinct lines intersect in no more than one point.) There is a plane α which contains A and n. Why? Plane α contains P and A and therefore contains m. Why? Hence there is a plane α which contains m and n.

Proof of 2: Suppose there is a second plane β that contains both m and n. Then plane β contains A because A is on m. Thus both α and β contain A and n. What theorem does this last statement contradict? Does this prove that there is no more than one plane α containing m and n?

EXERCISES 1.7

Refer to Figure 1-25 in working Exercises 1–12. Assume that points A, B, and C are in the plane of the paper and that point D is not.

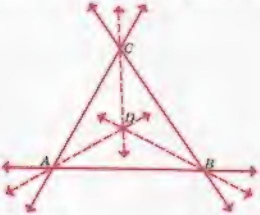


Figure 1-25

1. Name all the planes, such as ABC, that are determined by the points in the figure.
2. Name all the lines that are determined by the points in the figure. One of them is line $\overleftrightarrow{AB}$.
3. How many lines are there in each plane?
4. How many planes contain line $\overleftrightarrow{BD}$? Name them. Is this number of planes the same for every line?
5. How many lines are on every point? Name the lines that are on point D.
6. How many points do plane ABC and line $\overleftrightarrow{DC}$ have in common? What theorem justifies your conclusion?

7. Name two planes, and name a point that is contained in both planes. Which postulate tells us that these two planes have a line in common? Name the line.
8. There are at least how many points common to every pair of planes in the figure?
9. Two planes which do not intersect are said to be *parallel*. On the basis of our first eight postulates, do you think it can be proved that there must be two parallel planes in space?
10. Is it possible for three planes to have one and only one point in common? If your answer is "Yes," name three planes and the single point they have in common.
11. Two noncoplanar lines in space are said to be *skew* lines. Name three pairs of lines in the figure such that each pair is a pair of skew lines. Can a pair of skew lines intersect? Why?
12. Can three lines have a single point in common and be noncoplanar lines?

13. Let P be a point in plane α. On the basis of only our first eight postulates do you think that we can conclude that there are three distinct lines in plane α that contain point P? Give reasons for your answer.
14. How many points do the planes of the ceiling, front wall, and side wall of your classroom have in common? On the basis of only our first eight postulates do you think it can be proved that there are three distinct planes that have more than one point in common?
15. Using three pages of your book as physical examples of planes, what would you guess to be the intersection of three distinct planes if their intersection contained more than one point?
16. Lay the edge of your ruler along the top of your desk. Do all the points on the edge of the ruler seem to touch the desk? Which of our postulates does this illustrate?

In Exercises 17–25, write a short paragraph to show how our postulates or theorems can be used to prove the statement.

17. Every line is contained in at least one plane.
18. There are at least three distinct lines in every plane.
19. There are at least four distinct planes in space.
20. There are at least six distinct lines in space.
21. If four points are noncoplanar, then they are noncollinear.
22. If four points are noncoplanar, then any three of these points are noncollinear.
23. There are at least three distinct lines through every point.
24. Every line is contained in at least two distinct planes.
25. Every point is contained in at least three distinct planes.

26. CHALLENGE PROBLEM. Suppose, for this one exercise, that we replace our Postulate 8 with this postulate: "If two distinct planes have one point in common, then they have a second point in common." Prove our replaced Postulate 8 as a theorem.

27. CHALLENGE PROBLEM. Suppose, just for Exercises 27 and 28, that space consists of four distinct points A, B, C, D and no others. Suppose that there is one line a containing all four of the points A, B, C, D and that there are no other lines. Suppose that there are no planes. Does this structure satisfy Postulate 4? Since A, B, C, and D are not contained in any plane, it follows that they are noncoplanar, and hence Postulate 4 is satisfied.

    Does this structure satisfy Postulate 5? Yes, it does. To show this we must check each plane in the structure to see if it contains three noncollinear points. If there is no plane that violates the property of Postulate 5, then Postulate 5 is satisfied. Since there is no plane in the structure, it follows that there is no plane that violates Postulate 5, and hence Postulate 5 is satisfied. Show that Postulates 6, 7, and 8 are satisfied by this structure.

28. CHALLENGE PROBLEM. Does the structure of Exercise 27 satisfy Postulate 1? Postulate 2? Postulate 3? Your intuition might tell you that Postulate 4 includes Postulate 1 in the sense that Postulate 1 could be proved once Posulate 4 is assumed. The structure of Exercise 27 provides an answer to the question: Is it possible to satisfy Postulate 4, indeed all of the postulates from 2 through 8, without satisfying Postulate 1? In other words, is Postulate 1 independent of Postulates 2 through 8? Why?

CHAPTER SUMMARY

In this first chapter we compared informal geometry and formal geometry noting that geometry began informally many years ago as a collection of rules to solve practical problems related to "earth measurement." Although geometry as a school subject is both formal and informal, the approach in this book is predominantly formal.

The formal approach features carefully stated postulates, definitions, and theorems. Proving theorems involves deductive reasoning. We discussed examples of intuitive reasoning, inductive reasoning, and deductive reasoning.

We reviewed the language of sets. If A and B are sets, the INTERSECTION of A and B, denoted by $A \cap B$, is the set of elements contained both in A *and* in B. The UNION of A and B, denoted by $A \cup B$, is the set of elements contained in A *or* B. The intersection of two sets may be the EMPTY or NULL SET denoted by $\varnothing$. If two sets INTERSECT, they must have at least one element in common.

If p and q are statements, the CONJUNCTION of these two statements is the statement p and q. The conjunction of p and q is true if both p and q are true, but false in all other cases. The DISJUNCTION of the two statements is the statement p or q. The disjunction of p and q is false if both p and q are false, but true in all other cases.

In order to avoid circular definitions we accepted POINT, LINE, and PLANE as UNDEFINED TERMS in our formal geometry.

We defined SPACE to be the set of all points. The points of a set are COLLINEAR if and only if there is a line that contains all of them. The points of a set are COPLANAR if and only if there is a plane that contains all of them.

Our first eight postulates are called INCIDENCE postulates and are listed below by name only. Be sure that you can state each of them in your own words.

1. THE THREE-POINT POSTULATE
2. THE LINE-POINT POSTULATE
3. THE POINT-LINE POSTULATE
4. THE FOUR-POINT POSTULATE
5. THE PLANE-POINT POSTULATE
6. THE POINT-PLANE POSTULATE
7. THE FLAT-PLANE POSTULATE
8. THE PLANE-INTERSECTION POSTULATE

In addition to these eight postulates we stated and proved six theorems. Read these theorems again and study their proofs.

REVIEW EXERCISES

In Section 1.1 an example involving the area of a right triangle was given to illustrate intuition or intuitive reasoning. In Exercises 1–10, state whether or not you think the given example is a good example of intuitive reasoning. If you think the decision made is false, indicate it.

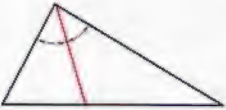
1. Examine a rectangular box and decide that the opposite faces of a rectangular solid have the same size and shape.
2. Examine a ball and decide that a plane through the center of a spherical region (a sphere and its interior) would separate it into two hemispherical regions with equal volumes.
3. Examine a ball and decide that the area of a sphere is four times the area of a circular cross section formed by a plane passing through the center of the sphere.
4. Examine three distinct points on a line and decide that if A, B, C are any three distinct points of a line with B between A and C, then the distance from A to B is less than the distance from A to C.

5. Think about a triangle T lying in a plane α. Think about a line in α that passes through a point P lying inside of T. Decide that this line must contain two distinct points of T.

6. Think about a triangle, a quadrilateral, and a pentagon. Decide that the sum of the measures of the angles is the same for each.

7. Think about three books of 300 pages each standing upright on a shelf as suggested in the figure. Decide that it is farther from page 1 of book I to page 300 of book III than it is from page 300 of book I to page 1 of book III.

8. Think about a plane figure consisting of a triangle and the bisector of one of its angles as suggested in the following figure. Decide that the bisector of an angle of a triangle divides the opposite side into two segments of equal length.



9. Think about a circle and a line that separates it into two arcs of equal length. Decide that the line passes through the center of the circle.

10. Draw a triangle. Mark one point on each side as close to the midpoint as you can without measuring. These points are vertices of a second triangle. Decide that the area of the original triangle is four times the area of the second triangle.

■ In Section 1.1 an example involving the sum of the measures of the angles of a triangle was given to illustrate inductive reasoning. In Exercises 11–20 draw one or more figures and make some measurements if you wish. Then draw an appropriate conclusion.

11. Is the following statement true or false? The bisectors of the angles of a triangle are concurrent, that is, they all pass through the same point.

12. What is the sum of the measures of the angles of a pentagon? (Consider only pentagons whose diagonals pass through the interior of the pentagons.)

13. If $ABCD$ is a quadrilateral with $AB = BC$, is it necessarily true that $CD = DA$?

14. If $ABCD$ is a quadrilateral with $AC = BD$, is it necessarily true that $AB = BC$?
15. If the sides of a quadrilateral have equal lengths, is it necessarily true that its diagonals have equal lengths?
16. If the diagonals of a quadrilateral bisect each other, is the quadrilateral necessarily a parallelogram?
17. If the sides of $\triangle ABC$ have equal lengths, if D is the midpoint of $\overline{BC}$, if E is on line $\overleftrightarrow{DA}$, if D is between A and E, and if $AD = DE$, is it true that $\overleftrightarrow{BE}$ is parallel to $\overleftrightarrow{AC}$?
18. If $\triangle ABC$ is any triangle and if D is the midpoint of $\overline{BC}$, is it necessarily true that $\triangle ABD$ and $\triangle ACD$ have equal areas?
19. If $\triangle ABC$ is a triangle and if D is the midpoint of $\overline{BC}$, is it true that
$$AD = \tfrac{1}{2}(AB + AC)?$$
20. If $\triangle ABC$ is a scalene triangle (no two of its sides have the same length), is it true that no two of its angles have equal measures?

Section 1.2 contains two examples of deductive reasoning. In Exercises 21–25, state whether you think the conclusion C ($C_1, C_2, \ldots,$ if more than one) follows logically from the hypothesis H ($H_1, H_2, \ldots,$ if more than one). Be prepared to explain what assumptions you are making, if any, in addition to those that are stated as hypotheses.

21. H: $x = 5$
    C: $2x + 3 = 13$
22. H: $2x + 3 = 13$
    C: $x = 5$
23. H_1: Every polygon is a plane figure.
    H_2: P is a polygon.
    C: P is a plane figure.
24. H: Let $\triangle ABC$ and $\triangle DEF$ be given. If $m\angle A = m\angle D$, $m\angle B = m\angle E$, and $AB = DE$, then $AC = DF$ and $BC = EF$.
    C_1: Let $\triangle ABD$ be given with C between A and D. If $m\angle ABC = m\angle DBC$ and $m\angle ACB = m\angle DCB$, then $AC = CD$.
    C_2: Let $\triangle ABC$ and $\triangle DEF$ be given. If $m\angle A = m\angle D$, $AB = DE$, and $AC = DF$, then $BC = EF$.
25. H_1: All cows eat grass.
    H_2: Bessy is not a cow.
    C: Bessy does not eat grass.

26. If A is the set of even integers and B is the set of odd integers, describe the union of A and B; the intersection of A and B. Do sets A and B intersect?

Exercises 27–38 refer to the three sets

$Q = \{-3, 0, 3, 6\}$,
$S = \{-3, 0, 3, 6, 9\}$,
$T = \{x : x \text{ is an integer, } -3 \leq x < 9, \text{ and } x \text{ is divisible by } 3\}$.

27. Is $Q \subset S$?
28. Is $S \subset Q$?
29. Is $Q \subset T$?
30. Is $T \subset Q$?
31. Does $Q = T$?
32. Is $S \subset T$?
33. Is $T \subset S$?
34. Does $S = T$?
35. Find $Q \cap S$.
36. Does $Q \cap S = Q$?
37. Find $Q \cup S$.
38. Does $Q \cup S = S$?

39. If $A = \{x : x \in R \text{ and } x < 7\}$ and $B = \{x : x \in R \text{ and } x > 3\}$, draw the graphs of A, B, and $A \cap B$ on three separate number lines.
40. Describe the set $A \cap B$ in Exercise 39 using set-builder notation.
41. If $S = \{x : x \in R \text{ and } x \geq 5\}$ and $T = \{x : x \in R \text{ and } x \leq 0\}$, draw the graphs of S, T, and $S \cup T$ on three separate number lines.
42. Describe the set $S \cup T$ in Exercise 41 using set-builder notation.

If p is the statement $7 = 7$, q is the statement $\sqrt{9} = 5$, and r is the statement $-5 < -2$, determine in Exercises 43–49 if the given "compound" statement is true or false.

43. p and q
44. p or q
45. p and r
46. p or r
47. q and r
48. q or r
49. (p and q) or r

50. State the Flat-Plane Postulate in your own words.
51. The following figure is a picture of two distinct "curves" that intersect in two distinct points. State a postulate or theorem that would settle an argument about whether these "curves" might be lines in our formal geometry.

52. If P and Q are distinct points and m and n are lines, and if $P \in m$, $Q \in m$, $P \in n$, and $Q \in n$, what conclusion can you draw? What postulate or theorem justifies this conclusion?
53. If m and n are distinct lines and P and Q are points, and if $m \cap n = P$ and $m \cap n = Q$, what can you conclude? What postulate or theorem justifies this conclusion?

54. Which postulate assures us that no plane contains all the points of space?
55. Prove, using only the postulates, that every point is contained in at least one plane.
56. Draw a single diagram which illustrates what all eight postulates assert. Label the points in your diagram and then name one pair of skew lines.
57. What does it mean to say that points A, B, and C are collinear?
58. Does the Three-Point Postulate eliminate the possibility of there being three points in a plane that are collinear?
59. Assume that A, B, C, D, and E are five distinct points in the same plane with no three of these points collinear.
    (a) How many distinct lines are there each containing two of the five points?
    (b) How many lines determined by A, B, C, D, and E are there on each point?
60. Let A, B, C, D, and E be five distinct points in the same plane.
    (a) If A, B, and C are on line l, and C, D, and E are on line m, and $l \neq m$, how many more lines are there each containing two of the five points? How many lines are there altogether?
    (b) How many distinct lines pass through point A? Point B? Point C?
61. If line l contains three distinct points A, B, and C and if points A and B lie in plane α, what can you conclude?
62. Planes RST and STU are distinct.
    (a) Name a line that lies in both of these planes.
    (b) Name two more planes determined by points R, S, T, U.
63. State what intersect means
    (a) with regard to two lines.
    (b) with regard to two planes.
64. Which of our Incidence Postulates "pushes" us off a plane into space?

■ Determine in Exercises 65–71 if the given statement is true or false.

65. It is a theorem that every line contains at least two distinct points.
66. The Line-Point Postulate assures us that there is just one line which contains two distinct points.
67. It is a theorem that there are at least three distinct lines in every plane.
68. If a line intersects a plane not containing the line, the intersection is a single point.
69. It is a theorem that there are at least four distinct planes in space.
70. Our Incidence Postulates assure us of at least twelve distinct lines in space.
71. The Three-Point Postulate assures us that there are at least three distinct points on every line.

Chapter 2

Syd Greenberg/D.P.I.

Separation and Related Concepts

2.1 INTRODUCTION

In this chapter we list our second group of postulates. These postulates are concerned with points between other points, how a point separates a line, how a line separates a plane, and how a plane separates space. The ideas these postulates convey are very simple ones, and the information they give can be easily determined by looking at pictures.

We list these postulates so that we can give a meaning to the relationships of betweenness and separation and to make it quite clear what statements we agree to use in our proofs. Later in the chapter we use these relationships to make several definitions.

2.2 THE BETWEENNESS POSTULATES

In this section we state the Betweenness Postulates. They are sometimes called **order** postulates because they tell us how points are arranged in order on a line. For example, if A, B, C are three distinct points on a line, and if B is *between* A and C, then we could say that the points are arranged on the line in the order A, then B, then C. We could also say that the order is C, then B, then A. (See Figure 2-1.)

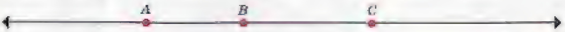


Figure 2-1

In our formal geometry *between* and *order*, as they apply to points, are undefined terms. We do not explain by a definition what it means to say that point B is between points A and C. Rather we accept the following Betweenness Postulates as formal statements of betweenness properties.

The Betweenness Postulates

POSTULATE 9 (***The A-B-C Betweenness Postulate***) If point B is between points A and C, then point B is also between C and A, and all three points are distinct and collinear.

Note that the postulate does not allow us to say that "point B is between points A and C" if the points are as shown in Figure 2-2a.

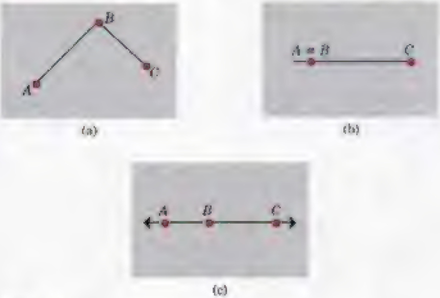


Figure 2-2

Why? Similarly, in Figure 2-2b point B is not between points A and C. Why? Figure 2-2c is a correct picture of what we mean when we say "point B is between points A and C." We use the symbol A–B–C or the symbol C–B–A when we wish to say that "point B is between points A and C."

POSTULATE 10 (***The Three-Point Betweenness Postulate***) If three distinct points are collinear, then one and only one is between the other two.

From this postulate it follows that if A, B, C are three distinct points on a line, then B–A–C, or A–B–C, or B–C–A; and if A–B–C, for example, then we cannot have B–A–C or B–C–A. From this postulate it also

follows that betweenness for points in our formal geometry is different from betweenness on a circle. For example, if three people are seated at a circular table, then most of us would agree that each of them is between the other two. But we should also now agree, after accepting Postulate 10, that betweenness around a table is different from betweenness in our formal geometry.

POSTULATE 11 (*The Line-Building Postulate*) If A and B are any two distinct points, then there is a point X_1 such that X_1 is between points A and B, a point Y_1 such that B is between A and Y_1, and a point Z_1 such that A is between Z_1 and B.

Figure 2-3 illustrates the meaning of Postulate 11. We call this postulate the Line-Building Postulate because it enables us to prove that there are infinitely many points on a line.

Figure 2-3

From this postulate it follows that there is a point Y_1 on line $\overleftrightarrow{AB}$ that is beyond B from A, a point Y_2 that is beyond Y_1 from B, a point Y_3 that is beyond Y_2 from Y_1, and so on (Figure 2-4).

Figure 2-4

Thus we are able to find on the line as many points as we choose, 3 or 30 or 3,000,000 or any natural number you wish, that are beyond B from A. Similarly, we can find as many points as we choose that are beyond A from B (Figure 2-5).

Figure 2-5

From this postulate it follows also that no matter how close together two distinct points may be there is always a point between them (Figure 2-6).

Figure 2-6

Does your intuition tell you that Postulate 11 builds every line completely, that no "holes" could possibly exist in a line? Actually, Postulate 11 assures us that there are infinitely many points on every line and furthermore, infinitely many points between every two points of a line. But it does not assure us that there are no holes in a line. The postulational basis for lines is completed in Chapter 3, where we adopt a Ruler Postulate. This postulate provides for the ideas of "continuity" and "infinite extent."

EXERCISES 2.2

In Exercises 1–22, indicate by T or F whether or not the statement is true or false.

In Exercises 1–4, R, S, T are points such that R–S–T.

1. R, S, T are collinear points.
2. R, S, T are distinct points.
3. T–S–R
4. S–R–T

In Exercises 5–10, points A and B are on line m, point C is between A and B and on line n, and m and n are distinct lines.

5. m and n intersect at A.
6. m and n intersect at B.
7. m and n intersect at C.
8. A, B, C are collinear points.
9. A–B–C
10. C–B–A

In Exercises 11–18, P, Q, R are distinct points and no one of them is between the other two.

11. P, Q, R are collinear points.
12. P and Q are collinear points.
13. P and R are collinear points.
14. Q and R are collinear points.
15. There is one and only one plane that contains P, Q, and R.

16. There is a point S such that Q–R–S.
17. There is a point T such that Q–T–R.
18. There is a point U such that U–Q–R.

In Exercises 19–22, R, S, T are three distinct collinear points, R–S–T is false, and S–R–T is false.

19. R–T–S
20. S–T–R
21. T–R–S
22. T–S–R

Exercises 23–32 refer to Figure 2-7 which shows three noncollinear points A, B, C and the three lines and the one plane that they determine.

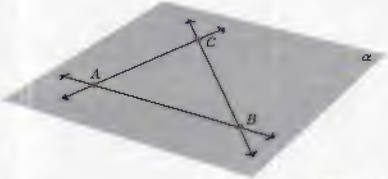


Figure 2-7

23. Which Incidence Postulate asserts the existence of three noncollinear points?
24. Which Incidence Postulate assures us, once we have points A, B, C, that lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ exist?
25. Which Incidence Postulate assures us, once we have points A, B, C, that there is a plane α containing them?
26. Which postulate assures us, once we have the points and the lines of the figure, that there are points D, E, F such that B–D–C, C–E–A, A–F–B?
27. Explain why D and E are distinct points.
28. Why is D–E–A a false statement?
29. Why is E–A–F a false statement?
30. Which postulate assures us that there are points G and H such that E–F–G and D–F–H?
31. Explain why H and G are distinct points.
32. Explain why $\overleftrightarrow{AB}$ and $\overleftrightarrow{HG}$ are distinct lines.

■ Exercises 33–35. The Incidence Postulates assert the existence of four noncoplanar points. Figure 2-8 suggests four noncoplanar points A, B, C, D and the four planes and six lines determined by them.

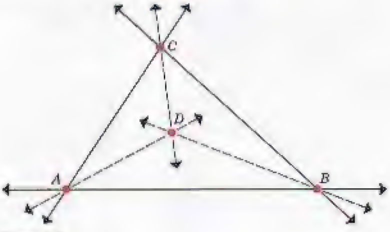


Figure 2-8

33. Deduce from the Incidence and Betweenness Postulates that there is at least one point in space that is not contained in any of the four planes of the figure.
34. Deduce from the postulates that there is at least one line in space that is not contained in any of the four planes of the figure.
35. Deduce from the postulates that there is at least one plane in space that is different from any of the four planes of the figure.

■ Exercises 36–42 refer to Figure 2-9 which shows a line and a portion of it (the heavier part including the points C and D) called a segment. (Segment is defined in the next section.) Points X, Y, Z as well as C and D are points of this line as indicated in the figure.

Figure 2-9

36. Are X and Y points of the segment?
37. Is Z a point of the segment?
38. Is C–X–D?
39. Is C–Y–D?
40. Is C–Z–D?
41. Is C–C–D?
42. Try to write a definition of segment.

43. CHALLENGE PROBLEM. Let a line l be given. Deduce from the postulates that there are at least three different planes containing l.
44. CHALLENGE PROBLEM. Let P be a point in a plane α. Deduce from the postulates that there are at least three different lines in α passing through P.

2.3 USING BETWEENNESS TO MAKE DEFINITIONS

We now list several definitions that make use of the betweenness relation. Most of these are definitions of terms you have met in your earlier work in geometry. However, it is the betweenness relation that enables us now to make these definitions precise.

Figure 2-10 is a picture of a set of points called a *segment*.

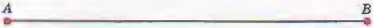


Figure 2-10

> ***Definition 2.1*** If A and B are any two distinct points, segment $\overline{AB}$ is the set consisting of points A, B, and all points between A and B. The points A and B are called endpoints of $\overline{AB}$.

Sometimes a segment is called a line segment to avoid possible confusion with a "segment" of a circle. Be careful not to confuse the symbol $\overline{AB}$ for a segment with the symbol $\overleftrightarrow{AB}$ for a line. The horizontal bar in $\overline{AB}$ reminds us that a segment has endpoints, whereas the bar with arrows in $\overleftrightarrow{AB}$ reminds us that a line has no endpoints.

We sometimes say that two distinct points determine a segment. Thus, if A and B are two distinct points, then the segment determined is $\overline{AB}$. Similarly, if A, B, C are three distinct points, then the segments determined are $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

Figure 2-11

Figure 2-11 is a picture of a set of points called a *ray*. As the figure indicates, a ray has just the one endpoint A. Speaking informally we might say that the ray continues without end from A through B in a straight line. We express the idea of a ray formally in our next definition.

> ***Definition 2.2*** If A and B are any two distinct points, ray $\overrightarrow{AB}$ is the union of segment $\overline{AB}$ and all points X such that A–B–X. The point A is called the endpoint of $\overrightarrow{AB}$.

In the symbol $\overrightarrow{AB}$, the arrow reminds us that it is a ray. The arrow in this symbol is always drawn from left to right regardless of the direction in which the ray points. However, do not confuse the symbol $\overrightarrow{AB}$ with the symbol $\overrightarrow{BA}$. Both represent rays, but they do not represent the same ray, as seen in Figure 2-12. Observe that ray $\overrightarrow{AB}$ has endpoint A, whereas ray $\overrightarrow{BA}$ has endpoint B.

Figure 2-12

It is often convenient to speak about *opposite rays*. Figure 2-13 is a picture of what we mean by opposite rays. The figure suggests the following definition.

Figure 2-13

***Definition* 2.3** If A is between B and C, then rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays.

The betweenness relation C–A–B in this definition requires opposite rays to be collinear. Why? They must also have a common endpoint. Thus the two rays shown in Figure 2-14 are not opposite rays even though they "point in opposite directions."

Figure 2-14

Note that the definition of opposite rays is equivalent to saying that two distinct rays are opposite rays if they are collinear and have the same endpoint.

We often use the symbol "$opp\,\overrightarrow{AB}$" for the ray opposite $\overrightarrow{AB}$. Thus, in Figure 2-13, $opp\,\overrightarrow{AB} = \overrightarrow{AC}$ since both symbols name the same set of points, namely, the ray that is opposite $\overrightarrow{AB}$.

The following summary should be helpful: If A and B are any two distinct points on a line, they determine the six subsets of the line shown in Figure 2-15, with subsets indicated by heavier marking. Be sure to notice the differences among the symbols $\overleftrightarrow{AB}$, $\overline{AB}$, $\overrightarrow{AB}$, and $\overrightarrow{BA}$.

Figure 2-15

A segment has two endpoints and a ray has one endpoint. Sometimes we wish to speak about those points of a segment or ray that are not endpoints. Accordingly, we state the following definition.

Definition 2.4 The **interior of a segment** is the set of all points of the segment except its endpoints. The **interior of a ray**, also called a **halfline**, is the set of all points of the ray except its endpoint.

EXERCISES 2.3

1. Use the A–B–C Betweenness Postulate to prove that every point of $\overline{AB}$ is in $\overleftrightarrow{AB}$.
2. Consider the following definition of a segment $\overline{AB}$:
$$\overline{AB} = \{X : X \in \overleftrightarrow{AB}; \text{ and } X = A, \text{ or } X = B, \text{ or } A\text{–}X\text{–}B\}.$$
Is this definition equivalent to Definition 2.1?
3. Consider the following definition of a ray $\overrightarrow{AB}$: If A and B are distinct points, then ray $\overrightarrow{AB}$ consists of point A, point B, all points X such that A–X–B, and all points Y such that A–B–Y. Is this definition equivalent to Definition 2.2?
4. If A and B are distinct points, what is the intersection of rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$?
5. If A and B are distinct points, what is the union of $\overrightarrow{AB}$ and $\overrightarrow{BA}$?
6. Write an alternate definition of segment in terms of intersecting rays.
7. Can two distinct rays have no point in common? Illustrate with a figure.
8. Can two distinct rays have exactly one point in common? Illustrate.
9. Can two distinct rays have infinitely many points in common? Illustrate.
10. Can the intersection of two distinct rays be a set containing two and only two distinct points?
11. Can the intersection of two distinct rays be a ray?
12. Can two collinear rays have no point in common?
13. Is there a segment with no endpoint?
14. Is there a ray with no endpoint?
15. Is there a segment with no interior point?
16. Is there a ray with no interior point?
17. Is there a segment with exactly one interior point?
18. Is there a ray with exactly one interior point?
19. Is there a segment with exactly two interior points?
20. Is there a ray with exactly two interior points?
21. If A–B–C, what is the intersection of $\overline{AB}$ and $\overline{AC}$?
22. If A–B–C, what is the union of $\overline{AB}$ and $\overline{AC}$?
23. If A–B–C, what is the intersection of $\overline{AB}$ and $\overline{BC}$?
24. If A–B–C, what is the union of $\overline{AB}$ and $\overline{BC}$?
25. If A–B–C, what is the intersection of $\overrightarrow{AB}$ and $\overrightarrow{CB}$?
26. If A–B–C, what is the union of $\overrightarrow{AB}$ and $\overrightarrow{CB}$?
27. If A–B–C, if S is the interior of $\overrightarrow{BA}$, and if T is the interior of $\overrightarrow{BC}$, what is the intersection of S and T?

28. If A-B-C, if S is the interior of $\overrightarrow{BA}$, and if T is the interior of $\overrightarrow{BC}$, describe the union of S and T.
29. If S is the interior of a segment $\overline{AB}$, what is the union of S and $\overline{AB}$?
30. If S is the interior of a segment $\overline{AB}$, what is the intersection of S and $\overline{AB}$?
31. If A and B are distinct points of the segment $\overline{RS}$, what is the union of $\overleftrightarrow{AB}$ and $\overline{RS}$?

In Exercises 32–41, a subset of a line l is named. Points on line l are indicated in Figure 2-16. Using this figure, write a simpler name for the set. Exercise 32 has been worked as a sample.

Figure 2-16

32. $\overrightarrow{CE} \cap \overline{CE} = \overline{CE}$
33. $\overrightarrow{CE} \cup \overline{CE}$
34. $\overrightarrow{DF} \cap \overrightarrow{DA}$
35. $opp\ \overrightarrow{DF} \cap \overrightarrow{CF}$
36. $\overrightarrow{CF} \cup \overrightarrow{CA}$
37. $\overrightarrow{BE} \cap \overrightarrow{ED}$
38. $\overline{BD} \cup \overline{CF}$
39. $\overline{BD} \cap \overline{CF}$
40. $opp\ \overrightarrow{EF} \cap opp\ \overrightarrow{CA}$
41. $\overrightarrow{DE} \cap opp\ \overrightarrow{BE}$

42. Prove that there is no segment on a line l which contains every point of line l. (*Hint:* Let $\overline{AB}$ be any segment on line l. Show that there is a point on line l that is not in $\overline{AB}$.)
43. If P and Q are two distinct points on a line, does $\overrightarrow{QP} = opp\ \overrightarrow{PQ}$? Explain.
44. How many segments do three distinct points on a line determine? Four distinct points? Five distinct points? (A given set of points determines a segment if the endpoints of the segment are in the given set.)
45. Use your answers to Exercise 44 to predict how many segments six distinct points on a line determine. Test your prediction by counting.
46. (a) A, B, C, D are four distinct points on a line l. The conjunction of the two sentences (1) and (2) is true if the appropriate symbol (—, →, ↔) is supplied over each letter pair (BA). Copy these statements and supply these symbols.
    (1) BA contains points C and D, but BA contains neither of these points.
    (2) D belongs to BA, but C does not.

    (b) Draw a sketch which shows the order of the four points A, B, C, D on l.
47. A, B, C, D are four distinct points on a line l. If $C \notin \overrightarrow{DB}$, $A \in \overrightarrow{DB}$, $C \in \overrightarrow{BD}$, $A \notin \overrightarrow{BD}$, draw a sketch which shows the order of the four points on l.

48. If $\overrightarrow{PQ}$ is opposite to $\overrightarrow{PR}$, which one of the three points P, Q, R is between the other two?

49. If $opp\ \overrightarrow{RQ} = \overrightarrow{RP}$, which one of the points P, Q, R is between the other two?

2.4 THE CONCEPT OF AN ANGLE

The concept of an angle is a fundamental one in mathematics as well as in the practical world of the house builder and the engineer. There are several ways to think of an angle. One way is to think of an angle as two noncollinear rays that have the same endpoint. A second way is to think of an angle as the points on two noncollinear rays that have the same endpoint. In one, an angle is a set of rays. In the second, an angle is a set of points. In the following formal definition we agree to think of an angle as a set of points.

> ***Definition 2.5*** An **angle** is the union of two noncollinear rays with the same endpoint. Each of the two rays is called a **side** of the angle. The common endpoint of the two rays is called the **vertex** of the angle.

Notation. The angle formed by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$. When using a symbol such as $\angle BAC$ it is important that the middle letter denote the vertex of the angle. Since the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is the same as the union of $\overrightarrow{AC}$ and $\overrightarrow{AB}$, it should be clear that $\angle BAC = \angle CAB$. (See Figure 2-17.) It should also be clear that many different points may be used in identifying the same angle. Thus, if

$$\overrightarrow{AB} = \overrightarrow{AC} = \overrightarrow{AD} \qquad \text{and} \qquad \overrightarrow{AE} = \overrightarrow{AF} = \overrightarrow{AG},$$

as indicated in Figure 2-18, then $\angle BAE = \angle CAE = \angle DAE = \angle BAF = \angle BAC$, and so on.

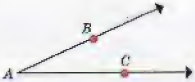


Figure 2-17

Figure 2-18

If a figure or other information makes clear which rays are the sides of the angle, we can denote $\angle BAE$ by simply writing $\angle A$. However,

if there is more than one angle with vertex A as in Figure 2-19, we would not know which angle is referred to by $\angle A$.

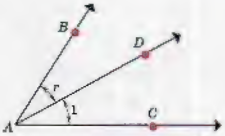


Figure 2-19

It is often convenient to label an angle by indicating a numeral or a lower case letter in its interior. Thus in Figure 2-19 we can write $\angle 1$ for $\angle DAC$ and $\angle r$ for $\angle BAD$. When used, the arcs in the figure indicate the sides of the angle and the interior of the angle.

Sometimes we speak of the angle determined by two noncollinear segments which have a common endpoint as indicated in Figure 2-20. If $\overline{AB}$ and $\overline{BC}$ are the segments, then the unique angle determined by them is $\angle ABC$.

Figure 2-20

Observe in the definition of an angle that the sides of an angle are noncollinear rays and therefore distinct rays. In some books this restriction is not made. A special case of an angle results when its two sides are "coincident." It is called the zero angle. Another special case results if the two sides are distinct but collinear. In this case the sides of the angle are opposite rays and the angle is called a straight angle. In Figure 2-21, $\angle ABC$ is a zero angle and $\angle$ DEF is a straight angle.

Figure 2-21

However, because zero angles and straight angles are not needed and it is simpler not to consider them, they have been excluded from the formal definitions in this book.

EXERCISES 2.4

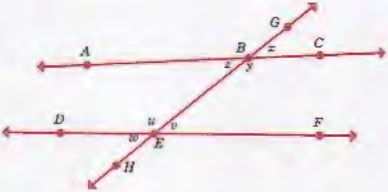
1. Copy and complete the following definition: An angle is the $\boxed{?}$ of two $\boxed{?}$ which have a common endpoint but do not lie on the same $\boxed{?}$.

In Exercises 2–9, draw a picture to illustrate a set satisfying each of the given descriptions.

2. Two distinct coplanar rays whose union is not an angle.
3. Two angles with the same vertex whose intersection is a ray.
4. Two angles with different vertices whose intersection is a ray.
5. Two angles whose intersection is a segment.
6. Two angles whose intersection is a set consisting of exactly one point.
7. Two angles whose intersection is a set consisting of exactly two points.
8. Two angles whose intersection is a set consisting of exactly three points.
9. Two angles whose intersection is a set consisting of exactly four points.

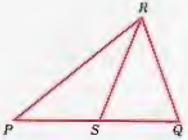
10. If A, B, C are three distinct points on $\overrightarrow{AB}$, if A, D, E are three distinct points on $\overrightarrow{AD}$, and if A, B, D are noncollinear points, then the union of $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is an angle which we may indicate as $\angle BAD$. Using A, B, C, D, E, write the other possible symbols for this angle.
11. In the following figure, the capital letters denote points and the lower case letters denote angles. Using three capital letters, write another name for each angle denoted by a lower case letter.

(a) $x = \angle\boxed{?}$
(b) $y = \angle\boxed{?}$
(c) $z = \angle\boxed{?}$
(d) $u = \angle\boxed{?}$
(e) $v = \angle\boxed{?}$
(f) $w = \angle\boxed{?}$



12. How many angles are determined by three distinct coplanar rays having a common endpoint if no two of the rays are opposite rays? By four distinct rays? By five distinct rays? (A given set of rays determines an angle if the sides of the angle are elements of the given set.)
13. Would your answers to Exercise 12 be different if the rays were not all in the same plane?
14. In the figure below, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays. How many angles do the four rays determine? Name them.

15. If two distinct lines intersect, how many angles are determined? (A given set of lines determines an angle if each side of the angle is contained in one of the lines of the given set.)
16. If three distinct coplanar lines intersect in a common point, how many angles do they determine?
17. Name all the angles determined by the segments shown in the figure. How many are there? Which angles can be named by using only the vertex letter?



18. Using letters in the figure in Exercise 17, name the
   (a) angle with vertex P in four different ways.
   (b) ray that is the intersection of $\angle PRS$ and $\angle QRS$.
   (c) ray that is the intersection of $\angle PSR$ and $\angle QSR$.
   (d) segment that is contained in the intersection of $\angle PQR$ and $\angle SRQ$.
   (e) point that is not in $\overline{PR}$ but is in the intersection of $\angle RPQ$ and $\angle PRS$.
   (f) three distinct rays contained in the union of $\angle PRS$ and $\angle QRS$.

Exercises 19–30 are informal geometry exercises. The formal development of angle measure appears in later sections. Using Figure 2-22, which shows degrees on a protractor, compute the number named in the exercise. Exercise 19 has been worked as a sample.

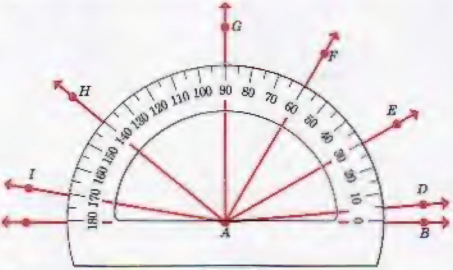


Figure 2-22

19. $m\angle DAF = 55$
20. $m\angle BAD$
21. $m\angle DAH$
22. $m\angle HAE$
23. $m\angle IAH$
24. $m\angle IAE$
25. $m\angle EAD$
26. $m\angle IAD$
27. $m\angle IAE + m\angle EAD$
28. $m\angle DAH - m\angle HAE$
29. $m\angle DAE$
30. $m\angle DAE + m\angle DAF$

31. Every triangle is the union of three segments, but not every union of three segments is a triangle. Write a definition of a triangle that you think will "hold water." (The formal definition of triangle appears in a later section.)

2.5 THE SEPARATION POSTULATES

In this section we state three more postulates involving the idea of betweenness, which we call the Separation Postulates. These postulates tell us how a point separates a line, how a line separates a plane, and how a plane separates space. Although the ideas are very simple, we cannot prove them from the postulates thus far agreed on. In order to facilitate their phrasing, we introduce the idea of a convex set.

> ***Definition 2.6*** A set of points is called **convex** if for every two points P and Q in the set, the entire segment $\overline{PQ}$ is in the set. The null set and every set that contains only one point are also called **convex sets.**

The definition implies that if P and Q are any two points of a convex set T, then $\overline{PQ} \subset T$. A line, a ray, and a segment are examples of convex sets of points as Figure 2-23 suggests. Is a plane a convex set of points?

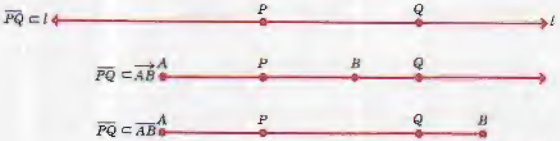


Figure 2-23

The interior of a triangle is a convex set. Figure 2-24 shows two choices, P_1, P_2 and Q_1, Q_2, for points P and Q, respectively, in the interior of the triangle and in each case the entire segment $\overline{PQ}$ is contained in the set. Is the interior of the circle shown in Figure 2-24 a convex set? (We define interior of triangle, circle, and interior of circle formally in later sections.)

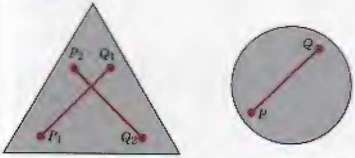


Figure 2-24

However, none of the sets shown in Figure 2-25 is a convex set. In each instance it is possible to find points P and Q such that not all of the segment $\overline{PQ}$ is contained in the set.

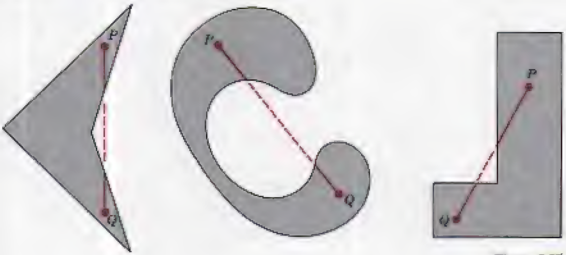


Figure 2-25

Let S be a convex set, T a convex set, and R the intersection of the two sets. If R is the null set or if R consists of a single point, why is R a convex set? Suppose, then, that R consists of more than one point. Let P and Q be any two distinct points in R. Draw an appropriate figure to suggest the sets S, T, R and the points P and Q. Then answer the following questions.

1. Why is P in the set S? In the set T?
2. Why is Q in the set S? In the set T?
3. Why is $\overline{PQ}$ in the set S? In the set T?
4. Why is $\overline{PQ}$ in the set R?
5. Does this prove that R is a convex set?
6. Does this prove the following theorem?

THEOREM 2.1 The intersection of two convex sets is a convex set.

Figure 2-26 suggests the first of our Separation Postulates. It shows point A dividing or separating line l into two convex sets, S and T.

Figure 2-26

No point of l is in both S and T. Point A is the only point on line l that is in neither S nor T. If points B and C are in different sets (that is, if $B \in S$ and $C \in T$, or if $C \in S$ and $B \in T$), then $\overline{BC}$ contains point A. On the other hand, if points C and D are in the same set (that is, if $C \in S$ and $D \in S$, or $C \in T$ and $D \in T$), then $\overline{CD}$ does not contain point A.

We state these ideas formally in our next postulate.

POSTULATE 12 (*The Line Separation Postulate*) Each point A on a line separates the line. The points of the line other than the point A form two distinct sets such that

1. each of the two sets is convex;
2. if two points are in the same set, then A is not between them;
3. if two points are in different sets, then A is between them.

Definition 2.7 Let a line l and a point A on l be given.

1. The two convex sets described in Postulate 12 are called **halflines** or **sides** of point A on line l; A is the **endpoint** of each of them.
2. If C and D are two points in one of these sets, we say that C and D are on the **same side** of A, or that C is on the **D-side** of A, or that D is on the **C-side** of A.
3. If B is a point in one of these sets and C is a point in the other set, we say that B and C are on **opposite sides** of A on line l or that B and C are in the **opposite halflines** of l determined by the point A.

Note that a halfline is a ray with the endpoint omitted. Although it is convenient to think of a halfline as *having* an endpoint, remember that a halfline *does not contain* its endpoint. (This use of "have" should not bother you since you *have* friends, but you *do not contain* them.) Note also that opposite halflines are collinear and that they have the same endpoint.

Since there are infinitely many lines in space that contain any given point, it follows that although a point has only two sides on any given line, it has infinitely many sides in a plane or in space. Normally, we do not speak of the sides of a point except when a given line contains the point and we are discussing the sides of the point on that line.

Postulate 13 describes how a line separates any plane containing the line. In Figure 2-27, we see that the points of plane α that do not lie on line l are separated into two sets:

(1) those points that are on the B-side of line l;
(2) those points that are on the C-side of line l.

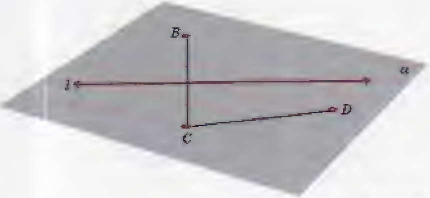


Figure 2-27

No point is in both of these sets and no point of line l is in either of the two sets. If points B and C are in different sets, then $\overline{BC}$ intersects l. On the other hand, if points C and D are in one set, then $\overline{CD}$ does not intersect line l. We state these ideas formally in our next postulate.

POSTULATE 13 (*The Plane Separation Postulate*) Each line l in a plane separates the plane. The points of the plane other than the points on line l form two distinct sets such that

1. each of the two sets is convex;
2. if two points are in the same set, then no point of line l is between them;
3. if two points are in different sets, then there is a point of line l between them.

Definition 2.8 Let a plane α and a line l in α be given.

1. The two convex sets described in Postulate 13 are called **halfplanes** or **sides** of l in plane α; l is the **edge** of each of them.
2. If C and D are two points in one of these sets, then we say that C and D are on the **same side** of l in plane α, or that C is on the **D-side** of l, or that D is on the **C-side** of l, or that C and D are in the **same halfplane.**
3. If B is a point in one of these sets and C is a point in the other set, we say that B and C are on **opposite sides** of l in plane α or that B and C are in the **opposite halfplanes** of α determined by the line l.

Note that a halfplane does not contain its edge and that opposite halfplanes are coplanar and have the same edge. Furthermore, since there are infinitely many planes in space that contain any given line, it follows that although a line has only two sides in any given plane, it has infinitely many sides in space. Normally, we do not speak of the sides of a line except when some given plane contains the line and we are discussing the sides of the line in that plane.

Figure 2-28 suggests the last of the Separation Postulates. The figure shows how a plane α separates the points of space that do not lie in plane α into two sets:

(1) those points that are on the B-side of plane α;
(2) those points that are on the C-side of plane α.

No point is in both of these sets and no point of plane α is in either of the two sets. If points B and C are in different sets, then $\overline{BC}$ inter-

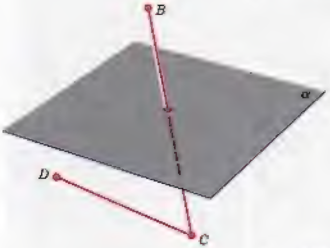


Figure 2-28

sects plane α. If points C and D are in the same set, then $\overline{CD}$ does not intersect plane α.

Although these space separation ideas can be proved using the postulates introduced up to this point we shall summarize and state these ideas formally as a postulate in order to simplify and shorten our formal development of geometry. Note the similarity of the Space Separation Postulate to the Plane Separation Postulate.

POSTULATE 14 (*The Space Separation Postulate*) Each plane α in space separates space. The points in space other than the points in plane α form two distinct sets such that

1. each of the two sets is convex;
2. if two points are in the same set, then no point of plane α is between them;
3. if two points are in different sets, then there is a point of plane α between them.

Definition 2.9 Let a plane α be given.

1. The two convex sets described in Postulate 14 are called halfspaces or sides of plane α and plane α is called the face of each of them.
2. If C and D are any two points in one of these sets, then we say that C and D are on the same side of α, or that C is on the D-side of α, or that D is on the C-side of α, or that C and D are in the same halfspace.
3. If B is a point in one of these sets and C is a point in the other set, then we say that B and C are on opposite sides of α or that B and C are in opposite halfspaces.

Note that a halfspace does not contain its face and that opposite halfspaces have the same face. Furthermore, although a point or a line has infinitely many sides in space, a given plane has only two sides in space.

The statements of the Separation Postulates and the definitions accompanying them are lengthy. However, the ideas they convey are simple and can be easily described by means of figures. Briefly, the Separation Postulates tell us that a point separates a line into two halflines; that a line separates a plane into two halfplanes; that a plane separates space into two halfspaces; and that these sets are convex.

Notation. Figure 2-29 shows point A on line l and two halflines determined by point A. We may denote the halfline on the C-side of A in line l by the symbol l_1 or by the symbol $\overset{\circ}{\overrightarrow{AC}}$. Similarly, the halfline on the B-side of A may be denoted by l_2 or by $\overset{\circ}{\overrightarrow{AB}}$. Be careful to note the difference between the symbol $\overrightarrow{AB}$ for ray $\overrightarrow{AB}$ and the symbol $\overset{\circ}{\overrightarrow{AB}}$ for halfline $\overset{\circ}{\overrightarrow{AB}}$. Does $\overrightarrow{AB} = \overset{\circ}{\overrightarrow{AB}}$? Explain.

Figure 2-29

Figure 2-30 shows line l in plane α and the two halfplanes which line l determines. We may denote the halfplane on the B-side of line l in plane α by α_1, and the halfplane on the C-side of line l in plane α by α_2. Similarly, if S represents the set of all points in space, we may denote the two halfspaces into which space is separated by a plane with the symbols S_1 and S_2.

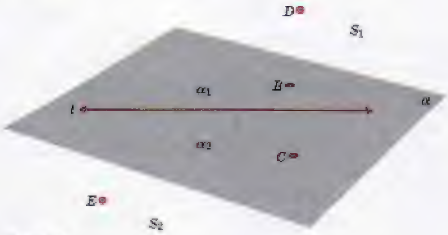


Figure 2-30

We make use of the Separation Postulates to prove the next two theorems.

THEOREM 2.2 If a segment has only one endpoint on a given line, then the entire segment, except for that endpoint, lies in one halfplane whose edge is the given line.

RESTATEMENT:

Given: Line l and segment $\overline{AB}$ in plane α such that l and $\overline{AB}$ have only the point A in common.

To Prove: If X is any point of $\overline{AB}$ such that A–X–B, then X is on the B-side of l in plane α.

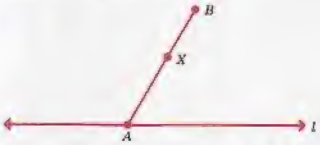


Figure 2-31

Proof:

Statement	Reason
1. X is a point such that A–X–B.	1. Given
2. A, X, B are distinct points.	2. Why?
3. l does not intersect $\overline{XB}$.	3. Why?
4. X and B are not on opposite sides of l.	4. Why?
5. X and B are on the same side of l.	5. Plane Separation Postulate

We have shown that X, which is any point of $\overline{AB}$ except point A or point B, is on the B-side of l in plane α; hence every point of $\overline{AB}$, except point A, is on the B-side of l in plane α.

THEOREM 2.3 If the intersection of a line and a ray is the endpoint of the ray, then the interior of the ray is contained in one half-plane whose edge is the given line.

RESTATEMENT:

Given: $\overrightarrow{AB}$ and line l intersect in just the point A in plane α.

To Prove: All points of $\overset{\circ}{\overrightarrow{AB}}$ are on the B-side of l in plane α.

Figure 2-32

Proof: Let X be any point of $\overrightarrow{AB}$ such that A–B–X.

1. All the points of $\overrightarrow{AB}$, except A, lie on the B-side of l in plane α. Why?
2. $X \notin l$. Why?
3. l does not intersect $\overline{BX}$. Why?
4. X is not on the opposite side of l from B. Why?
5. X is on the same side of l as B. Why?

Since X is any point such that A–B–X, it follows that *opp* $\overrightarrow{BA}$ lies entirely on the B-side of l. Since $\overset{\circ}{\overrightarrow{AB}}$ is the union of *opp* $\overrightarrow{BA}$ and $\overline{AB}$ with A deleted, it follows that $\overset{\circ}{\overrightarrow{AB}}$ is on the B-side of l.

EXERCISES 2.5

1. Prove that the intersection of any three convex sets is a convex set.
2. There is a theorem which "extends" Theorem 2.1 to any number of sets. State this theorem.
3. Is a ray a convex set?
4. Is the interior of a ray a convex set?
5. Consider the following definition of a ray: Ray $\overrightarrow{AB}$ consists of point A and all the points on the B-side of A on line $\overleftrightarrow{AB}$. Define *opp* $\overrightarrow{AB}$ in a similar manner.
6. Complete the following: Let $\overline{AB}$ be a segment. Then $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$. Since $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are convex sets it follows from $\boxed{?}$ that $\boxed{?}$ is a convex set.
7. Let l be any line and let P and Q be any two distinct points on l. If X is any point between P and Q, is X on line l? Why? Does this prove that $\overline{PQ}$ is contained in l? Does this prove that every line is a convex set?
8. Let α be any plane and let P and Q be any two distinct points in plane α.
   (a) Is $\overleftrightarrow{PQ}$ in plane α? Why?
   (b) Is $\overline{PQ}$ a subset of $\overleftrightarrow{PQ}$? Why?

(c) Is $\overline{PQ}$ a subset of plane α? Why?
(d) Does this prove that every plane is a convex set?

9. The interior of each circle shown in the figure below is a convex set.
(a) Is the intersection of these interiors a convex set? Why?
(b) Is the union of these interiors a convex set?

10. Draw two circles in such a way that the union of their interiors is a convex set.

Exercises 11–15. In Figure 2-33, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E so that A–E–B and C–E–D. The D-side of $\overleftrightarrow{AB}$ has been shaded.

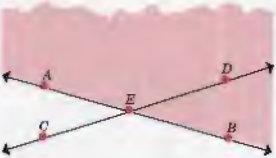


Figure 2-33

11. Copy the figure and shade the B-side of $\overleftrightarrow{CD}$.
12. Why are the two shaded halfplanes coplanar?
13. Why is the intersection of these two halfplanes a convex set?
14. Describe in your own words the intersection of the two shaded halfplanes.
15. Does your description in Exercise 14 suggest a definition of "interior of an angle?"

16. Let P, Q, and R be distinct points on a line l, with R and Q in the same halfline with endpoint P. On the basis of the given information is it possible that P is between Q and R? That Q is between R and P? That R is between P and Q?
17. Let P, Q, R be distinct points on a line l, with R and Q in opposite halflines with endpoint P. On the basis of the given information, is it possible that P is between Q and R? That Q is between R and P? That R is between P and Q?

18. Is the union of two opposite halflines a line? Explain.
19. Is the union of two opposite halflines a convex set?
20. Is the union of two opposite rays a line? Is their union a convex set?
21. In what respect does the set of points in ray $\overrightarrow{AB}$ differ from the set of points in halfline $\overset{\circ\!\!\rightarrow}{AB}$?
22. Is the interior of ray $\overrightarrow{AB}$ the same set of points as halfline $\overset{\circ\!\!\rightarrow}{AB}$?
23. Is the union of two opposite halfplanes a plane? Is their union a convex set?
24. Line l lies in plane α. Point A is on one side of l in plane α. Point B is in plane α and is not on the A-side of l and is not on the opposite side of l from A. Make a deduction.
25. A, B, C, X are four distinct points on line m. B and A are on opposite sides of X and C is on the A-side of X. Draw a conclusion about points B and C.
26. E, F, G are three distinct points in plane α. E and F are on opposite sides of line n in plane α. If E and G are on opposite sides of line n, what conclusion can you draw with regard to points G and F?
27. From which postulate may we infer that a given line in a given plane has only two sides?
28. Explain why the following statement is true: If P and Q are any two distinct points in halfplane α_1, then $\overline{PQ}$ is in α_1.
29. A halfplane is an example of a "connected region." A line in a plane separates the points of the plane not on this line into two connected regions. Into how many distinct connected regions do two distinct intersecting lines separate the remaining points of the plane that contains them?
30. Into how many distinct connected regions do three distinct coplanar lines separate the remaining points of the plane that contains them if no point lies on all three lines and if each two of the lines intersect?
31. Into how many distinct connected regions do four distinct coplanar lines separate the remaining points of the plane that contains them if no three of the lines contain the same points and if each two of the lines intersect?
32. Use your answers to Exercises 29–31 to predict the number of distinct connected regions into which five distinct coplanar lines separate the remaining points of the plane that contains them if no three of the lines contain the same point and if each two of the lines intersect.
33. Can three distinct coplanar lines be situated so as to separate the remaining points of the plane that contains them into three distinct connected regions? Four distinct connected regions? Five distinct connected regions? Six distinct connected regions? Seven distinct connected regions? More than seven distinct connected regions?

34. Draw a figure for each part of Exercise 33 to which you answered "Yes."

35. Figure 2-34 shows two distinct planes α and β intersecting in a line l. The line l is the edge of how many different halfplanes represented in the figure?

36. Name two distinct halfplanes represented in Figure 2-34 that are coplanar.

37. Name two distinct halfplanes represented in Figure 2-34 that are not coplanar.

38. How many different pairs of halfplanes in Figure 2-34 are not coplanar?

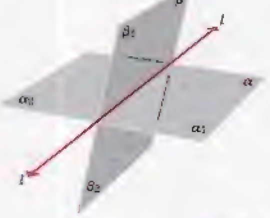


Figure 2-34

39. One plane separates the rest of space into two connected regions. Into how many distinct connected regions do two distinct intersecting planes separate the rest of space?

40. Into how many distinct connected regions do three distinct intersecting planes separate the rest of space if no line lies in all three of the planes, if every two of the planes intersect, and if each plane intersects the line of intersection of the other two planes?

41. Use your answers to Exercises 39 and 40 to predict the number of distinct connected regions formed by four distinct intersecting planes if no three of these planes contain the same line, if each two of these planes intersect, and if each plane intersects each line of intersection formed by two of the other planes.

42. CHALLENGE PROBLEM. Construct a model to represent the situation of Exercise 41. Count the number of distinct connected regions formed. How does this number compare with your prediction?

43. CHALLENGE PROBLEM. Extend the result of Exercise 41 to five planes.

2.6 INTERIORS AND EXTERIORS OF ANGLES

In Section 2.5 we introduced the concept of separation. A line in a plane separates the points of the plane not on the line into two halfplanes. A picture of an angle suggests that an angle separates its plane. Indeed, if plane α contains $\angle ABC$, then all the points of α that are not points of $\angle ABC$ make up two sets, one called the interior and the other the exterior of $\angle ABC$. We shall state carefully what we mean by these terms.

One of the simplest ways to think of the interior of an angle is as the intersection of two halfplanes associated with the angle. For $\angle ABC$

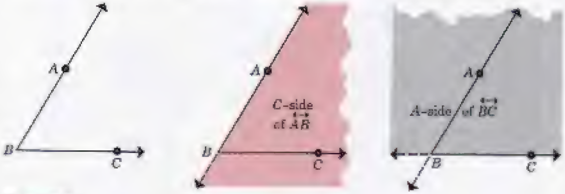


Figure 2-35

these halfplanes are the C-side of $\overleftrightarrow{AB}$ and the A-side of $\overleftrightarrow{BC}$, as indicated in Figure 2-35. Our formal definition is as follows:

Definition 2.10 The interior of an angle, say $\angle ABC$, is the intersection of two halfplanes, the C-side of $\overleftrightarrow{AB}$ and the A-side of $\overleftrightarrow{BC}$.

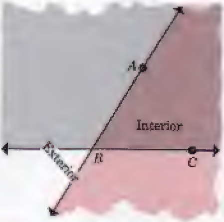


Figure 2-36

Definition 2.11 The exterior of an angle is the set of all points in the plane of the angle except those points on the sides of the angle and in its interior.

Figure 2-36 illustrates both Definitions 2.10 and 2.11.

The following theorem is easy to prove using Definition 2.10 and Theorem 2.3.

THEOREM 2.4 If P is any point in the interior of $\angle ABC$, then the interior points of ray $\overrightarrow{BP}$ are points of the interior of $\angle ABC$.

RESTATEMENT:

Given: $\angle ABC$ with P a point in the interior of $\angle ABC$.

To Prove: $\overset{\circ}{\overrightarrow{BP}}$ is in the interior of $\angle ABC$.

Proof: By definition of the interior of $\angle ABC$, P is on the C-side of $\overleftrightarrow{AB}$ and on the A-side of $\overleftrightarrow{BC}$. By Theorem 2.3, $\overset{\circ}{\overrightarrow{BP}}$ is on the C-side of $\overleftrightarrow{AB}$ and on the A-side of $\overleftrightarrow{BC}$. Therefore $\overset{\circ}{\overrightarrow{BP}}$ is in the intersection of these two halfplanes which, by definition, is the interior of $\angle ABC$.

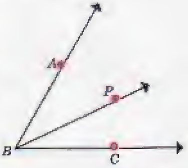


Figure 2-37

EXERCISES 2.6

1. Copy and complete the following definition.

   The interior of $\angle PQR$ is the $\boxed{?}$ of the halfplane that is the P-side of $\boxed{?}$ and the halfplane that is the $\boxed{?}$ of $\overleftrightarrow{PQ}$.

Exercises 2–7 refer to Figure 2-38.

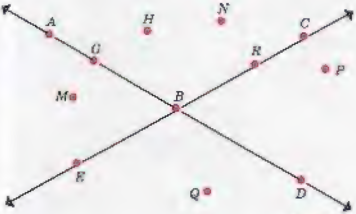


Figure 2-38

2. Which of the labeled points are in the interior of $\angle ABC$?
3. Which of the labeled points are in the exterior of $\angle ABC$?
4. Which of the labeled points are not in the interior of $\angle ABC$?
5. Which of the labeled points are in the interior of $\angle GBR$?
6. Which of the labeled points are in the interior of $\angle CBD$?
7. Which of the labeled points are in the interior of $\angle CBG$?

8. Is the vertex of an angle a point of the interior of the angle? Explain.
9. Is the vertex of an angle a point of the exterior of the angle? Explain.
10. Is B a point of the interior of $\angle ABC$? Explain.
11. Is B a point of the exterior of $\angle ABC$? Explain.
12. Suppose that A, B, C, D are four noncoplanar points. Is it possible that D is a point of the interior of $\angle ABC$? Explain.
13. Suppose that A, B, C, D are four noncoplanar points. Is it possible that D is a point of the exterior of $\angle ABC$? Explain.

Exercises 14–20 refer to Figure 2-39.

Figure 2-39

14. Make a copy of Figure 2-39 and shade the A-side of $\overleftrightarrow{EB}$ with vertical rays $\uparrow\uparrow\uparrow\uparrow$ and the E-side of $\overleftrightarrow{AB}$ with horizontal rays $\rightrightarrows$
15. Which angle has only vertical shading in its interior?
16. Which angle has only horizontal shading in its interior?
17. Which angle has both kinds of shading in its interior?
18. Which angle has no shading in its interior?
19. Is the intersection of the two shaded portions the interior of any of the angles shown in the figure? If so, name the angle(s).
20. Is the union of the two shaded portions the exterior of any of the angles shown in the figure? The interior?

21. Does an angle separate the points of its plane not on the angle into two connected regions that do not intersect?
22. What are the connected regions of Exercise 21 called?
23. Draw a picture of an angle. Mark two points P and Q in the exterior of the angle. Does $\overline{PQ}$ intersect the angle?
24. Does your answer in Exercise 23 depend on your choice of P and Q or would the answer be the same for every choice of P and Q?
25. Is the exterior of an angle a convex set?
26. Draw a picture of an angle. Mark two points P and Q such that P is in the interior of the angle and Q is in the exterior. Does $\overline{PQ}$ intersect the angle?

27. Does your answer in Exercise 26 depend on your choice of P and Q, or would the answer be the same for every choice of P and Q if P is chosen in the interior of the angle and Q in the exterior?

28. CHALLENGE PROBLEM. It can be proved that if P is a point in the interior of an angle and Q is a point in the exterior, then $\overline{PQ}$ intersects the angle. The proof is difficult and there are essentially five cases to consider as indicated in Figure 2-40 where $\angle ABC$ is the angle, P is a point in the interior of the angle, and Q_1, Q_2, Q_3, Q_4, Q_5 are points in the exterior of the angle such that

(1) Q_1 is any point on the A-side of $\overleftrightarrow{BC}$ and on the opposite side of $\overleftrightarrow{AB}$ from C.

(2) Q_2 is any point on *opp* $\overrightarrow{BC}$.

(3) Q_3 is any point on the opposite side of $\overleftrightarrow{AB}$ from C and on the opposite side of $\overleftrightarrow{BC}$ from A.

(4) Q_4 is any point on *opp* $\overrightarrow{BA}$.

(5) Q_5 is any point on the C-side of $\overleftrightarrow{AB}$ and on the opposite side of $\overleftrightarrow{BC}$ from A.

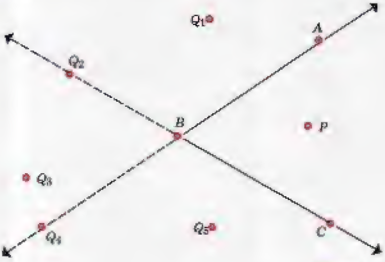


Figure 2-40

Following is an outline of a proof for case 1, a proof that if Q_1 is any point on the A-side of $\overleftrightarrow{BC}$ and on the opposite side of $\overleftrightarrow{AB}$ from C, then $\overline{PQ_1}$ intersects $\angle ABC$. Answer the "Whys" in this outline.

1. P and C are on the same side of $\overleftrightarrow{AB}$. Why?
2. C and Q_1 are on opposite sides of $\overleftrightarrow{AB}$. Why?
3. P and Q_1 are on opposite sides of $\overleftrightarrow{AB}$. Why?
4. There is a point R between P and Q_1 on line $\overleftrightarrow{BA}$. Why?

This means that $\overline{PQ_1}$ intersects $\overleftrightarrow{AB}$ in an interior point of $\overline{PQ_1}$. We wish now to prove that $\overline{PQ_1}$ intersects $\overrightarrow{BA}$ rather than *opp* $\overrightarrow{BA}$.

5. P is on the A-side of $\overleftrightarrow{BC}$. Why?
6. Q_1 is on the A-side of $\overleftrightarrow{BC}$. Why?
7. The A-side of $\overleftrightarrow{BC}$ is a convex set. Why?
8. All points of $\overline{PQ_1}$ are on the A-side of $\overleftrightarrow{BC}$. Why?

Therefore the point R is on the A-side of $\overleftrightarrow{BC}$ and on $\overleftrightarrow{BA}$. Therefore R is on $\overrightarrow{BA}$, and hence R is a point of $\angle ABC$. Therefore $\overline{PQ_1}$ intersects $\angle ABC$.

29. CHALLENGE PROBLEM. Let the situation of Exercise 28 be given. Prove that $\overline{PQ_2}$ intersects $\overrightarrow{BA}$.
30. CHALLENGE PROBLEM. Let the situation of Exercise 28 be given. Prove that $\overline{PQ_3}$ intersects $\angle ABC$.
31. Draw a picture of an angle. Mark two points P and Q in the interior of the angle. Does $\overline{PQ}$ intersect the angle?
32. Does your answer in Exercise 31 depend on your particular choice of P and Q?
33. Is the interior of an angle a convex set?
34. CHALLENGE PROBLEM. Use Theorem 2.1, the Plane Separation Postulate, and the definition of interior of an angle to prove that the interior of an angle is a convex set.
35. On the basis of your experiences in informal geometry try to write a definition of a triangle, thinking of it as a set of points. (Remember that a segment is a set of points and that the union of several sets of points is a set of points.)
36. On the basis of your experiences in informal geometry try to write a definition of a quadrilateral.

2.7 TRIANGLES AND QUADRILATERALS

Next to segments and angles perhaps the simplest geometrical figures are the polygons, and the simplest polygons are the triangles. You all know what a triangle looks like. It has three sides and three angles. A drawing of a triangle (Figure 2-41) shows its three sides, which are the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

Figure 2-41

On the other hand, the angles of a triangle are not shown completely in a picture of the triangle. In Figure 2-42, however, there are pictures showing the angles of a triangle. A picture that does show all the angles of a triangle is not a picture of what is usually meant by a triangle.

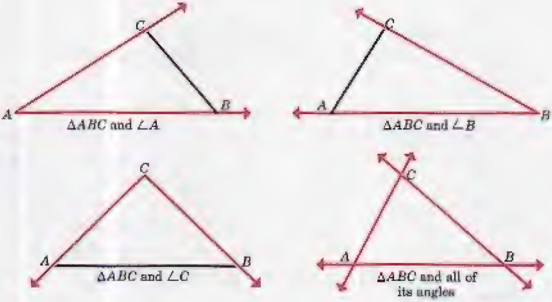


Figure 2-42

The following definition is worded carefully so that it says exactly what we want it to say.

> *Definition 2.12* If A, B, C are three noncollinear points, then the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is a **triangle**.

Notation. The triangle which is the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is denoted by $\triangle ABC$.

An alternate definition which you might prefer is as follows.

► A **triangle** is the union of the three segments determined by three noncollinear points.

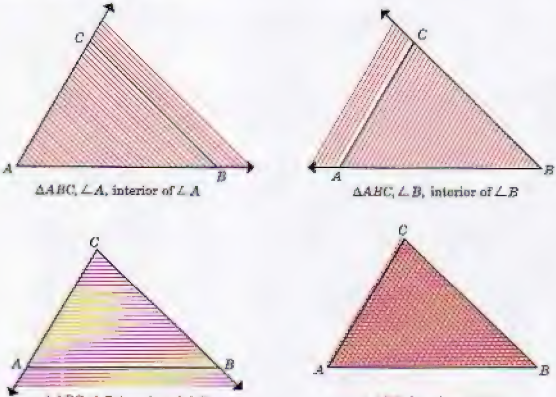
This is an acceptable definition since we have agreed (in Section 2.3) that "if A, B, C are three distinct points, then the segments determined are $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$."

Definition 2.13 Let $\triangle ABC$ be given.

1. Each of the points A, B, C is a **vertex** of $\triangle ABC$.
2. Each of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is a **side** of $\triangle ABC$.
3. Each of the angles $\angle ABC$, $\angle BCA$, $\angle CAB$ is an **angle** of $\triangle ABC$.
4. A side and a vertex not on that side are **opposite** to each other.
5. A side and an angle are **opposite** to each other if that side and the vertex of that angle are opposite to each other.

Note that a triangle contains its vertices and its sides but that it does not contain its angles. An angle of a triangle is not a subset of the triangle. It is customary to speak of the "angles of a triangle" or "the angles determined by a triangle," but it is incorrect to refer to them as "the angles contained in the triangle." Remember that whereas a triangle has angles, it does not contain them. (In this connection it might be helpful to think of a farmer who has farms and barns but does not contain them, or of a man who has a car and a house and lot but does not contain them!)

Figure 2-43 illustrates the interior of a triangle and the interiors of the angles of a triangle.



$\triangle ABC$, $\angle A$, interior of $\angle A$

$\triangle ABC$, $\angle B$, interior of $\angle B$

$\triangle ABC$, $\angle C$, interior of $\angle C$

$\triangle ABC$, interior $\triangle ABC$

Figure 2-43

Definition 2.14 The intersection of the interiors of the three angles of a triangle is the **interior of the triangle.**

Definition 2.15 The **exterior of a triangle** is the set of all points in the plane of the triangle that are neither points of the triangle nor points of the interior of the triangle.

In Figure 2-44 the interiors of $\angle A$ and $\angle B$ are shaded. Note that the intersection of these interiors is the interior of the triangle.

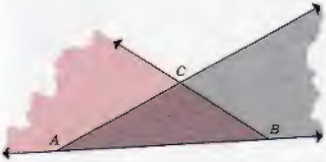


Figure 2-44

This suggests the following theorem.

THEOREM 2.5 The intersection of the interiors of two angles of a triangle is the interior of the triangle.

Proof: Let $\triangle ABC$ be given. Let X be any point in the intersection of the interiors of any two of its angles, say $\angle A$ and $\angle B$.

X is on the B-side of $\overleftrightarrow{AC}$ because it is in the interior of $\angle A$.

X is on the A-side of $\overleftrightarrow{BC}$ because it is in the interior of $\angle B$.

X is in the interior of $\angle C$ because it is on the B-side of $\overleftrightarrow{AC}$ and the A-side of $\overleftrightarrow{BC}$.

This proves that if a point is in the interiors of two angles of a triangle, then it is also in the interior of the third angle. Therefore the intersection of the interiors of two angles of a triangle is contained in the

interior of the triangle. Since the interior of the triangle is the intersection of the interiors of all three angles, it follows that the interior of the triangle is contained in the intersection of the interiors of any two of its angles. Therefore the interior of the triangle is contained in, and also contains, the intersection of the interiors of any two of its angles. Therefore the interior of the triangle is the intersection of the interiors of any two of its angles.

Consider any three noncollinear points A, B, C in a plane α and a line l in α which does not contain A or B or C but which does contain an interior point of $\overline{AC}$ as shown in Figure 2-45.

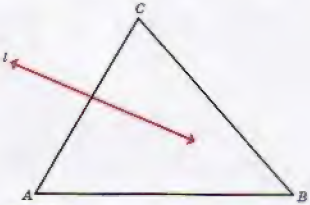


Figure 2-45

1. B is either on the C-side of l or on the A-side of l in plane α. Why?
2. If B is on the C-side of l, then l intersects $\overline{AB}$ in a point in the interior of $\overline{AB}$. Why? Does l intersect $\overline{BC}$ in this case? Why?
3. If B is on the A-side of l, then l intersects $\overline{BC}$ in a point in the interior of $\overline{BC}$. Why? Does l intersect $\overline{AB}$ in this case? Why?
4. Does this prove the following theorem?

THEOREM 2.6 If a line and a triangle are coplanar, if the line does not contain a vertex of the triangle, and if the line intersects one side of the triangle, then it also intersects just one of the other two sides.

Triangles have three sides. Quadrilaterals have four sides. By quadrilaterals we mean some, but not all, plane figures that are made up of four segments as suggested by Figure 2-46.

Try writing a definition of quadrilateral before reading further. Then compare your definition with the following definition which we adopt for our formal geometry.

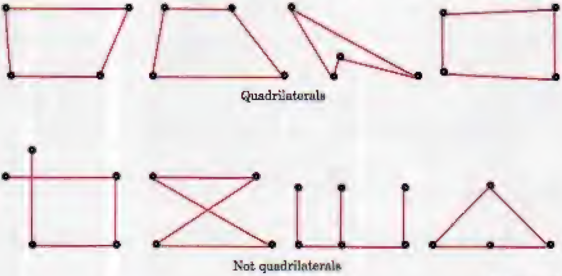


Figure 2-46

Definition 2.16 Let A, B, C, D be four coplanar points such that no three of them are collinear and such that none of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersects any other at a point which is not one of its endpoints. Then the union of the four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ is a **quadrilateral**. Each of the four segments is a **side** of the quadrilateral and each of the points A, B, C, D is a **vertex** of the quadrilateral.

Is it possible that four given points are the vertices of more than one quadrilateral? Figure 2-47 shows that this is indeed possible. Think of the figure as four different pictures of the same four points. The second, third, and fourth pictures show different quadrilaterals with the same vertices.

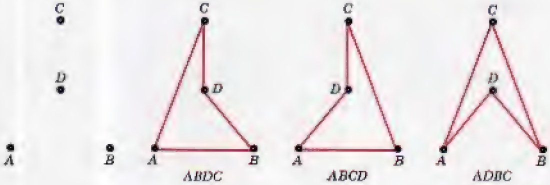


Figure 2-47

To name a quadrilateral using the names of its vertices, and to do it so that we know which segments are its sides, the names of the vertices are so arranged that (1) letters adjacent to each other in the name

of the quadrilateral are names of the endpoints of a side of the quadrilateral and (2) the first and last letters in the name of the quadrilateral are names of the endpoints of a side of the quadrilateral.

Note that if $ABCD$ is a quadrilateral, then $BCDA$ is the same quadrilateral. Give several additional names for this quadrilateral. Notice also that $ABCD$ and $ACBD$ are not names for the same quadrilateral.

EXERCISES 2.7

1. Copy and complete the definition: If A, B, C are three $\boxed{?}$ points, then $\boxed{?}$ is the triangle denoted by $\triangle ABC$.
2. Name the side which is opposite to $\angle ABC$ in $\triangle ABC$.
3. Name the angle which is opposite to side $\overline{AB}$ in $\triangle ABC$.

In Exercises 4–10, let $\triangle ABC$ be given. State whether or not the given set is a subset of $\triangle ABC$.

4. $\triangle ABC$
5. Interior of $\angle ABC$
6. Interior of $\triangle ABC$
7. $\overline{AB}$
8. $\overline{AB} \cup \overline{BC} \cup \overline{CA}$
9. $\overline{AB} \cap \overline{BC} \cap \overline{CA}$
10. $\{A, B, C\}$

In Exercises 11–15, let $\triangle ABC$ be given. Let S denote the union of $\triangle ABC$ and its interior. State whether or not the given set is a subset of S.

11. Interior of $\angle ABC$
12. (Interior of $\angle ABC$) $\cap$ (Interior of $\angle BCA$)
13. $\{\overline{AB}, \overline{BC}, \overline{CA}\}$
14. $\overline{AB} \cap \overline{BC} \cap \overline{CA}$
15. $\overline{AB} \cap$ (Interior of $\triangle ABC$)

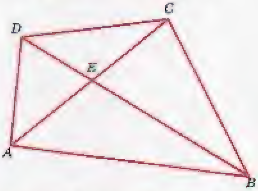
In Exercises 16–19, let $\triangle ABC$ be given. Let H_1, H_2, H_3 denote the following halfplanes, respectively: the A-side of $\overleftrightarrow{BC}$, the B-side of $\overleftrightarrow{CA}$, and the C-side of $\overleftrightarrow{AB}$. Express the given set in terms of these halfplanes.

16. Interior of $\angle ABC$
17. Interior of $\angle BCA$
18. Interior of $\angle CAB$
19. Interior of $\triangle ABC$

In Exercises 20–24, let the same situation as in Exercises 16–19 be given. State whether or not the given set is a subset of the exterior of $\triangle ABC$.

20. The interior of the ray opposite to $\overrightarrow{BC}$
21. The ray opposite to $\overrightarrow{BC}$
22. $\overrightarrow{BC}$
23. $\overline{BC}$
24. The opposite halfplane from H_1

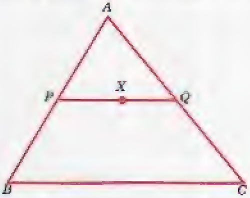
25. Draw a picture of a triangle and a line so that
    (a) their intersection is one point;
    (b) their intersection is exactly two points;
    (c) their intersection consists of more than two points.
26. Can a line intersect a triangle in exactly one point which is not a vertex? Illustrate.
27. What is your answer to Exercise 26 if the line and triangle are contained in the same plane?
28. Can a line intersect a triangle in exactly three points?
29. Name all of the triangles shown in the figure.



30. If A, B, C are the vertices of a triangle, prove that there is a point of $\angle A$ that is not a point of the triangle.
31. If A, B, C are noncollinear points, is the following statement true?

$$\triangle ABC = (\angle A \cap \angle B) \cup (\angle A \cap \angle C) \cup (\angle B \cap \angle C).$$

32. Let $\triangle ABC$ with points P and Q such that A–P–B and A–Q–C as in the figure be given. To prove that $\triangle$ABC is not a convex set of points, it is sufficient to show that there is a point X of $\overline{PQ}$ which is not contained in $\triangle ABC$. Let X be a point of $\overline{PQ}$ such that P–X–Q.



    (a) $X \notin \overline{AC}$. Why?
    (b) $X \notin \overline{AB}$. Why?
    (c) $X \notin \overline{BC}$. Why? (*Hint:* Use Theorem 2.6.)
    Does this prove that a triangle is not a convex set?
33. Is the exterior of a triangle a convex set? Illustrate with a figure.
34. Is the interior of a triangle a convex set? Explain why.

For Exercises 35–38 draw a triangle, $\triangle ABC$.

35. Mark a point that is in the interior of $\angle A$ and in the exterior of the triangle.
36. Mark a point that is in the interior of $\angle A$ but is neither in the interior nor the exterior of the triangle.
37. Mark a point in the exterior of the triangle that is not in the interior of any of the angles of the triangle.
38. Can you mark a point that is in the interior of $\angle A$ and is in the interior of $\angle B$ but is not in the interior of $\angle C$? Why?

In Exercises 39–42, let quadrilateral $ABCD$ be given.

39. Is it possible that D is an element of $\overline{BC}$?
40. Is it possible that D is an element of the interior of $\angle BAC$?
41. Is it possible that D is an element of the exterior of $\angle BAC$?
42. Is it possible that the interior of $\triangle ABC$ is a subset of the interior of $\triangle ADC$?

43. Prove that every plane contains a quadrilateral.
44. Prove that any three vertices of a quadrilateral are the vertices of a triangle.

2.8 PROPERTIES OF EQUALITY AND NUMBER OPERATIONS

In preparation for Chapter 3, which involves some algebra, we conclude Chapter 2 with several sections devoted to a review of elementary algebra. Chapter 3 contains many equations, that is, statements of equality. Following are some basic properties of equality and number operations that are useful in working with equations involving numbers.

Substitution Property of Equality. In any statement about some thing (physical object, number, point, line, idea, etc.) one of the names for that thing may be replaced by another name for the same thing. If the original statement is true, then the statement with the substitution made is also true. If the original statement is false, then the statement with the substitution made is false.

Recall that $A = B$ means that A and B are names for the same thing. The substitution property tells us that we may replace A by B in any statement about A without changing the truth or falsity of the statement.

Example Consider these two statements:

$$(1)\quad 7 = 2 + 5$$
$$(2)\quad 3 + 4 = 7$$

From (1) and (2) it follows by the substitution property of equality that

$$3 + 4 = 2 + 5.$$

Reflexive Property of Equality. For any thing a, $a = a$.

Symmetric Property of Equality. For any thing a and for any thing b, if $a = b$, then $b = a$.

Transitive Property of Equality. For any things a, b, c, if $a = b$ and $b = c$, then $a = c$.

Addition Property of Equality for Numbers. If a, b, c are real numbers, and if $a = b$, then

$$a + c = b + c.$$

More generally, if a, b, c, d are numbers such that $a = b$ and $c = d$, then

$$a + c = b + d \qquad \text{and} \qquad a - c = b - d.$$

(It is appropriate to include subtraction here since each difference can be expressed as an addition. Thus $a - c = a + (-c)$. Also if $c = d$, then $-c = -d$.)

Multiplication Property of Equality for Numbers. If a, b, c are real numbers and if $a = b$, then

$$ac = bc.$$

More generally, if a, b, c, d are numbers, and if $a = b$ and $c = d$, then

$$ac = bd.$$

and if $c \neq 0$, $d \neq 0$, then

$$\frac{a}{c} = \frac{b}{d}.$$

(It is appropriate to include division here since each quotient can be expressed as a product . Thus $\frac{a}{c} = a \cdot \frac{1}{c}$. Also, if $c = d$, $c \neq 0$, $d \neq 0$, then $\frac{1}{c} = \frac{1}{d}$.)

Commutative Property of Addition. If a and b are real numbers, then

$$a + b = b + a.$$

Commutative Property of Multiplication. If a and b are real numbers, then

$$ab = ba.$$

Associative Property of Addition. If a, b, c are real numbers, then

$$(a + b) + c = a + (b + c).$$

Associative Property of Multiplication. If a, b, c are real numbers, then

$$(ab)c = a(bc).$$

Distributive Property of Multiplication over Addition. If a, b, c are real numbers, then

$$a(b + c) = ab + ac \qquad \text{and} \qquad (a + b)c = ac + bc.$$

EXERCISES 2.8

In Exercises 1–20, name the property that justifies the given statement.

1. If $x = 5$ and $x = y$, then $y = 5$.
2. If $x = 5$ and $5 = y$, then $x = y$.
3. $7 = 7$
4. If $AB = CD$, then $CD = AB$.
5. If $3 = x$ and $4 = y$, then $3 + 4 = x + y$.
6. If $3 = x$ and $4 = y$, then $12 = xy$.
7. If $\mathcal{S}$ is a coordinate system, then $\mathcal{S} = \mathcal{S}$.
8. If $x + y = z$ and if $x = a + b$, then $(a + b) + y = z$.
9. If $uv = w$ and if $a + b = u$ and $c + d = v$, then $(a + b)(c + d) = w$.
10. If $\dfrac{a}{b} = c$, then $a = cb$.
11. If $a + b = c$, then $(a + b) + (-b) = c + (-b)$.
12. If $\dfrac{x - 5}{k - 0} = 6$, then $x - 5 = 6(k - 0)$.
13. $3(20 + 5) = 3 \cdot 20 + 3 \cdot 5$
14. $3(xy) = (3x)y$
15. $(2 + 3)5 = 2 \cdot 5 + 3 \cdot 5$
16. $(2 + 3)(4 + 7) = (2 + 3)4 + (2 + 3)7$
17. $(3 + 2 \cdot 3) + 7 = 3 + (2 \cdot 3 + 7)$

18. $5 + x = x + 5$
19. $3 + (4 + 7) = 3 + (7 + 4)$
20. $3(7 + x) = (7 + x)3$

2.9 SOLVING EQUATIONS

The properties of equality and number operations are useful in solving elementary equations.

Example 1 Solve for x: $x + 3 = 52$.

Solution:

$x + 3 = 52$	Given
$x = 49$	Addition property (Add -3)

Example 2 Solve for x: $3x - 5 = 7x + 2$.

Solution:

$3x - 5 = 7x + 2$	Given
$3x = 7x + 7$	Addition property (Add 5)
$-4x = 7$	Addition property (Add $-7x$)
$x = -\dfrac{7}{4}$	Multiplication property (Mult. by $-\frac{1}{4}$)

Example 3 Solve for x: $\dfrac{x-5}{10} = \dfrac{x+8}{12}$.

Solution:

$\dfrac{x-5}{10} = \dfrac{x+8}{12}$	Given
$12(x - 5) = 10(x + 8)$	Multiplication property (Mult. by 120)
$12x - 60 = 10x + 80$	Distributive property
$12x = 10x + 140$	Addition property (Add 60)
$2x = 140$	Addition property (Add $-10x$)
$x = 70$	Multiplication property (Mult. by $\frac{1}{2}$)

Example 4 Solve for x: $\dfrac{x-5}{x-7} = \dfrac{5}{8}$. $(x \neq 7)$

Solution:

$\dfrac{x-5}{x-7} = \dfrac{5}{8}$	Given
$8(x-5) = 5(x-7)$	Multiplication property (Mult. by $8(x-7)$)
$8x - 40 = 5x - 35$	Distributive property
$3x = 5$	Addition property (Add $(-5x + 40)$)
$x = \dfrac{5}{3}$	Multiplication property (Mult. by $\frac{1}{3}$)

Example 5 Solve for x: $\dfrac{x+5}{2} = \dfrac{x-1}{3} + 5x$.

Solution:

$\dfrac{x+5}{2} = \dfrac{x-1}{3} + 5x$	Given
$3(x+5) = 2(x-1) + 30x$	Multiplication property (Mult. by 6)
$3x + 15 = 2x - 2 + 30x$	Distributive property
$-29x = -17$	Addition property (Add $(-32x - 15)$)
$x = \dfrac{17}{29}$	Multiplication property (Mult. by $-\frac{1}{29}$)

Although the list of properties does not explicitly include the division and subtraction properties of equality for numbers, they are included implicitly. For example, dividing by 7 is the same as multiplying by $\frac{1}{7}$, and subtracting 5 is the same as adding -5. In Example 1 you could think of subtracting 3 and justifying it by the subtraction property instead of adding -3 and justifying it by the addition property. In Example 2 you could obtain the last step from the preceding step by dividing by -4 and justifying it by the division property instead of multiplying by $-\frac{1}{4}$ and justifying it with the multiplication property. Note, however, that the multiplication property does not permit us to divide by 0, since 0 is not the reciprocal of any number.

Example 6 Solve for x: $a(x + x_1) = 5$, where $a \neq 0$.

Solution:

$a(x + x_1) = 5$	Given
$x + x_1 = \frac{5}{a}$	Division property (Divide by a)
$x = \frac{5}{a} - x_1$	Subtraction property (Subtract x_1)

EXERCISES 2.9

In Exercises 1–20, solve for x. Name the properties that justify the steps in your solution. Express your answer in simplest form.

1. $3x + 3 = 2x + 4$
2. $3x - 4 = 4x + 3$
3. $\frac{1}{2}x = \frac{1}{4}x + \frac{1}{2}$
4. $0.75x = 100$
5. $4(x - 64) = 28$
6. $\frac{x + 17}{5} = 3$
7. $\frac{3(x - 1)}{5} = \frac{1}{8}$
8. $\frac{x - 5}{8 - 5} = \frac{7 - (-6)}{-9 - (-6)}$
9. $\frac{x - 1}{7 - 1} = \frac{y - 3}{15 - 3}$
10. $\frac{x - 10}{-11 - 10} = \frac{k - 0}{1 - 0}$
11. $4x - 5 = 14x - 75$
12. $\frac{x - 3}{-1 - 3} = \frac{x' - (-8)}{-10 - (-8)}$
13. $x + 4 = 40 - k$
14. $\frac{x - 7}{5 - 1} = \frac{7 - (-3)}{1 - 0}$
15. $3(x + 15) = 2(x + 15)$
16. $3(x + 15) + 2(x + 16)$
17. $\frac{x + 100}{x - 100} = 0 \ (x \neq 100)$
18. $\frac{x + 50}{5} = 7 + \frac{x + 50}{10}$
19. $-1 - (-x) = -x - (-1)$
20. $0.3x + 0.8x = 220$

In Exercises 21–40, solve for x. Express answers in simple form.

21. $3x + 5 = 0$
22. $3(x - 1) + 2(x - 1) = 4x$
23. $5x + 7 = 3(x + 4) - 5$
24. $\frac{x - 1}{2} = \frac{x - 3}{3}$
25. $\frac{2x - 3}{7} = \frac{5x - 1}{8}$
26. $\frac{x - 1}{x - 5} = \frac{7}{8} \ (x \neq 5)$
27. $5x - 1 = \frac{1}{2}x - \frac{1}{4}$
28. $\frac{x - 1}{1} = \frac{5x - 1}{7 - 3}$
29. $\frac{x - 1}{2} = \frac{x - 2}{3}$

30. $\dfrac{5x-1}{6x-7} = \dfrac{5x-2}{6x-8}$

31. $x + 3y = 0$

32. $3x + y = 0$

33. $\dfrac{1}{10}x + \dfrac{1}{5}y = 1$

34. $x - 3 = k + 4$

35. $5(x - 3) = 3(k + 4)$

36. $5x - 3 = 3k + 4$

37. $\dfrac{x-1}{2} = \dfrac{k+2}{3}$

38. $4x - 5 = 3x_1 + 4x_2$

39. $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{5}{7}$

40. $x + 1 = \dfrac{8-5}{7-5}$

2.10 EQUIVALENT EQUATIONS

In solving an equation we find the number (or numbers) that "satisfies" the equation. Such a number is a root of the equation. In Example 1 of Section 2.9 the root of the equation

$$x + 3 = 52$$

is 49. Note that 49 satisfies the equation since $49 + 3 = 52$ is a true sentence and that no other number satisfies the equation. The set of all roots of the equation $x + 3 = 52$ is

$$\{49\}$$

since 49 is the one and only root of the equation. $\{49\}$ is the solution set of the equation $x + 3 = 52$.

In each example of Section 2.9 we used properties of equality and number operations to obtain other equations that have the same solution set. Equations that have the same solution set are called equivalent equations. Note in Example 2, for instance, that

$$3x - 5 = 7x + 2,$$

$$3x = 7x + 7,$$

$$-4x = 7,$$

and

$$x = -\frac{7}{4}$$

are four equivalent equations. The solution set of each of them is

$$\left\{-\frac{7}{4}\right\}.$$

Sometimes the properties of equality and number operations are used to produce equations not equivalent to the given equation.

Example 1

$x = 5$	Given
$0 = 0$	Multiplication property (Mult. by 0)

It is true that

$$\text{if } x = 5, \text{ then } 0 = 0.$$

Indeed $0 = 0$ is a true sentence regardless of what may be true about x. The solution set of $x = 5$ is obviously $\{5\}$. The solution set of $0 = 0$ is the set of all real numbers. Even though it is true that

$$\text{if } x = 5, \text{ then } 0 = 0,$$

it is not true that

$$x = 5 \text{ and } 0 = 0 \text{ are equivalent equations.}$$

When we went from $x = 5$ to $0 = 0$, we went from an equation with one root to an equation with infinitely many roots. We did not lose any roots, but we certainly gained many of them!

Example 2

$$\begin{aligned}
x &= \frac{x-2}{x-2} + 1 \\
x(x-2) &= (x-2) + 1(x-2) \\
x^2 - 2x &= x - 2 + x - 2 \\
x^2 - 2x &= 2x - 4 \\
x^2 - 4x + 4 &= 0 \\
(x-2)^2 &= 0 \\
x - 2 &= 0 \\
x &= 2
\end{aligned}$$

In this example we gained a root somewhere along the way. The solution set of $x = 2$ is obviously $\{2\}$, but 2 is not a root of the original equation. Why? The solution set of the given equation is the null set. If we try to reverse the steps in this "solution," we can justify each step except the last one. Given $x = 2$, we cannot justify going from

$$x(x-2) = (x-2) + 1(x-2)$$

to $x = \dfrac{x-2}{x-2} + 1$ by dividing by $x - 2$. The multiplication property permits us to divide by any number except 0 and $x - 2 = 0$ if $x = 2$.

Example 3

$$x^2 = 3x$$
$$x = 3$$

In this example we multiplied both sides of $x^2 = 3x$ by $\frac{1}{x}$ (or divided both sides by x). In doing this we lost a root. How did this happen? Dividing by x is legal except if $x = 0$. Is 0 a root of the given equation? Is $0^2 = 3 \cdot 0$? Yes, it is. The solution set of the given equation is $\{0, 3\}$. The solution set of $x = 3$ is $\{3\}$. The equations $x^2 = 3x$ and $x = 3$ are not equivalent equations.

In solving an equation through a sequence of equations it is advisable to check each step for a possible loss of roots. If there is a value of x that might be lost as a root (as 0 is lost in going from $x^2 = 3x$ to $x = 3$ in Example 3), it should be identified and checked by substitution in the original equation. Of course, it is a root if and only if it satisfies the original equation.

To make sure that no roots are gained in the solution process you may (1) check each root of the final equation by substitution in the original equation or (2) check to see if the equations can be obtained in reverse order without a loss of roots at any step. If there is no loss going backward, there is no gain going forward. If a root is lost going backward, it was gained going forward, and hence is not a root of the given equation. In Example 2 we have a sequence of equations and there is no loss of roots in going forward. When we reverse the steps, there is no loss of roots at any step until the last one. 2 is a root of

$$x(x - 2) = x - 2 + 1(x - 2)$$

but not a root of

$$x = \frac{x - 2}{x - 2} + 1.$$

2 is lost as a root going backward; it was gained as a root going forward.

Example 4

$\frac{x - 7}{x - 7} = 1$	Given
$x - 7 = x - 7$	Multiply by $x - 7$
$0 = 0$	Subtract $x - 7$

In this example we have a sequence of three equations and the solution set of the last one is the set of all real numbers. In reversing the

steps we do not lose any roots until the last step. To get

$$\frac{x-7}{x-7}=1$$

from $x - 7 = x - 7$ we divide both sides by $x - 7$, and this is not permissible if $x = 7$. Checking, we see that 7 is not a root of the given equation. The solution set of the given equation is the set of all real numbers except 7.

Example 5 (1) $\quad \dfrac{y-2}{x-2}=7$

(2) $\quad y - 2 = 7(x - 2)$

In this example there are two variables, x and y. In Equation (2) we may wish to think of x as the independent variable and y as the dependent variable. A solution of Equation (2) is an ordered pair of numbers (a, b) such that Equation (2) is satisfied when x is replaced by a and y is replaced by b. Thus (3, 9) is a solution of Equation (2) since

$$9 - 2 = 7(3 - 2)$$

is a true sentence.

If we solve Equation (2) for y, we get

$$y = 2 + 7(x - 2) \qquad \text{or} \qquad y = 7x - 12.$$

Let a be any real number whatsoever. Then $x = a$ and $y = 7a - 12$ satisfy Equation (2). For upon substituting a for x and $7a - 12$ for y in (2) we get

$$(7a - 12) - 2 = 7(a - 2),$$

which is a true sentence. Indeed, the set of all ordered pairs $(a, 7a - 12)$, where a is a real number, is the solution set of Equation (2). Any other letter can be used for the symbol a here. Thus we could say, if we wish, that the solution set of (2) is the set of all ordered pairs $(x, 7x - 12)$, where x is real.

Is the set of all ordered pairs $(a, 7a - 12)$ also the solution set of Equation (1)? Let us check. Substituting a for x and $7a - 12$ for y in Equation (1), we get

$$\frac{(7a-12)-2}{a-2}=7,$$

or after simplifying, $\dfrac{7(a-2)}{a-2} = 7$. This is a true statement for every a

with one exception. It is not true if $a = 2$. The solution set of (1) is the set of all ordered pairs $(a, 7a - 12)$ for $a \neq 2$. Equations (1) and (2) are not equivalent.

Some of you have graphed equations like (2). You know that its graph is a line. What is the graph of (1)? The graph of (1) includes all the points of the graph of (2) except the point with abscissa 2, that is, the point (2, 2). The graph of (1) is the union of two opposite halflines, or to use a bit of informal language, a line with a hole in it.

EXERCISES 2.10

In Exercises 1–20, check by substitution to see if the given value of x is a root of the given equation.

1. $x^2 + 5x + 6.25 = 0,\ x = 1$
2. $x^2 + 5x + 6.25 = 0,\ x = -2$
3. $x^2 + 5x + 6.25 = 0,\ x = -2.5$
4. $3(x - 5) + 7(x - 5) = 10(x - 5),\ x = 5$
5. $3(x - 5) + 7(x - 5) = 10(x - 5),\ x = 5.1$
6. $3(x - 5) + 7(x - 5) = 10(x - 5),\ x = 137.3$
7. $\dfrac{x - 2}{3x - 6} = \dfrac{2x + 3}{6x + 9},\ x = 1$
8. $\dfrac{x - 2}{3x - 6} = \dfrac{2x + 3}{6x + 9},\ x = 2$
9. $\dfrac{x - 2}{3x - 6} = \dfrac{2x + 3}{6x + 9},\ x = -1.5$
10. $\dfrac{x - 2}{3x - 6} = \dfrac{2x + 3}{6x + 9},\ x = 1000$
11. $\dfrac{1}{x + 2} = \dfrac{2}{x - 2},\ x = -6$
12. $\dfrac{1}{x + 2} = \dfrac{2}{x - 2},\ x = 2$
13. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1} = \dfrac{2x}{x^2 - 1},\ x = 3$
14. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1} = \dfrac{2x}{x^2 - 1},\ x = 10$
15. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1} = \dfrac{2x}{x^2 - 1},\ x = -1$
16. $\dfrac{2x + 3}{x - 3} = \dfrac{7 - 5}{8 - 6},\ x = -6$

17. $\frac{x-2}{x-2} + \frac{x+3}{x+3} + \frac{x-4}{x-4} = 3, x = 2$

18. $\frac{x-2}{x-2} + \frac{x+3}{x+3} + \frac{x-4}{x-4} = 3, x = 3$

19. $\frac{x-2}{x-2} + \frac{x+3}{x+3} + \frac{x-4}{x-4} = 3, x = 4$

20. $\frac{x-2}{x-2} + \frac{x+3}{x+3} + \frac{x-4}{x-4} = 3, x = -3$

In Exercises 21–30, check to see if the two given equations are equivalent. If they are not equivalent, state whether there is a loss or gain of roots in going from the first equation to the second one.

21. $x^2 = -2x, x = -2$
22. $2x = 3, x = 1.5$
23. $x = 5, x^2 = 25$
24. $x = 5, x^2 = 5x$
25. $x = 5, x - 1 = 4$
26. $x = 5, 2x + 1 = 11$
27. $3x + 2x = 6x, x = 0$
28. $\frac{x-2}{3} = \frac{x-3}{4}, 4(x-2) = 3(x-3)$
29. $\frac{x-2}{3} = \frac{x-3}{4}, \frac{x-2}{x-3} = \frac{3}{4}$
30. $\frac{x-2}{3} = \frac{x-3}{4}, \frac{x-3}{x-2} = \frac{4}{3}$

In Exercises 31–35, find an ordered pair of real numbers that satisfies the second of the given equations but not the first.

31. $\frac{y-2}{x} = 5, y - 2 = 5x$
32. $\frac{y-2}{x-3} = \frac{7-2}{5-3}, y - 2 = \frac{5}{2}(x-3)$
33. $\frac{y-4}{x-4} = \frac{14-4}{9-4}, 2x - y - 4 = 0$
34. $\frac{y-10}{2x} = 1, 2x - y + 10 = 0$
35. $\frac{x-2}{y-1} = \frac{2}{3}, \frac{x-2}{2} = \frac{y-1}{3}$

In Exercises 36–45, find the solution set of the given equation.

36. $3x - 4x = 8 - 5$

37. $3(x - 2) + 7(x - 3) = 50$

38. $x + \frac{x}{x} + 1 = \frac{x}{x}$

39. $\frac{x}{x} = x$

40. $x - 1 = x - 2$

41. $2x - 2 = 2(x - 1)$

42. $\frac{2x - 2}{x - 1} = 2$

43. $\frac{x - 1}{x - 1} + \frac{x - 2}{x - 2} + \frac{x - 3}{x - 3} = 3$

44. $\frac{x - 2}{x - 3} = 5$

45. $\frac{x - 2}{x - 3} + \frac{x - 2}{x - 3} = 2\left(\frac{x - 2}{x - 3}\right)$

CHAPTER SUMMARY

There were three BETWEENNESS POSTULATES and three SEPARATION POSTULATES in this chapter. We list them below by name only. Try to state each of them in your own words. Draw a picture illustrating what each postulate says.

9. THE A–B–C BETWEENNESS POSTULATE.
10. THE THREE-POINT BETWEENNESS POSTULATE.
11. THE LINE-BUILDING POSTULATE.
12. THE LINE SEPARATION POSTULATE.
13. THE PLANE SEPARATION POSTULATE.
14. THE SPACE SEPARATION POSTULATE.

The following concepts were defined in this chapter. Be sure that you know all of them.

SEGMENT
INTERIOR OF A SEGMENT
RAY
INTERIOR OF A RAY
OPPOSITE RAYS
ANGLE
INTERIOR OF AN ANGLE
EXTERIOR OF AN ANGLE
TRIANGLE
INTERIOR OF A TRIANGLE
EXTERIOR OF A TRIANGLE
QUADRILATERAL
TWO SIDES OF A POINT ON A LINE
TWO SIDES OF A LINE IN A PLANE
TWO SIDES OF A PLANE IN SPACE
HALFLINE
OPPOSITE HALFLINES
HALFPLANE
OPPOSITE HALFPLANES
HALFSPACE
OPPOSITE HALFSPACES

A set of points is called CONVEX if for every two points P and Q in the set, the entire segment $\overline{PQ}$ is in the set. The null set and every set of points that contains just one point are also said to be convex. Each of the following sets is a convex set: segment, ray, line, plane, halfline, halfplane, halfspace, interior of an angle, and interior of a triangle.

Six theorems were stated and proved in this chapter. Study them again so that you know and understand what they mean.

The last part of this chapter contains a review of elementary algebra. You should know and be able to use the properties of equality and number operations that are useful in solving equations.

REVIEW EXERCISES

In Exercises 1–15, indicate whether the statement is true or false.

1. If points A, B, C and line l are in the same plane, and if A and B are on the opposite sides of l, then C must be either on the A-side of l or on the B-side of l.
2. If B and C are two distinct points on the same side of line n in plane α, then every point of $\overline{BC}$ is on the B-side of n.
3. If A–B–C (point B is between points A and C) and B–D–C, then A–D–C and A–B–D.
4. If R–S–T and Q–S–T, then R–Q–S and R–Q–T.
5. If points R and S are on opposite sides of line m in plane α and points R and T are on opposite sides of m, then S and T are on the same side of m.
6. The betweenness relations R–S–T and R–U–T uniquely determine the order of the points R, S, T, U on a line.
7. If two rays intersect, they have one and only one point in common.
8. $\overline{AB} = \overline{BA}$
9. $\overrightarrow{AB} = \overrightarrow{BA}$
10. $opp\ \overrightarrow{AB} = \overrightarrow{BA}$
11. $\overleftrightarrow{AB} = \overleftrightarrow{BA}$
12. $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$
13. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$
14. $opp\ \overrightarrow{AB} \cap opp\ \overrightarrow{BA} = \overline{AB}$
15. If point B is between points A and C, then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays.

Exercises 16–20 refer to Figure 2-48.

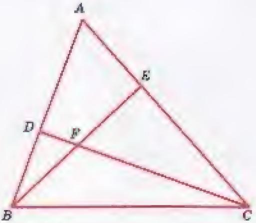


Figure 2-48

16. How many angles are determined by the segments shown in the figure?
17. How many triangles are determined by the segments shown in the figure? Name them.
18. Point E is in the interior of two angles. Name them.
19. Only one of the labeled points in the figure is in the interior of each of three different angles. Name the point and name the three angles.
20. Two of the labeled points in the figure are not in the interior of any of the angles. Name them.

21. Draw $\triangle ABC$. Mark a point D such that D is between A and B. Mark a point E such that B is between C and E.
    (a) Does $\overleftrightarrow{ED}$ intersect $\overline{AB}$? Why?
    (b) Does $\overleftrightarrow{ED}$ intersect $\overline{BC}$? Why?
    (c) Does $\overleftrightarrow{ED}$ intersect $\overline{AC}$? Why?
22. Define the interior of $\angle PQR$.
23. *Given:* Line $l \neq$ line m.
    Point C is between points A and B on line l.
    Point C is between points D and E on line m.
    Point G is between points C and E.
    Point F is between points A and E.
    (a) Which point (C, G, or F) is in the interior of $\angle ACE$?
    (b) Indicate whether each of the following is true or false.
        (1) Point F is on line l.
        (2) Point F is on line m.
        (3) Point F is a point of $\angle ACE$.
        (4) Point G is a point of $\angle BCE$.
        (5) Line $\overleftrightarrow{FG}$ does not intersect $\overline{AC}$.

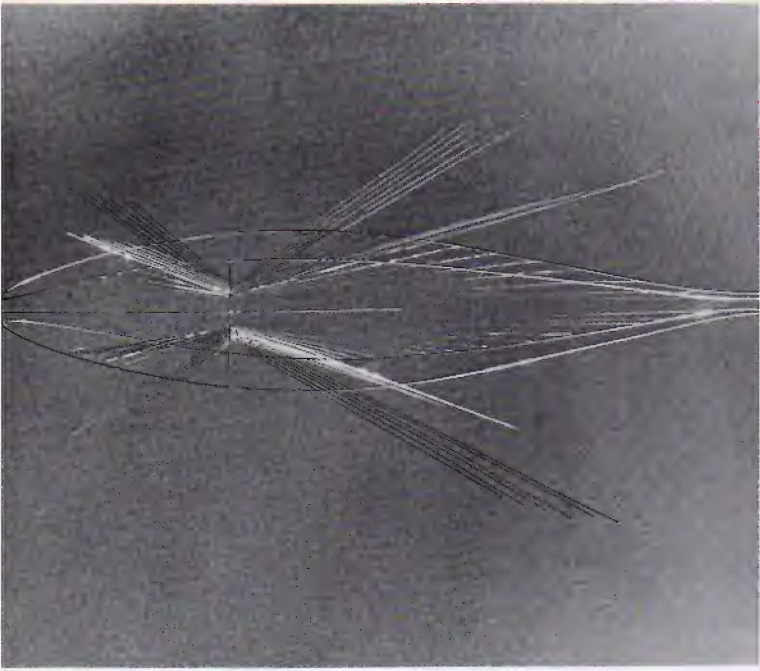
24. If r, s, t, u are four distinct coplanar rays having a common endpoint and if no two of the rays are collinear, how many angles are formed by these rays?
25. Which of our postulates guarantees that a halfplane is a convex set?
26. Is the intersection of two convex sets always a convex set?
27. Explain why the interior of an angle is a convex set.
28. Is a line with one point deleted a convex set?
29. Is the union of two convex sets always a convex set?
30. Describe two convex sets whose union is a convex set.

In Exercises 31–35, solve for x.

31. $3x - 5 = 7x - 25$
32. $2(x - 5) = 3x - 5$
33. $\frac{x - 1}{1} + 1 = \frac{x - 2}{3}$
34. $1.75x = 17.50$
35. $x - 1 + 2(x - 1) = 17(x - 1) + 14$

In Exercises 36–40, find the solution set.

36. $x + x = 2x$
37. $x + 1 = x + 2$
38. $\frac{x}{x} = 1$
39. $\frac{x - 1}{2x - 2} = \frac{1}{2}$
40. $x - 1 = \frac{1}{2}(2x - 2)$



Chapter 3

Hella Hammid/Rapho Guillumette

Distance and Coordinate Systems

3.1 INTRODUCTION

At this point in our formal geometry we have no postulates concerning the sizes of objects. We have no basis for saying how big an object is or even for saying that one object is bigger than another. Although the word "size" is often used in informal speech, it is not in the official vocabulary of formal geometry. Instead, we talk about the length of a segment, the measure of an angle, the area of a rectangle, the volume of a sphere, the distance between two points, and so on.

In elementary mathematics it is customary to draw a horizontal number line with numbers increasing from left to right and a vertical number line with numbers increasing in the upward direction. In some applications it may be more convenient to order the numbers from right to left or from up to down. A number line, such as the one in Figure 3-1 in which the numbers increase from left to right, is a good device for illustrating certain relationships among real numbers. One of them is the order relation. The fact that 3 is greater than 1 is consistent with the fact that 3 is to the right of 1 in Figure 3-1. The fact that 1 is to the right of -2 agrees with the fact that 1 is greater than -2.

Figure 3-1

Another relationship is based on the betweenness relation for real numbers. Since $\frac{1}{3}$ is between $\frac{1}{5}$ and $\frac{2}{5}$, the point marked $\frac{1}{3}$ lies between the points marked $\frac{1}{5}$ and $\frac{2}{5}$ on the number line shown in Figure 3-2.

Figure 3-2

This chapter is about distance and coordinate systems. We begin by considering the distance between two points. The idea of a coordinate system on a line is an extension of our ideas about a number line. Coordinate systems are useful in developing the properties of distance.

3.2 DISTANCE

Asking "How long is a certain segment?" is equivalent to asking, "How far apart are the endpoints of that segment?" In the world of real objects we can answer the question, "How far apart?" by using a physical ruler. We might determine that two points P and Q are 2 yd. apart, or 6 ft. apart, or 72 in. apart.

To make a physical ruler graduated, say, in inches, we must know what 1 in. is. We must have a segment 1 in. long. In the United States the accepted relation between inches and meters is 39.37 in. = 1 meter. For many years the meter was described officially by two marks on a platinum-iridium bar kept in France. These marks represented the endpoints of the segment that was the official meter. Although the modern standard for measuring distances is now based on the wavelength of a certain kind of light, the idea that a given segment may be a standard or unit for measuring distance is important in our geometry. In the remainder of this section we discuss informally some of the basic properties of distance and then state these properties formally as postulates.

Given any segment, say $\overline{PQ}$, we could agree that this segment is the unit of distance. Thus we might start with a given stick and say, "Let the distance from one end of this stick to the other end be 1, and let us call this unit of distance the "stick." Then the given stick or unit is the basis of a system of distances which we might call the stick-system. In this system the distance between any two different points is a positive number. In particular, the distance between the endpoints of the

unit stick is 1. If the distance between points R and S is 3 in this system, this means that R and S are 3 times as far apart as the endpoints of the unit stick. If the distance between points U and V is 1 in this system, this means that U and V are just as far apart as the endpoints of the unit stick.

Speaking more formally, each segment $\overline{PQ}$ determines a distance function. The domain of this function is the set of all segments, or if you prefer, the set of all pairs of distinct points. The range of this function is a set of positive numbers. It is convenient in developing the concept of distance to take the distance between a point and itself to be 0. Then the domain of a distance function is the set of all pairs (not necessarily distinct) of points and the range is a set of nonnegative numbers including 0. Later, after we adopt the Ruler Postulate, it will be obvious that the range is the set of *all* nonnegative numbers including 0.

Our first Distance Postulate is related to the idea that any stick can serve as a unit of distance. We might develop a formal geometry with one distance function. This would be like using inches for all distances whether they are thicknesses of paper or distances between stars. Our first postulate, the Distance Existence Postulate, reveals our preference for recognizing the possibility of various units of distance in our formal geometry. The other Distance Postulates are motivated also by physical experiences with distance, betweenness, and separation.

If A, B, C are three distinct collinear points with B between A and C, then we want the distance between A and B plus the distance between B and C to be equal to the distance between A and C as illustrated in Figure 3-3.

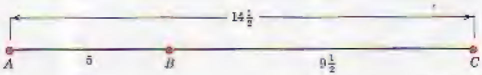


Figure 3-3

If A, B, C are three noncollinear points, then we want the distance from A to B (the same as the distance between A and B) to be less than the distance from A to C plus the distance from C to B as illustrated in Figure 3-4.

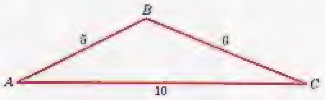


Figure 3-4

We are now ready to state our Basic Distance Postulates in which we exercise extreme care in the choice of words so that we agree on exactly what they mean. As always we need to know precisely what we are accepting without proof in the building of our formal geometry. We use "unique" to mean "one and only one" or "exactly one."

Basic Distance Postulates

POSTULATE 15 ***(Distance Existence Postulate)*** If $\overline{AB}$ is any segment, there is a correspondence which matches with every segment $\overline{CD}$ in space a unique positive number, the number matched with $\overline{AB}$ being 1.

If C and D are distinct points, there is a unique positive number matched with $\overline{CD}$ according to Postulate 15. We may also think of this number as matched with the set $\{C, D\}$. If we associate this number with the segment $\overline{CD}$, we think of it as a length. If we associate the number with the set $\{C, D\}$, we think of it as a distance. Postulate 15, however, says nothing about the distance from C to D if $C = D$; in other words, it says nothing about the distance between a point and itself. We take care of this with a definition.

Definition 3.1 The distance between any point and itself is 0.

Definition 3.2

1. The correspondence that matches a unique positive number with each pair of distinct points C and D, as in Postulate 15, and the number 0 with the points C and D if $C = D$, as in Definition 3.1, is called the distance function determined by $\overline{AB}$ or the distance function based on $\overline{AB}$.
2. The segment $\overline{AB}$ that determines a distance function is the unit segment for that distance function.
3. The number matched with C and D, as in Postulate 15, is the distance from C to D or the distance between C and D or the length of $\overline{CD}$.

Notation. If P and Q are any points, not necessarily distinct, then the distance between P and Q in the distance function based on $\overline{AB}$ is denoted by PQ(in $\overline{AB}$ units) or simply by PQ if the unit is understood.

Example 1 (*Informal*) Suppose $\overline{AB}$ is a segment 1 in. long, $\overline{CD}$ is a segment 1 ft. long, and $\overline{EF}$ is a segment 6 in. long. Then

$$EF \text{ (in } \overline{AB} \text{ units)} = 6$$

and

$$EF \text{ (in } \overline{CD} \text{ units)} = \tfrac{1}{2}.$$

Although Postulates 16 and 17 could be omitted and proved as theorems after the Ruler Postulate is adopted, they are included in our formal geometry in order to simplify the development.

POSTULATE 16 (***Distance Betweenness Postulate***) If A, B, C are collinear points such that A–B–C, then for any distance function we have $AB + BC = AC$. (See Figure 3-5.)

Figure 3-5

Example 2 If A–B–C, $AB = 6$, $BC = 8$, then $AC = 14$.

POSTULATE 17 (***Triangle Inequality Postulate***) If A, B, C are noncollinear points, then for distances in any system we have $AB + BC > AC$. (See Figure 3-6.)

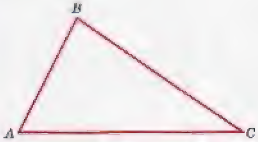


Figure 3-6

Example 3 If A, B, C are noncollinear points and if $BC = 10$, then

$$BA + AC > 10.$$

Example 4 If the lengths of the sides of a triangle are x, y, z, then

$$x + y > z, \qquad y + z > x, \qquad \text{and} \qquad z + x > y.$$

Suppose that $\overline{PQ}$ is a segment 1 in. long and that $\overline{RS}$ is a segment 1 ft. long. Suppose that A and B are two different points and that C and

D are two different points. We use distances to compare how far A is from B with how far C is from D. If we say that it is 3 times as far from A to B as from C to D, we mean that $AB = 3\ CD$, or that $\frac{AB}{CD} = 3$.

Figure 3-7

(See Figure 3-7.) Our experience with physical measurements tells us that we should get the same comparison regardless of which distance function we use. Thus

$$\text{if} \quad \frac{AB \text{ (in } \overline{PQ} \text{ units)}}{CD \text{ (in } \overline{PQ} \text{ units)}} = 3, \quad \text{then} \quad \frac{AB \text{ (in } \overline{RS} \text{ units)}}{CD \text{ (in } \overline{RS} \text{ units)}} = 3 \text{ also.}$$

For example, if $AB = 72$ (in inches) and $CD = 24$ (in inches), then $AB = 6$ (in feet) and $CD = 2$ (in feet). Thus

$$\frac{AB \text{ (in inches)}}{CD \text{ (in inches)}} = \frac{72}{24} = 3 \qquad \text{and} \qquad \frac{AB \text{ (in feet)}}{CD \text{ (in feet)}} = \frac{6}{2} = 3.$$

This is the basis for our next postulate.

POSTULATE 18 (*Distance Ratio Postulate*) If $\overline{PQ}$ and $\overline{RS}$ are unit segments and A, B, C, D are points such that $A \neq B$, $C \neq D$, then

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{CD \text{ (in } \overline{PQ} \text{ units)}} = \frac{AB \text{ (in } \overline{RS} \text{ units)}}{CD \text{ (in } \overline{RS} \text{ units)}}$$

or, equivalently,

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{AB \text{ (in } \overline{RS} \text{ units)}} = \frac{CD \text{ (in } \overline{PQ} \text{ units)}}{CD \text{ (in } \overline{RS} \text{ units)}}.$$

In making physical measurements we recognize that there are many accurate foot rulers. Measurements made with an official foot ruler and an accurate copy of one should agree. Let us see how Postulate 18 is concerned with this.

Suppose that $\overline{PQ}$ and $\overline{RS}$ are unit segments and RS (in $\overline{PQ}$ units) $= 1$. This means that the length of $\overline{RS}$ in $\overline{PQ}$ units is 1 or, informally, that $\overline{RS}$ is a copy of $\overline{PQ}$. If A and B are any points, it follows from Postulate 18 that

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{RS \text{ (in } \overline{PQ} \text{ units)}} = \frac{AB \text{ (in } \overline{RS} \text{ units)}}{RS \text{ (in } \overline{RS} \text{ units)}}.$$

Since we assumed that RS (in $\overline{PQ}$ units) $= 1$ and since RS (in $\overline{RS}$ units) $= 1$ by the Distance Existence Postulate, it follows that

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{1} = \frac{AB \text{ (in } \overline{RS} \text{ units)}}{1}$$

or
$$AB \text{ (in } \overline{PQ} \text{ units)} = AB \text{ (in } \overline{RS} \text{ units)}.$$

This proves the following theorem.

THEOREM 3.1 If $\overline{PQ}$ and $\overline{RS}$ are segments such that the length of $\overline{RS}$ in $\overline{PQ}$ units is 1, then for all points A and B it is true that

$$AB \text{ (in } \overline{RS} \text{ units)} = AB \text{ (in } \overline{PQ} \text{ units)}.$$

EXERCISES 3.2

1. If A and B are points and if $\overline{CD}$ is a segment, which of the following are necessarily true about the number AB (in $\overline{CD}$ units)?
   (a) It is a real number.
   (b) It is a positive number.
   (c) It is a nonnegative number.
   (d) It is an irrational number.
2. If you know that A and B are distinct points and that $\overline{RS}$ is a segment, what can you say about the number AB (in $\overline{RS}$ units)?
3. If you know that A–B–C, what can you say about the number $\dfrac{AB + BC}{AC}$?
4. If you know that A, B, C are noncollinear points, what can you say about the number $\dfrac{AB + BC}{AC}$?
5. If you know that A, B, C are points, that $A \neq C$, and that $\overline{PQ}$ and $\overline{RS}$ are segments, what can you say about the numbers

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{AC \text{ (in } \overline{PQ} \text{ units)}} \quad \text{and} \quad \frac{AB \text{ (in } \overline{RS} \text{ units)}}{AC \text{ (in } \overline{RS} \text{ units)}}?$$

   Why is it not necessary to say $A \neq C$ in Exercise 4?
6. If you know that A, B, C are noncollinear points and that $\overline{PQ}$ and $\overline{RS}$ are segments, how does the number

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)} + BC \text{ (in } \overline{PQ} \text{ units)}}{AC \text{ (in } \overline{PQ} \text{ units)}}$$

   compare with the number

$$\frac{AB \text{ (in } \overline{RS} \text{ units)} + BC \text{ (in } \overline{RS} \text{ units)}}{AC \text{ (in } \overline{RS} \text{ units)}}?$$

7. Let A, B, C, D, E be distinct points ordered on a line l as shown in the figure, and equally spaced so that AB (in $\overline{AB}$ units) $= BC$ (in $\overline{AB}$ units) $= CD$ (in $\overline{AB}$ units) $= DE$ (in $\overline{AB}$ units). Find the following: AB (in $\overline{BC}$ units), BC (in $\overline{BC}$ units), CD (in $\overline{BC}$ units), and DE (in $\overline{BC}$ units).

8. Given the situation of Exercise 7, find the distances AB, BC, CD, DE, all in $\overline{AC}$ units.
9. Given the situation of Exercise 7, find the numbers AB (in $\overline{AB}$ units), AB (in $\overline{AC}$ units), AB (in $\overline{BE}$ units), and AB (in $\overline{CE}$ units).
10. Given the situation of Exercise 7, find AD (in $\overline{AB}$ units), AD (in $\overline{BC}$ units), AD (in $\overline{AC}$ units), AD (in $\overline{BE}$ units), and AD (in $\overline{AE}$ units).
11. Given the situation of Exercise 7, copy and complete the following proof that BD (in $\overline{AC}$ units) $= 1$.

    Proof: Expressing all distances in $\overline{AC}$ units we have the following:

1. $BD = BC + CD$	1. Distance Betweenness Postulate
2. $\dfrac{CD}{AB} = 1$	2. Exercise 7; Distance Ratio Postulate
3. $CD = AB$	3. Multiplication Property of Equality
4. $BD = BC + AB$	4. Steps 1, 3; Substitution
5. $BD = AB + BC$	5. $\boxed{?}$
6. $AC = AB + BC$	6. $\boxed{?}$
7. $BD = AC$	7. $\boxed{?}$
8. $AC = 1$	8. $\boxed{?}$
9. $BD = 1$	9. $\boxed{?}$

12. In the Distance Ratio Postulate, take $\overline{PQ} = \overline{CD}$ and $\overline{RS} = \overline{AB}$. Using this special case of the postulate, prove that

    $$AB \text{ (in } \overline{CD} \text{ units)} \cdot CD \text{ (in } \overline{AB} \text{ units)} = 1.$$

    (Does this result seem reasonable? Think of $\overline{AB}$ as a stick 2 ft. long and $\overline{CD}$ as a stick 3 ft. long. Then AB (in $\overline{CD}$ units) $= \frac{2}{3}$, CD (in $\overline{AB}$ units) $= \frac{3}{2}$, and $\frac{2}{3} \cdot \frac{3}{2} = 1$.)

13. Given A, B, C, D such that AB (in $\overline{CD}$ units) $= \dfrac{3}{4}$, find CD (in $\overline{AB}$ units).

    (Does your answer seem reasonable? The given information means that $\overline{AB}$ is $\dfrac{3}{4}$ as long as $\overline{CD}$. Your answer means that $\overline{CD}$ is how many times as long as $\overline{AB}$?)

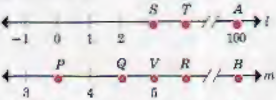
14. CHALLENGE PROBLEM. Given segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, prove that

    $$EF \text{ (in } \overline{AB} \text{ units)} = EF \text{ (in } \overline{CD} \text{ units)} \cdot CD \text{ (in } \overline{AB} \text{ units)}.$$

    (A simple example is: The number of feet from E to F equals the number of yards from E to F times the number of feet in a yard.)

15. (*An Informal Geometry Exercise.*) In the figure, suppose that the part of l between P and M is on the edge of a ruler. On this ruler 0 is assigned to P and 1 to Q. What numbers should be assigned to C, D, E, and M?

16. (*An Informal Geometry Exercise.*) Two number lines l and m are placed parallel to each other as shown in the figure below. What numbers should be assigned to S and T on l? What numbers should be assigned to P, Q, and R on m?



17. Refer to Exercise 16. If A on l is directly above B on m and if 100 is assigned to A, what number is assigned to B?

18. CHALLENGE PROBLEM. Refer to Exercise 16. If a number x is assigned to a point on l and if the number x' is assigned to the point directly below it on m, express x' in terms of x.

19. In the Distance Ratio Postulate we stated that the two equations were equivalent. (a) Derive the second equation from the first one using some of the properties of equality. (b) Derive the first equation from the second one using some of the properties of equality.

3.3 LINE COORDINATE SYSTEMS

A key feature of a number line is that different points are matched with different numbers. In fact, we consider all the points of the line as matched with all the real numbers. Such a matching is called a one-to-one correspondence between the set of all points on the line and the set of all real numbers.

Another key feature of a number line is that the distance between any two points is the absolute value of the difference of the numbers matched with those two points. A unit segment for these distances is the segment whose endpoints are matched with 0 and 1.

Example In Figure 3-8,

$AB = \|-1 - (-2)\| = 1.$	$BC = \|0 - (-1)\| = \|1\| = 1.$
$CE = \|2 - 0\| = 2.$	$AC = \|0 - (-2)\| = \|2\| = 2.$
$BD = \|1 - (-1)\| = 2.$	$BE = \|2 - (-1)\| = \|3\| = 3.$
$AE = \|2 - (-2)\| = 4.$	$AE = \|2 - (-2)\| = \|4\| = 4.$

Figure 3-8

The concept in formal geometry that corresponds to a number line in informal geometry is the concept of a coordinate system on a line. Coordinate systems are useful tools in our formal development of geometry for planes and spaces as well as for lines.

Definition 3.3 Let $\overline{PQ}$ be a unit segment and l a line. A coordinate system on l relative to $\overline{PQ}$ is a one-to-one correspondence between the set of all points of l and the set of all real numbers such that if points A, B, C are matched with the real numbers a, b, c, respectively, then

1. B is between A and C if and only if b is between a and c and
2. AB (in $\overline{PQ}$ units) $= |a - b|$.

Definition 3.4

1. The origin of a coordinate system on a line is the point matched with 0.
2. The unit point is the point matched with 1.
3. The number matched with a point is its coordinate.

These definitions describe and help us to think about a coordinate system. But a very important issue must not be overlooked here. How do we know there are things such as coordinate systems in our formal geometry? To answer this question we return to our experiences with number lines and physical measurements.

Let us suppose that $\overline{PQ}$ is a unit segment, say a segment 1 cm. long, and that A and B are any two distinct points on l. From our experience with physical rulers we know that we can lay a ruler graduated in centimeters alongside l and measure distances starting from A and extending toward B as shown in Figure 3-9.

Figure 3-9

Of course, we can just as well start at B and measure toward A as shown in Figure 3-10. These ideas suggest our next postulate.

Figure 3-10

POSTULATE 19 (***Ruler Postulate***) If $\overline{AB}$ is a unit segment and if P and Q are distinct points on a line l, then there is a unique coordinate system on l relative to $\overline{AB}$ such that the origin is P and the coordinate q of Q is a positive number. (See Figure 3-11.)

Figure 3-11

We are now ready for a theorem.

THEOREM 3.2 (***The Origin and Unit Point Theorem***) If P and Q are any two distinct points, then there is a unique coordinate system on $\overleftrightarrow{PQ}$ with P as origin and Q as unit point.

Proof:

1. There is a unique coordinate system on $\overleftrightarrow{PQ}$ relative to $\overline{PQ}$, with the coordinate of P equal to 0 and the coordinate q of Q a positive number.	1. Ruler Postulate
2. PQ (in $\overline{PQ}$ units) $= q - 0 = q$.	2. $\boxed{?}$
3. PQ (in $\overline{PQ}$ units) $= 1$.	3. $\boxed{?}$
4. $q = 1$	4. $\boxed{?}$
5. Q is the unit point.	5. $\boxed{?}$

3.4 RAYS, SEGMENTS, AND COORDINATES

Let A and B be two distinct points on a line l. We know from the Origin and Unit Point Theorem that there is a coordinate system $\mathcal{S}$ with A as origin and B as unit point. Let X be any point of l and let x be its coordinate in $\mathcal{S}$. Then it follows from the definition of a coordinate system that A is between X and B if and only if 0 is between x and 1; X is between A and B if and only if x is between 0 and 1; B is between A and X if and only if 1 is between 0 and x. (See Figure 3-12.)

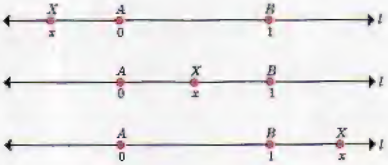


Figure 3-12

Now 0 is between x and 1 if and only if $x < 0$; x is between 0 and 1 if and only if $0 < x < 1$; and 1 is between 0 and x if and only if $x > 1$. It follows from the definitions of ray, ray opposite to a ray, and segment that:

$opp\ \overrightarrow{AB} = \{X : x \leq 0\}$

$\overline{AB} = \{X : 0 \leq x \leq 1\}$

$\overrightarrow{AB} = \{X : x \geq 0\}$

$\overrightarrow{BA} = \{X : x \leq 1\}$

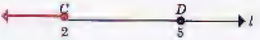
Similarly, if C and D are two distinct points on a line l with coordinates 2 and 5, respectively, and if X is a (variable) point on l with coordinate x, then:

$\overline{CD} = \{X : 2 \leq x \leq 5\}$

$\overrightarrow{CD} = \{X : x \geq 2\}$

$\overrightarrow{DC} = \{X : x \leq 5\}$

$opp\ \overrightarrow{CD} = \{X : x \leq 2\}$



$opp\ \overrightarrow{DC} = \{X : x \geq 5\}$

$\overleftrightarrow{CD} = \{X : x \text{ is real}\}$

More generally, if the coordinates of two distinct points A and B on a line l are a and b, respectively, then it is convenient to consider two cases in expressing subsets of $\overleftrightarrow{AB}$ using set-builder symbols as follows.

If $a < b$	If $a > b$
$\overline{AB} = \{X : a \leq x \leq b\}$	$\overline{AB} = \{X : b \leq x \leq a\}$
$\overrightarrow{AB} = \{X : x \geq a\}$	$\overrightarrow{AB} = \{X : x \leq a\}$
$\overrightarrow{BA} = \{X : x \leq b\}$	$\overrightarrow{BA} = \{X : x \geq b\}$
$opp\ \overrightarrow{AB} = \{X : x \leq a\}$	$opp\ \overrightarrow{AB} = \{X : x \geq a\}$
$opp\ \overrightarrow{BA} = \{X : x \geq b\}$	$opp\ \overrightarrow{BA} = \{X : x \leq b\}$

EXERCISES 3.4

In Exercises 1–5, a line l with a coordinate system $\mathcal{S}$ is given. The coordinates of points A, B, C are -3, 0, 4, respectively. In each exercise draw the graph of the set and express it in terms of coordinates using a set-builder symbol. Exercise 1 has been completed as a sample.

1. $\overrightarrow{BC}$

   Solution: $\overrightarrow{BC} = \{X : x \geq 0\}$

2. $\overrightarrow{BA}$
3. $\overline{AC}$
4. $\overline{CB}$
5. $\overleftrightarrow{AB}$

In Exercises 6–10, a line l and a coordinate system $\mathcal{S}$ are given. In each exercise, given the coordinates of two points, find the coordinate of a third point. Exercise 6 has been worked as a sample. (*Note: cd A* means "coordinate of A.")

6. $cd\ A = 2$, $cd\ B = 5$, $cd\ X = x$. If $X \in \overrightarrow{BA}$ and $AX = 2 \cdot AB$, find x.

   Solution: $x < 2$, $AX = 2 - x$, $AB = 3$, $2 - x = 6$, $x = -4$

7. $cd\ C = 1$, $cd\ D = 7$, $cd\ P = p$. If $P \in \overrightarrow{CD}$ and $\dfrac{CP}{CD} = \dfrac{1}{2}$, find p.
8. $cd\ E = -5$, $cd\ F = 0$, $cd\ Q = q$. If $Q \in opp\ \overrightarrow{FE}$ and $EF = FQ$, find q.
9. $cd\ G = 29$, $cd\ H = 129$, $cd\ I = i$. If $GI = HI$, find i.
10. $cd\ J = 15$, $cd\ K = 0$, $cd\ R = r$. If $JR = 2 \cdot RK$, find the two possible values of r.

In Exercises 11–15, given the coordinates of two points on a line, find the length of the segment joining these two points.

11. $cd\ A = 5$, $cd\ B = 173$
12. $cd\ C = -5$, $cd\ B = 173$
13. $cd\ C = -5$, $cd\ D = -173$
14. $cd\ E = 147.5$, $cd\ F = 237.6$
15. $cd\ G = 19\frac{1}{3}$, $cd\ H = -17\frac{2}{5}$

In Exercises 16–20, given the coordinates of two points on a line, find the coordinate p of a third point P satisfying the stated condition.

16. $cd\ A = 5$, $cd\ B = 10$, $P \in \overrightarrow{AB}$ and $AP = 5$

17. $cd\ A = 5$, $cd\ B = 10$, $P \in opp\ \overrightarrow{BA}$ and $AP = 5$

18. $cd\ A = 5$, $cd\ B = 10$, $P \in \overrightarrow{BA}$ and $AP = 5$

19. $cd\ A = 5$, $cd\ B = 10$, $P \in opp\ \overrightarrow{AB}$ and $AP = 5$

20. $cd\ C = -10$, $cd\ D = 0$, $P \in \overrightarrow{CD}$ and $PC = 17\frac{2}{3}$

In Exercises 21–25, given a line l and a coordinate system on it in which x is the coordinate of a variable point X, draw a sketch showing l and the given subset of l.

21. $\{X : -6 \leq x \leq -2\}$

22. $\{X : x \leq -6\}$

23. $\{X : x \geq -2\}$

24. $\{X : x \leq -6 \text{ or } x \geq -2\}$

25. $\{X : x \leq 10\}$

In Exercises 26–35, given a line with points and coordinates as marked in Figure 3-13, find the given distance.

Figure 3-13

26. CE

27. DG

28. HP

29. IC

30. FQ

31. PQ

32. ST

33. SF

34. MQ

35. PS

In Exercises 36–45, given a line and a coordinate system in which $cd\ A = 12$, $cd\ B = -8$, $cd\ C = 0$, $cd\ D = -4.2$, and $cd\ E = \sqrt{5}$, determine if the given betweenness relation is true or if it is false.

36. A–B–C

37. A–B–D

38. D–C–E

39. C–E–A

40. E–C–B

41. B–E–A

42. D–C–A

43. D–B–A

44. B–E–D

45. C–A–B

In Exercises 46–50, given that $\overline{AB}$ is a segment 1 ft. long and that $\overline{CD}$ is a segment 1 yd. long, find the given distance.

46. EF (in $\overline{AB}$ units), if EF (in $\overline{CD}$ units) $= 5$
47. GH (in $\overline{CD}$ units), if GH (in $\overline{AB}$ units) $= 5$
48. CD (in $\overline{CD}$ units)
49. CD (in $\overline{AB}$ units)
50. AB (in $\overline{CD}$ units)

51. Copy and complete the proof of Theorem 3.2.

3.5 SEGMENTS AND CONGRUENCE

In Chapter 5 we develop in considerable detail an idea called the congruence idea. Informally speaking, two figures are *congruent* if they have the "same size and shape." The terms "size" and "shape" are not considered as a part of our formal geometry. In elementary geometry, we develop the concept of congruence for segments, for angles, and for triangles. The concept of congruence for segments is easy and is appropriate to include here. Intuitively, we feel that all segments have the same shape; hence, they have the same size and shape if they have the same length. We make this formal in the following definition.

> ***Definition 3.5*** Two segments (distinct or not) are congruent if and only if they have the same length. If two segments are congruent, we say that each of them is congruent to the other one and we refer to them as congruent segments.

It is convenient to have a special symbol for congruence; thus $\overline{AB} \cong \overline{CD}$ means $\overline{AB}$ is congruent to $\overline{CD}$.

It may be helpful to compare the words "congruent and congruence" with the words "equal and equality."

$7 = 3 + 4$ may be read as "7 is equal to 3 plus 4."
$7 = 3 + 4$ is an example of an equality.
$\overline{AB} \cong \overline{CD}$ may be read as "$\overline{AB}$ is congruent to $\overline{CD}$."
$\overline{AB} \cong \overline{CD}$ is an example of a congruence.

When working with segments it is important to note carefully the difference between equality and congruence. The statement $\overline{AB} = \overline{CD}$ means that $\overline{AB}$ and $\overline{CD}$ are the same set of points, that is, that "$\overline{AB}$"

and "$\overline{CD}$" are different names for the same segment. This statement is true if and only if $A = C$ and $B = D$, or $A = D$ and $B = C$. (See Figure 3-14.)

Figure 3-14

The statement $\overline{AB} \cong \overline{CD}$ means that $\overline{AB}$ and $\overline{CD}$ have the same length, that is, $AB = CD$. Therefore, if $\overline{AB} \cong \overline{CD}$, then $AB = CD$; and if $AB = CD$, then $\overline{AB} \cong \overline{CD}$. Note that $\overline{AB} = \overline{CD}$ implies $\overline{AB} \cong \overline{CD}$ but that $\overline{AB} \cong \overline{CD}$ does not imply $\overline{AB} = \overline{CD}$.

We have emphasized the difference between congruence and equality as these concepts apply to segments. We note next some similarities. Recall the equivalence properties of equality, the reflexive, symmetric, and transitive properties. (See Section 2.8.) Congruence for segments has the same properties, as stated in the following theorem.

THEOREM 3.3 Congruence for segments is reflexive, symmetric, and transitive.

Proof: Let $\overline{AB}$ be any segment. Then its length AB is a number, and $AB = AB$ by the reflexive property of equality. But $AB = AB$ implies that $\overline{AB} \cong \overline{AB}$. Therefore congruence for segments is reflexive. (The remainder of this proof is assigned as an exercise.)

Our next theorem expresses formally a simple idea concerned with adding or subtracting lengths. The theorem expresses this idea formally in terms of congruences. Theorem 3.4 is followed by Corollary 3.4.1. A corollary is a theorem associated with another theorem from which it follows rather easily.

THEOREM 3.4 ***(The Length-Addition Theorem for Segments)*** If distinct points B and C are between points A and D and if $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

Proof: There are two possibilities as suggested in Figure 3-15.

Figure 3-15

Case 1. If A–B–C, then B–C–D, and it follows from the Distance Betweenness Postulate that

$$AB + BC = AC \qquad \text{and} \qquad BC + CD = BD.$$

Since $AB = CD$ (hypothesis) and since $BC = BC$ (Why?), it follows from the addition property of equality that

$$AB + BC = CD + BC.$$

But

$$CD + BC = BC + CD. \qquad \text{(Why?)}$$

Therefore

$$AB + BC = BC + CD \qquad \text{(Why?)}$$

and $AC = BD$. This proves that $\overline{AC} \cong \overline{BD}$.

Case 2. If A–C–B, then C–B–D, and it follows from the Distance Betweenness Postulate that

$$AC + CB = AB \qquad \text{and} \qquad CB + BD = CD.$$

(The remainder of this proof is assigned as an exercise. Note in Case 1 that the idea, or strategy, of the proof is the addition of lengths. The proof of Case 2 involves the subtraction of lengths.)

COROLLARY 3.4.1 If distinct points B and C are between points A and D and if $\overline{AC} \cong \overline{BD}$, then $\overline{AB} \cong \overline{CD}$.

Proof: If B and C are between A and D, then C and B are between A and D. The corollary follows immediately from Theorem 3.4 by interchanging B and C, that is, by renaming point B as point C and renaming point C as point B.

COROLLARY 3.4.2 If A, B, C, D, E, F, are points such that A–B–C, D–E–F, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

Proof: Assigned as an exercise.

COROLLARY 3.4.3 If A, B, C, D, E, F are points such that A–B–C, D–E–F, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, then $\overline{BC} \cong \overline{EF}$.

Proof: Assigned as an exercise.

We now raise a question that leads to the final theorem of this section. Suppose that a distance function determined by $\overline{PQ}$, say, and a ray $\overrightarrow{AB}$ are given. Is there a point C on $\overrightarrow{AB}$ such that $AC = 7$? Is there only one such point? We know from the Ruler Postulate that there is a unique coordinate system on $\overleftrightarrow{AB}$ relative to $\overline{PQ}$ such that the origin is A and the coordinate b of B is a positive number. Then

$$\overrightarrow{AB} = \{X : x \geq 0\}.$$

(See Figure 3-16.)

Figure 3-16

This means that there is a one-to-one correspondence between the set of all points of $\overrightarrow{AB}$ and the set of all nonnegative numbers. Therefore there is exactly one point C on $\overrightarrow{AB}$ such that the coordinate of C is 7. Then

$$AC = 7 - 0 = 7.$$

If D is any other point of $\overrightarrow{AB}$, then the coordinate d of D is not 7 and $AD = d - 0 \neq 7$. Therefore there is one and only one point C on $\overrightarrow{AB}$ such that $AC = 7$.

We know from our experience with rulers that we can start at any point on a line and "lay off" in either direction (along either of the two rays on the line that have the starting point as its endpoint) a segment of any desired length. The example we discussed shows that in our formal geometry, given any ray, we can "lay off," or "construct," a segment which is 7 units long such that the endpoint of the ray is one of the endpoints of the segment. Of course, we could start with any positive number other than 7 and the same reasoning would apply. We could start also with a given segment and "lay off" on $\overrightarrow{AB}$ a segment whose length is the length of the given segment, that is, a segment congruent to the given segment. The "lay off" or "construct" language is informal. All we are really saying is that there is such a segment and that it is unique. These ideas lead to our next theorem.

THEOREM 3.5 ***(Segment Construction Theorem)*** Given a segment $\overline{CD}$ and a ray $\overrightarrow{AB}$, there is exactly one point P on $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.

Proof: Given a ray $\overrightarrow{AB}$, a segment $\overline{CD}$ with $CD = p$, and a distance function, there is a unique coordinate system $\mathcal{S}$ on $\overleftrightarrow{AB}$ such that A is the origin and such that the coordinate b of B is a positive number. (See Figure 3-17.) Then

$$\overrightarrow{AB} = \{X : x \geq 0\}.$$

Figure 3-17

Let P be the unique point of $\overrightarrow{AB}$ such that the coordinate of P is p. Then $AP = p - 0 = p$. If Q is any point of $\overrightarrow{AB}$ other than P, then its coordinate q is different from p and

$$AQ = q - 0 = q \neq p.$$

Therefore there is only one P on $\overrightarrow{AB}$ such that $AP = p$, that is, such that

$$\overline{AP} \cong \overline{CD}.$$

EXERCISES 3.5

In Exercises 1–10, a line l with a coordinate system $\mathcal{S}$ is given. The coordinates of points A, B, C, D are -5, 3, 5, 13, respectively. (See Figure 3-18.) In each exercise, determine whether the given statement is true or false.

Figure 3-18

1. $\overline{AB} \cong \overline{CD}$
2. $\overline{BC} \cong \overline{BC}$
3. $\overline{AC} \cong \overline{BD}$
4. $AB + BC = AC$
5. $\overline{AB} \cup \overline{BC} = \overline{AC}$
6. $\overline{AB} \cong \overline{BC}$
7. $\overline{AB} = \overline{BA}$
8. $\overline{AB} = \overline{CD}$
9. $\overline{AC} \cap \overline{BD} = \overline{BC}$
10. $AC + BD = AB + CD + 2 \cdot BC$

In Exercises 11–15 name the property of congruence for segments which justifies the given statement.

11. If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
12. If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{DE}$, then $\overline{AB} \cong \overline{DE}$.
13. $\overline{PQ} \cong \overline{QP}$
14. If $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.
15. If $\overline{XY}$ is a segment, then $\overline{XY} \cong \overline{XY}$.

16. Complete the proof of Theorem 3.3.
17. Complete the proof of Theorem 3.4.
18. Draw an appropriate figure and prove Corollary 3.4.2.
19. Draw an appropriate figure and prove Corollary 3.4.3.

In Exercises 20–25, the coordinates of points B and C in a coordinate system on $\overleftrightarrow{BC}$ are given. If A-B-C and $\overline{AB} \simeq \overline{BC}$, find the coordinate of A.

20. $cd\ B = 0,\ cd\ C = 5$
21. $cd\ B = 5,\ cd\ C = 0$
22. $cd\ B = -234,\ cd\ C = -198$
23. $cd\ B = \frac{2}{3},\ cd\ C = \frac{2}{5}$
24. $cd\ B = 159,\ cd\ C = -159$
25. $cd\ B = 27,\ cd\ C = 102$

3.6 TWO COORDINATE SYSTEMS ON A LINE

Let A and B be the origin and the unit point, respectively, of a coordinate system $\mathcal{S}$ on a line l. What is the coordinate of the point P if P is between A and B and two-thirds of the way from A to B? (See Figure 3-19.) Obviously, $cd\ P = \frac{2}{3}$ since $AP = \frac{2}{3}$ and $AB = 1$.

Figure 3-19

Let C and D be points with coordinates 35.0 and 39.8, respectively, on a line l. What is the coordinate q of point Q if it is between C and D and two-thirds of the way from C to D? (See Figure 3-20.)

Figure 3-20

We know that there is another coordinate system on l in which the coordinate of C is 0 and the coordinate of D is 1. If Q is thought of as a variable point on this line with q as its coordinate in one system and x as its coordinate in the other system, then the particular point Q we want has $\frac{2}{3}$ as its x-coordinate. In this section, we learn how to express the q-coordinates in terms of the x-coordinates. In this example, the relation is

$$q = 35.0 + 4.8x.$$

Substituting $\frac{2}{3}$ for x we get

$$q = 35.0 + (4.8)(\tfrac{2}{3}) = 35.0 + 3.2 = 38.2.$$

Therefore the point that is two-thirds of the way from the point with coordinate 35.0 to the point with coordinate 39.8 is the point with coordinate 38.2.

The relationship between two coordinate systems on a line is useful in solving exercises involving points of division like the "two-thirds of the way from C to D" exercise as well as some exercises appearing later in this chapter. This relationship is also useful in gaining new algebraic insights. In studying equations like

$$y = 2 + 3x \qquad \text{and} \qquad \frac{x-3}{2} = \frac{k-0}{1},$$

it is sometimes helpful to think of them geometrically in terms of coordinates on a line.

In later chapters we consider coordinate systems in a plane and in space. In studying a line in a plane or in space you will find it helpful to think of several coordinate systems associated with the line. What we do later is a natural extension of the groundwork we are laying in this chapter.

We begin with a lemma, a "little theorem," that is useful in proving a "big theorem." Lemma 3.6.1 plays a key role in the proof of Theorem 3.6, which is followed by Corollary 3.6.1.

LEMMA 3.6.1 Let x_1 and x_2 be the coordinates of distinct points X_1 and X_2, respectively, on a line l. If x is the coordinate of a point X on l, then

$$\frac{XX_1}{X_2X_1} = \frac{x - x_1}{x_2 - x_1} \qquad \text{if} \qquad X \in \overrightarrow{X_1X_2}$$

and

$$\frac{XX_1}{X_2X_1} = -\frac{x - x_1}{x_2 - x_1} \qquad \text{if} \qquad X \in opp\ \overrightarrow{X_1X_2}.$$

Proof: First, suppose that $X \in \overrightarrow{X_1X_2}$; then either $x_1 < x_2$ or $x_2 < x_1$. (See Figure 3-21.) If $x_1 < x_2$, then $x_1 \le x$, $XX_1 = x - x_1$, $X_2X_1 = x_2 - x_1$, and

$$\frac{XX_1}{X_2X_1} = \frac{x - x_1}{x_2 - x_1}$$

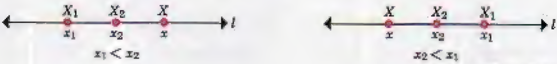


Figure 3-21

If $x_2 < x_1$, then $x \le x_1$, $XX_1 = x_1 - x$, $X_2X_1 = x_1 - x_2$, and

$$\frac{XX_1}{X_2X_1} = \frac{x_1 - x}{x_1 - x_2} = \frac{x - x_1}{x_2 - x_1}.$$

Next suppose that $X \in opp\ \overrightarrow{X_1X_2}$; then either $x_1 < x_2$ or $x_2 < x_1$. (See Figure 3-22.) If $x_1 < x_2$, then $x_1 \geq x$, $XX_1 = x_1 - x$, $X_2X_1 = x_2 - x_1$, and

$$\frac{XX_1}{X_2X_1} = \frac{x_1 - x}{x_2 - x_1} = -\frac{x - x_1}{x_2 - x_1}.$$

Figure 3-22

If $x_2 < x_1$, then $x \geq x_1$, $XX_1 = x - x_1$, $X_2X_1 = x_1 - x_2$, and

$$\frac{XX_1}{X_2X_1} = \frac{x - x_1}{x_1 - x_2} = -\frac{x - x_1}{x_2 - x_1}.$$

Therefore, if $X \in \overrightarrow{X_1X_2}$, then

$$\frac{XX_1}{X_2X_1} = \frac{x - x_1}{x_2 - x_1}$$

regardless of whether $x_1 < x_2$ or $x_1 > x_2$. Also, if $X \in opp\ \overrightarrow{X_1X_2}$, then

$$\frac{XX_1}{X_2X_1} = -\frac{x - x_1}{x_2 - x_1}$$

regardless of whether $x_1 < x_2$ or $x_1 > x_2$.

THEOREM 3.6 (*The Two Coordinate Systems Theorem*) If X_1 and X_2 are two distinct points of a line l, if the coordinates of X_1 and X_2 are x_1 and x_2, respectively, in a coordinate system $\mathcal{S}$, and x_1' and x_2', respectively, in a coordinate system $\mathcal{S}'$, then for every point X on l, it is true that

$$\frac{x - x_1}{x_2 - x_1} = \frac{x' - x_1'}{x_2' - x_1'}$$

where x and x' are the coordinates of X in $\mathcal{S}$ and in $\mathcal{S}'$, respectively.

Proof: Suppose that we are given a line l and points X_1 and X_2 on l with coordinates x_1 and x_2 in $\mathcal{S}$ and coordinates x_1' and x_2' in $\mathcal{S}'$ as in the statement of the theorem. (See Figure 3-23.)

Figure 3-23

Suppose that $X \in \overrightarrow{X_1X_2}$; then it follows from Lemma 3.6.1 that

$$\frac{XX_1}{X_2X_1} = \frac{x - x_1}{x_2 - x_1}, \quad \frac{XX_1}{X_2X_1} = \frac{x' - x_1'}{x_2' - x_1'}, \quad \text{and} \quad \frac{x - x_1}{x_2 - x_1} = \frac{x' - x_1'}{x_2' - x_1'}.$$

Suppose that $X \in opp\ \overrightarrow{X_1X_2}$; then it follows from Lemma 3.6.1 that

$$\frac{XX_1}{X_2X_1} = -\frac{x - x_1}{x_2 - x_1},$$

$$\frac{XX_1}{X_2X_1} = -\frac{x' - x_1'}{x_2' - x_1'},$$

and

$$\frac{x - x_1}{x_2 - x_1} = \frac{x' - x_1'}{x_2' - x_1'}.$$

Therefore for every X on l it is true that

$$\frac{x - x_1}{x_2 - x_1} = \frac{x' - x_1'}{x_2' - x_1'}$$

and the proof is complete.

Theorem 3.6 is closely related to the Distance Ratio Postulate. According to this postulate the ratio $\frac{XX_1}{X_2X_1}$ is independent of the distance function. If we use the distance function of the coordinate system $\mathcal{S}$ in Theorem 3.6, we have

$$\frac{XX_1}{X_2X_1} = \frac{|x - x_1|}{|x_2 - x_1|}.$$

If we use the distance function of the coordinate system $\mathcal{S}'$ in Theorem 3.6, we have

$$\frac{XX_1}{X_2X_1} = \frac{|x' - x_1'|}{|x_2' - x_1'|}.$$

Therefore it follows directly from the Distance Ratio Postulate that

$$\left|\frac{x - x_1}{x_2 - x_1}\right| = \left|\frac{x' - x_1'}{x_2' - x_1'}\right|.$$

It follows from the Two Coordinate Systems Theorem that the equation obtained from this one by omitting the absolute value symbols is also true.

COROLLARY 3.6.1 Let X_1 and X_2 be the origin and unit point, respectively, in a coordinate system $\mathcal{S}_k$, on a line l. Let x_1 and x_2 be the coordinates of X_1 and X_2, respectively, in a coordinate system $\mathcal{S}_x$ on l. Let k and x be the coordinates of a point X on l in the systems $\mathcal{S}_k$ and $\mathcal{S}_x$, respectively. Then

$$(1) \quad \frac{XX_1}{X_2X_1} = |k|$$

and

$$(2) \quad \frac{x - x_1}{x_2 - x_1} = k, \text{ that is, } x = x_1 + k(x_2 - x_1).$$

Proof: (See Figure 3-24.) It follows from Lemma 3.6.1 that

$$\frac{XX_1}{X_2X_1} = \frac{k-0}{1-0} = k$$

or

$$\frac{XX_1}{X_2X_1} = -\frac{k-0}{1-0} = -k.$$

But

$$\frac{XX_1}{X_2X_1} \geq 0. \qquad \text{Why?}$$

Therefore

$$\frac{XX_1}{X_2X_1} = |k|$$

and (1) is proved.

Figure 3-24

It follows from Theorem 3.6 that

$$\frac{x - x_1}{x_2 - x_1} = \frac{k-0}{1-0}.$$

Hence it follows that

$$\frac{x - x_1}{x_2 - x_1} = k,$$

$$x = x_1 + k(x_2 - x_1),$$

and so (2) is proved.

Example 1 (See Figure 3-25.) Think of A as X_1, B as X_2. Then $x_1 = 1$, $x_2 = 7$, $x_1' = 3$, $x_2' = 15$. Then it follows from the Two Coordinate Systems Theorem that

$$\frac{x - 1}{7 - 1} = \frac{x' - 3}{15 - 3}$$

for every point X on $\overleftrightarrow{AB}$. This equation may be solved for x in terms of x' or for x' in terms of x. The resulting equations are

$$x = \tfrac{1}{2}x' - \tfrac{1}{2} \qquad \text{and} \qquad x' = 2x + 1.$$

Figure 3-25

These equations are useful in finding x' if you know x or in finding x if you know x'. If you are given that $x' = -6$, you can use the first of these equations to get $x = -3\frac{1}{2}$. If you are given $x = 2$, you can use the second of these equations to get $x' = 5$.

Example 2 (See Figure 3-26.) Think of P as X_1, Q as X_2. Then

$$\frac{x' - 0}{5 - 0} = \frac{x - (-7)}{3 - (-7)} \qquad \text{and} \qquad x' = \frac{1}{2}(x + 7).$$

Figure 3-26

(Or we may think of Q as X_1 and P as X_2. Then the theorem gives us $\dfrac{x' - 5}{0 - 5} = \dfrac{x - 3}{-7 - 3}$, which simplifies to $x' = \frac{1}{2}(x + 7)$, the same equation obtained by the first method.)

Example 3 Given three coordinate systems with points and coordinates as marked in Figure 3-27, express x in terms of k. Then express x' in terms of k and, finally, express x' in terms of x.

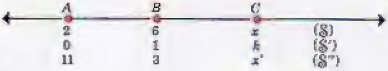


Figure 3-27

Solution:

$$(1)\quad \frac{x-2}{6-2} = \frac{k-0}{1-0} \qquad \text{or} \qquad x = 4k + 2$$

$$(2)\quad \frac{x'-11}{3-11} = \frac{k-0}{1-0} \qquad \text{or} \qquad x' = -8k + 11$$

$$(3)\quad \frac{x'-11}{3-11} = \frac{x-2}{6-2} \qquad \text{or} \qquad x' = -2x + 15$$

In working an example like this one you may need to write more details than we have shown. A more complete version of (3), for example, might be as follows:

$$\frac{x'-11}{3-11} = \frac{x-2}{6-2}$$

$$\frac{x'-11}{-8} = \frac{x-2}{4}$$

$$x' - 11 = -2(x-2)$$

$$x' = -2x + 4 + 11$$

$$x' = -2x + 15$$

Example 4 Given two coordinate systems with points and coordinates as marked in Figure 3-28, express x' in terms of x.

Figure 3-28

Solution:

$$\frac{x'-(-8)}{-10-(-8)} = \frac{x-3}{-1-3}$$

$$\frac{x'+8}{-10+8} = \frac{x-3}{-4}$$

$$\frac{x'+8}{-2} = \frac{x-3}{-4}$$

$$x' + 8 = \frac{1}{2}(x - 3)$$

$$x' + 8 = \frac{1}{2}x - \frac{3}{2}$$

$$x' = \frac{1}{2}x - \frac{19}{2} \qquad \text{or} \qquad x' = \frac{x - 19}{2}$$

Example 5 Given two coordinate systems with points and coordinates as marked in Figure 3-29, find x.

Figure 3-29

Solution:

$$\frac{x - (-23)}{-38 - (-23)} = \frac{19 - 0}{10 - 0}$$

$$\frac{x + 23}{-38 + 23} = \frac{19}{10}$$

$$\frac{x + 23}{-15} = 1.9$$

$$x + 23 = -28.5$$

$$x = -51.5$$

Example 6 A train traveled at a uniform speed on a trip from Chicago to New Orleans. If it was 180 miles from Chicago at 7:00 P.M. and 320 miles from Chicago at 9:00 P.M., how far from Chicago was it at 10:20 P.M.?

Solution: (See Figure 3-30.) Think of hours past noon as forming one coordinate system and miles from Chicago another one. Suppose the train is x miles from Chicago at $10\frac{1}{3}$ hours past noon, that is, at 10:20 P.M. Then

$$\frac{x - 180}{320 - 180} = \frac{10\frac{1}{3} - 7}{9 - 7} \qquad \text{or} \qquad \frac{x - 180}{140} = \frac{3\frac{1}{3}}{2};$$

so

$$x - 180 = 140\left(\frac{10}{6}\right) = \frac{1400}{6}, \qquad x = 180 + \frac{700}{3} = 413\tfrac{1}{3}.$$

Figure 3-30

Therefore the train is $413\frac{1}{3}$ miles from Chicago at 10:20 P.M.

Example 7 Given that $7, -12, p$ are the respective coordinates of points A, B, P on a line l, that $P \in \overrightarrow{AB}$, and that $AP = 3 \cdot AB$, find p.

Solution: (See Figure 3-31.) In the coordinate system with A as origin and B as unit point the coordinate of P is 3. Then

$$\frac{p-7}{-12-7} = \frac{3-0}{1-0}, \quad \frac{p-7}{-19} = 3, \quad p - 7 = -57, \quad p = -50.$$

Figure 3-31

Example 8 Given that $7, -12, p$ are the respective coordinates of points A, B, P on a line l, that $P \in opp\,\overrightarrow{AB}$, and that $AP = 3 \cdot AB$, find p.

Solution: (See Figure 3-32.) In the coordinate system with A as origin and B as unit point the coordinate of P is -3. Then

$$\frac{p-(-12)}{7-(-12)} = \frac{-3-1}{0-1}, \quad \frac{p+12}{19} = 4, \quad p + 12 = 76, \quad p = 64.$$

Figure 3-32

EXERCISES 3.6

In Exercises 1–5, a, b, c are the respective coordinates of points A, B, C on a line l. State which point, A, B, or C, is between the other two.

1. $a = 0, b = 5, c = 100$
2. $a = 0, b = 5, c = -100$
3. $a = -3, b = -7, c = 7$
4. $a = 0.500, b = 0.050, c = 0.005$
5. $a = \frac{2}{3}, b = \frac{1}{3}, c = -\frac{1}{3}$

In Exercises 6–15, a, b, c are the respective coordinates of points A, B, C on a line l. From the given information determine whether $b < c$ or $b > c$.

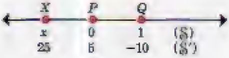
6. A–B–C, $a = 10, b = 20$
7. A–B–C, $a = 20, b = 10$
8. B–A–C, $a = 10, b = 20$
9. B–A–C, $a = 20, b = 10$
10. A–C–B, $a = 0, b = 8$
11. A–C–B, $a = 8, b = 0$
12. B–C–A, $a = 8, b = 0$
13. B–C–A, $a = -\frac{2}{3}, b = -0.66$
14. B–C–A, $a = \frac{2}{3}, b = 0.66$
15. B–C–A, $a = -\frac{2}{3}, b = 0$

In Exercises 16–20, a, b, c are the respective coordinates of points A, B, C on a line l. From the given information determine the number c.

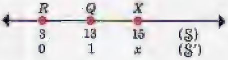
16. A–B–C, $a = 0$, $b = 1$, $\overline{AB} \cong \overline{BC}$
17. A–C–B, $a = 0$, $b = 1$, $\overline{AC} \cong \overline{CB}$
18. B–A–C, $a = 0$, $b = 1$, $\overline{AB} \cong \overline{AC}$
19. A–B–C, $a = -17$, $b = -8$, $\overline{AB} \cong \overline{BC}$
20. A–B–C, $a = -17$, $b = 8$, $\overline{AB} \cong \overline{BC}$

In Exercises 21–25, there is a sketch of a line with some points and coordinates marked. Find the coordinate x.

21.

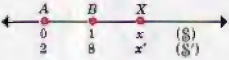

22.


23.

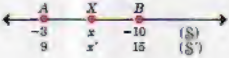

24.


25.

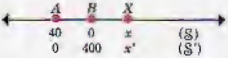

In Exercises 26–30, there is a sketch of a line with some points and coordinates marked. In each exercise write an equation relating x and x' and simplify it to the form $x' = ax + b$. (Note that a must be a number different from 0. For if $a = 0$, then $x' = b$ and every point X would be matched with the same number in the system $\mathcal{S}'$. Since $\mathcal{S}'$ is a coordinate system, we know that different points must be matched with different numbers.) Check your answers by substituting the values of x at A and B in the equation to see if you get the corresponding values of x' at A and B.

26.


27.


28.


29.


30.


In Exercises 31–35, a subset S of a line l is given in set-builder notation. Sketch the graph of the set and mark the x-coordinates and k-coordinates of three points of S. Exercise 31 has been worked as a sample.

31. $S = \{X : x = 6k + 2, k \leq 2\}$

Solution:
$$S = \left\{X : \frac{x-2}{6} = k, k \leq 2\right\}$$
$$= \left\{X : \frac{x-2}{8-2} = \frac{k-0}{1-0}, k \leq 2\right\}$$

k	2	1	0
x	14	8	2

32. $S = \{X : x = 3k + 6, 0 \leq k \leq 1\}$
33. $S = \{X : x = -3k + 9, 0 \leq k \leq 1\}$
34. $S = \{X : x = -3k - 6, k \geq 0\}$
35. $S = \{X : \dfrac{x-3}{-7-3} = \dfrac{k-0}{1-0}$, k is any real number$\}$

In Exercises 36–40, A and B are points on a line with coordinates 7 and 12, respectively. Find the coordinate of the point P subject to the given condition.

36. $P \in \overrightarrow{AB}$ and $AP = 3 \cdot AB$
37. $P \in \overrightarrow{AB}$ and $AP = \frac{1}{5} \cdot AB$
38. $P \in opp\ \overrightarrow{AB}$ and $AP = 3 \cdot AB$
39. $P \in \overrightarrow{AB}$ and $\dfrac{AP}{PB} = 2$ (two possibilities)
40. $P \in \overrightarrow{BA}$ and $AP = \frac{1}{2} \cdot PB$ (two possibilities)

In Exercises 41–50, A, B, P are points on a line l with coordinates 0, 1, k, respectively, in a system $\mathcal{S}$ and with coordinates -3, 7, x, respectively, in a system $\mathcal{S}'$. Figure 3-33 is an appropriate one for Exercise 41.

Figure 3-33

41. If $P \in \overrightarrow{AB}$ and $\dfrac{AP}{AB} = \frac{2}{3}$, find k and x.

42. If $P \in opp\ \overrightarrow{AB}$ and $\dfrac{AP}{AB} = \frac{2}{5}$, find k and x.

43. If $k = -2$, find $\dfrac{AP}{AB}$. Is $P \in \overrightarrow{AB}$, or is $P \in opp\ \overrightarrow{AB}$?

44. If $k = 5$, find $\dfrac{AP}{AB}$. Is $P \in \overrightarrow{AB}$, or is $P \in opp\ \overrightarrow{AB}$?

45. If $x = 17$, find k and $\dfrac{AP}{AB}$.

46. If $x = 27$, find k and $\dfrac{AP}{AB}$.

47. If $x = 27$, find $\dfrac{AP}{PB}$.

48. If $x = 37$, find $\dfrac{AP}{PB}$.

49. If $x = 47$, find $\dfrac{AP}{PB}$.

50. If $x = 0$, find $\dfrac{AP}{PB}$.

51. Given the situation of Example 6 on page 123, express the number x of miles from Chicago in terms of the number t of hours past noon of the day the trip began.
52. Given the situation of Example 6, find the "departure" time at Chicago.
53. On a car trip across the country Mr. X stopped at Ridgeville and "filled it up"; his odometer reading was 35378. Sometime later the gasoline gauge read $\frac{5}{8}$ full and the odometer read 35513. Assuming that there is a constant "gasoline mileage," what does the odometer read when the gauge reads $\frac{1}{8}$?
54. Given the situation of Exercise 53, express the odometer reading m in terms of the amount x of gasoline in the tank, where $x = 1$ when the tank is full and $x = 0$ when the tank is empty.
55. Mr. X recently completed a 180-day weight reduction program. If he weighed 200 lb. on the tenth day and 180 lb. on the 100th day, how much did he weigh on the 150th day? Assume that he loses the same weight each day.
56. Given the situation of Exercise 55, express Mr. X's weight w (in pounds) in terms of the number n of days, where $n = 1$ means the first day of the program, $n = 2$ means the second day of the program, and so on.
57. Under certain standard conditions the freezing point of water is 0° Centigrade and 32° Fahrenheit; the boiling point of water is 100° Centigrade and 212° Fahrenheit. What is the temperature in degrees Centigrade when the Fahrenheit reading is 77°?
58. Given the situation of Exercise 57, let C denote the number of degrees Centigrade and F the number of degrees Fahrenheit. Obtain an equation that relates C and F by substituting appropriate numbers for F_1, F_2, C_1, C_2 in the following equation: $\dfrac{F - F_1}{F_2 - F_1} = \dfrac{C - C_1}{C_2 - C_1}$.

59. Starting with the equation obtained in Exercise 58, derive an equation in simplified form that expresses F in terms of C.
60. Starting with the equation obtained in Exercise 58, derive an equation in simplified form that expresses C in terms of F.

In Exercises 61–65, a line with points and coordinates as marked in Figure 3-34 is given. Using the Two Coordinates Systems Theorem, x' may be expressed in terms of x, first with A as X_1 and B as X_2, and then with B as X_1 and A as X_2. The results are

(a) $$\frac{x' + 1}{6 + 1} = \frac{x - 2}{5 - 2}$$ (b) $$\frac{x' - 6}{-1 - 6} = \frac{x - 5}{2 - 5}$$

Figure 3-34

If $X = A$, then $x = 2$, $x' = -1$, and Equation (a) becomes

$$\frac{-1 + 1}{6 + 1} = \frac{2 - 2}{5 - 2} \text{ which reduces to } 0 = 0.$$

61. Simplify Equation (a) if $X = B$.
62. Simplify Equation (b) if $X = A$.
63. Simplify Equation (b) if $X = B$.
64. The sum of the left members of Equations (a) and (b) is

$$\frac{x' + 1}{6 + 1} + \frac{x' - 6}{-1 - 6}$$

which simplifies to

$$\frac{x' + 1}{7} + \frac{x' - 6}{-7} = \frac{x' + 1}{7} - \frac{x' - 6}{7} = \frac{x' + 1 - (x' - 6)}{7} = \frac{7}{7} = 1.$$

Add the right members of Equations (a) and (b) and simplify.

65. If you multiply the left member of Equation (a) by -1 and add 1 to the product, the result is $\frac{x' + 1}{6 + 1} \cdot (-1) + 1$. This simplifes to

$$\frac{x' + 1}{7} \cdot \frac{-1}{1} + 1 = \frac{-x' - 1}{7} + \frac{7}{7}$$
$$= \frac{-x' + 6}{7} = \frac{x' - 6}{-7} = \frac{x' - 6}{-1 - 6},$$

which is the left member of Equation (b). Multiply the right member of Equation (a) by -1 and add 1 to the product. Show that the result simplifies to the right member of Equation (b).

3.7 POINTS OF DIVISION

In this section we use the Two Coordinate Systems Theorem to help us find the coordinates of the points on a given segment which divide it into a given number of congruent parts or divide it in some other specified way.

Definition 3.6 The midpoint of a segment $\overline{AB}$ is the point P on $\overline{AB}$ such that

$$AP = PB = \tfrac{1}{2}AB.$$

The midpoint of a segment is said to bisect the segment or to divide it into two congruent parts.

Definition 3.7 The trisection points of a segment $\overline{AB}$ are the two points P and Q on $\overline{AB}$ such that

$$AP = PQ = QB = \tfrac{1}{3}AB.$$

The trisection points of a segment are said to divide the segment into three congruent parts. Similarly, points C, D, and E on $\overline{AB}$ such that

$$AC = CD = DE = EB = \tfrac{1}{4}AB$$

are said to divide $\overline{AB}$ into four congruent parts. This idea may be extended to any number of congruent parts.

A segment $\overline{AB}$ is a set of points. You might think of a "path" from A to B if you were to draw a picture of the segment. Although you may think of $\overline{AB}$ as a path, be careful to remember that the segment from A to B is the same as the segment from B to A. Indeed, $\overline{AB} = \overline{BA}$.

Sometimes, however, we want to consider B to A as different from A to B. It might be helpful to think of trips and to consider the trip from A to B as different from the trip from B to A. The point which is one-third of the way from A to B is different from the point which is one-third of the way from B to A. This leads us to the idea of a directed segment. We think of a *directed segment* as a segment with one endpoint designated as the starting point. A directed segment is known as soon as the segment is known and the starting point is known. Our formal definition follows.

Definition 3.8 The directed segment from A to B, denoted by $\overrightharpoon{AB}$, is the set $\{\overline{AB}, A\}$.

It is important to note the difference between the symbol for a ray and the symbol for a directed segment. The ray symbol, as in $\overrightarrow{AB}$, has a complete arrowhead, whereas the directed segment symbol, as in $\overrightharpoon{AB}$, has a half arrowhead. Directed segments are related to vectors, and half arrows are frequently used in vector notation. Vectors are very useful in many branches of higher mathematics. Note that whereas $\overline{AB} = \overline{BA}$, it is not true that $\overrightharpoon{AB} = \overrightharpoon{BA}$. Note that

$$\overrightharpoon{AB} = \{\overline{AB}, A\} = \{\overline{BA}, A\},$$

whereas

$$\overrightharpoon{BA} = \{\overline{AB}, B\} = \{\overline{BA}, B\}.$$

Definition 3.9 Let a directed segment $\overrightharpoon{AB}$ and two points P and Q on $\overleftrightarrow{AB}$ be given. If $P \in \overline{AB}$, $Q \notin \overline{AB}$, and $\dfrac{AP}{PB} = \dfrac{AQ}{QB}$, as shown in Figure 3-35, then P and Q are said to divide $\overrightharpoon{AB}$ in the same ratio, P dividing it internally and called an internal point of division, Q dividing it externally and called an external point of division. The ratio $\dfrac{AP}{PB}$ is the ratio of division.

Figure 3-35

Example 1 Given two points A and B on a line l with coordinates 4 and 20, respectively, as indicated in Figure 3-36, find the coordinate of the point P on $\overline{AB}$ if P divides AB into two congruent parts.

Figure 3-36

Solution: Let x be the coordinate of the desired point P. Then $4 < x < 20$, $x - 4 = 20 - x$, $2x = 24$, and $x = 12$.

Alternate Solution: Set up two coordinate systems as indicated in Figure 3-37.

Figure 3-37

Then

$$\frac{x-4}{20-4} = \frac{k-0}{1-0}$$

and $x = 4 + 16k$ for every P on l. Then P divides $\overline{AB}$ into two congruent parts if $k = \frac{1}{2}$ and

$$x = 4 + 16 \cdot \tfrac{1}{2} = 4 + 8 = 12.$$

Example 2 Given two points A and B on a line l with coordinates -3 and 21, respectively, find the coordinates of the four points which divide it into five congruent parts.

Solution: Let $\mathcal{S}$ and $\mathcal{S}'$ be two coordinate systems on l with coordinates of several points as marked in Figure 3-38.

Figure 3-38

Then

$$\frac{x-(-3)}{21-(-3)} = \frac{x+3}{24} = \frac{k-0}{1-0} = k$$

and $x = 24k - 3$ for every point P on l. The required points are the points with k-coordinates $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ or, in decimal form, 0.2, 0.4, 0.6, 0.8. Now we compute the x-coordinates of the division points using the equation $x = 24k - 3$. They are

$24(0.2) - 3 = 4.8 - 3 = 1.8$, $\quad 24(0.6) - 3 = 14.4 - 3 = 11.4$.
$24(0.4) - 3 = 9.6 - 3 = 6.6$, $\quad 24(0.8) - 3 = 19.2 - 3 = 16.2$.

Example 3 Given two points A and B on a line l with coordinates 3 and -21, respectively, find the coordinates of the points P and Q on $\overleftrightarrow{AB}$ which divide $\overline{AB}$ internally and externally, respectively, in the ratio $\frac{7}{8}$.

Solution: Let $\mathcal{S}$ and $\mathcal{S}'$ be two coordinate systems on l with coordinates of several points as marked in Figure 3-39.

Figure 3-39

Then, as in Example 2,

$$\frac{x-3}{-24} = k \qquad \text{and} \qquad x = -24k + 3$$

for every point P on l. Since $\dfrac{AQ}{QB} = \dfrac{7}{8}$, AQ is less than QB and Q is a point on *opp* $\overrightarrow{AB}$. The coordinates x and x' of P and Q are computed as follows:

$$\begin{aligned} AP &= k \\ PB &= 1 - k \\ \frac{AP}{PB} &= \frac{7}{8} = \frac{k}{1-k} \\ 7 - 7k &= 8k \\ 15k &= 7 \\ k &= \frac{7}{15} \\ x &= -24 \cdot \frac{7}{15} + 3 = -8.2 \end{aligned}$$

$$\begin{aligned} AQ &= 0 - k' \\ QB &= 1 - k' \\ \frac{AQ}{QB} &= \frac{7}{8} = \frac{-k'}{1-k'} \\ 7 - 7k' &= -8k' \\ k' &= -7 \\ x' &= -24(-7) + 3 = 171 \end{aligned}$$

Check (using x-coordinates).

$$\frac{AP}{PB} = \frac{3 - (-8.2)}{-8.2 - (-21)} = \frac{11.2}{12.8} = \frac{112}{128} = \frac{7}{8}$$

$$\frac{AQ}{QB} = \frac{171 - 3}{171 - (-21)} = \frac{168}{192} = \frac{42}{48} = \frac{7}{8}$$

Example 4 Given two points A and B on a line l with coordinates 3 and -21, respectively, find the coordinates of the points R and T on $\overleftrightarrow{AB}$ which divide $\overline{BA}$ internally and externally, respectively, in the ratio $\frac{7}{8}$. (Compare with Example 3.)

Solution: Let $\mathcal{S}$ and $\mathcal{S}'$ be the two coordinate systems on l with coordinates of several points as marked in Figure 3-40. Then, as in the preceding examples, $\dfrac{x+21}{24} = k$ and $x = 24k - 21$ for every point R on l. Since $\dfrac{BT}{TA} = \dfrac{7}{8}$, BT is less than TA and T is a point on *opp* $\overrightarrow{BA}$.

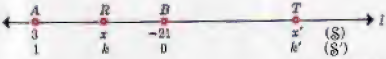


Figure 3-40

Then

$BR = k$

$RA = 1 - k$

$\dfrac{BR}{RA} = \dfrac{7}{8} = \dfrac{k}{1-k}$

$k = \dfrac{7}{15}$ (Why?)

$x = -9.8$ (Why?)

$BT = 0 - k'$

$TA = 1 - k'$

$\dfrac{BT}{TA} = \dfrac{7}{8} = \dfrac{-k'}{1-k'}$

$k' = -7$ (Why?)

$x' = -189$ (Why?)

Check (using x-coordinates).

$$\frac{BR}{RA} = \frac{-9.8+21}{3+9.8} = \frac{11.2}{12.8} = \frac{112}{128} = \frac{7}{8}$$

$$\frac{BT}{TA} = \frac{-21+189}{3+189} = \frac{168}{192} = \frac{7}{8}$$

EXERCISES 3.7

In Exercises 1–5, A and B are points on a line l with given coordinates a and b. In each exercise, find the coordinate of the midpoint of $\overline{AB}$.

1. $a = 5, b = 27$
2. $a = -5, b = 27$
3. $a = -5, b = -27$
4. $a = 5, b = -27$
5. $a = 0, b = -4.8$

In Exercises 6–10, A and B are points on a line l with given coordinates a and b. In each exercise, find the coordinates of the points which divide $\overline{AB}$ into the given number, n, of congruent parts.

6. $a = -3, b = -7, n = 2$
7. $a = 3, b = 0, n = 5$
8. $a = -1, b = 79, n = 3$
9. $a = 8, b = -8, n = 8$
10. $a = 0, b = 10, n = 4$

In Exercises 11–15, A and B are points with given coordinates a and b. In each exercise, find the coordinates of the points P and Q which divide $\overline{AB}$ internally and externally in the given ratio r.

11. $a = 10, b = 20, r = \frac{3}{5}$
12. $a = 20, b = 10, r = \frac{3}{5}$
13. $a = 26, b = 0, r = \frac{5}{1}$
14. $a = \frac{2}{3}, b = \frac{2}{5}, r = \frac{1}{3}$
15. $a = \frac{2}{5}, b = \frac{2}{3}, r = \frac{3}{1}$

16. Prove the following theorem.

    THEOREM If the coordinates of A and B are a and b, then the coordinate of the midpoint of $\overline{AB}$ is $\dfrac{a+b}{2}$.

17. Prove the following theorem.

    THEOREM If the coordinates of A and B are a and b, then the coordinates of the trisection points of $\overline{AB}$ are $\dfrac{2a+b}{3}$ and $\dfrac{a+2b}{3}$.

18. A and B are points on a line l with coordinates 0 and 1, respectively. Find the coordinate of the point P which divides $\overline{AB}$ externally in the ratio $\frac{999}{1000}$.
19. A and B are points on a line l with coordinates 0 and 1, respectively. Find the coordinate of the point Q which divides $\overline{AB}$ externally in the ratio $\frac{1000}{999}$.
20. A and B are distinct points on a line l. Is there a point P on $\overleftrightarrow{AB}$, but not on $\overline{AB}$, such that P is the same distance from A as it is from B? That is, is there a point P on $\overleftrightarrow{AB}$ which divides $\overline{AB}$ externally in the ratio $\frac{1}{1}$?

In Exercises 21–26, A and B are points on a line l with given coordinates a and b, respectively. P is the point which divides $\overline{AB}$ internally in the given ratio r. In each exercise, select the statement from the right-hand column that is true.

21. $a = 0, b = 1, r = \frac{1}{2}$
22. $a = 0, b = 10, r = \frac{1}{1}$
23. $a = 10, b = 209, r = \frac{99}{100}$
24. $a = 10, b = 211, r = \frac{101}{100}$
25. $a = -13, b = 0, r = \frac{1}{12}$
26. $a = 0, b = 13, r = \frac{12}{1}$

(A) P is between the midpoint of $\overline{AB}$ and B
(B) $AP = 1$
(C) P is a trisection point of $\overline{AB}$
(D) $BP = 1$
(E) P is the midpoint of $\overline{AB}$
(F) P is between the midpoint of $\overline{AB}$ and A

In Exercises 27–30, A and B are points on a line l with coordinates a and b, respectively. P is the point which divides $\overline{AB}$ externally in the given ratio r. In each exercise, select the statement from the right-hand column that is true.

27. $a = 0, b = 1, r = \frac{999}{1000}$ (A) P–A–B and $BP = 1000$
28. $a = 0, b = 1, r = \frac{1001}{1000}$ (B) A–B–P and $BP = 0.001$
29. $a = 0, b = 1, r = \frac{1001}{1}$ (C) A–B–P and $BP = 1000$
30. $a = 0, b = 1, r = \frac{1}{1001}$ (D) P–A–B and $AP = 0.001$

CHAPTER SUMMARY

In this chapter we have developed the concept of DISTANCE between two points and the concept of a COORDINATE SYSTEM on a line. We introduced five postulates: DISTANCE EXISTENCE POSTULATE, DISTANCE BETWEENNESS POSTULATE, TRIANGLE INEQUALITY POSTULATE, DISTANCE RATIO POSTULATE, and RULER POSTULATE. This chapter contains many definitions and theorems. A key definition is the definition of a coordinate system. The climax of the chapter is the TWO COORDINATE SYSTEMS THEOREM. In Section 3.7 we applied the tools of this chapter to find points of division which divide a segment internally and externally in a given ratio.

REVIEW EXERCISES

In Exercises 1–5, name the property that justifies the given statement.

1. $3 + 4 = 4 + 3$
2. $(AB + CD) + EF = AB + (CD + EF)$
3. If $a = b$, then $b = a$.
4. $(50 + 5) \cdot 3 = 50 \cdot 3 + 5 \cdot 3$
5. $(5 + 3) \cdot 7 = (3 + 5) \cdot 7$

In Exercises 6–11, name the postulate which justifies the given statement.

6. If P, Q, R are distinct collinear points with P-Q-R, then $PQ + QR = PR$.
7. If X and Y are distinct points and $\overline{AB}$ is any segment, then the distance between X and Y in the distance function based on $\overline{AB}$ is a positive number.
8. If R, W, K are three noncollinear points, then $RK + KW > RW$.

9. If A, B, C, D, E, F, G, H are eight distinct points, then
$$\frac{AB \text{ (in } \overline{EF} \text{ units)}}{CD \text{ (in } \overline{EF} \text{ units)}} = \frac{AB \text{ (in } \overline{GH} \text{ units)}}{CD \text{ (in } \overline{GH} \text{ units)}}.$$

10. If A, B, C, D, E, F, G, H are eight distinct points, then
$$\frac{AB \text{ (in } EF \text{ units)}}{AB \text{ (in } \overline{GH} \text{ units)}} = \frac{CD \text{ (in } EF \text{ units)}}{CD \text{ (in } \overline{GH} \text{ units)}}.$$

11. If A, B, C, D are four distinct points, then there is a unique coordinate system on $\overleftrightarrow{AB}$ relative to $\overline{CD}$ such that the origin is B and the unit point is A.

In Exercises 12–20, A, B, and X are points on a line l; a, b, x are their respective coordinates in one system; a', b', x' are their respective coordinates in another system. In each exercise, (*a*) draw and label an appropriate figure, (*b*) express x' in terms of x and simplify, and (*c*) express x in terms of x' and simplify.

12. $a = 0, b = 1, a' = 5, b' = 8$
13. $a = 0, b = 1, a' = -8, b' = 5$
14. $a = 1, b = 0, a' = 5, b' = 8$
15. $a = 17, b = 16, a' = 5, b' = 8$
16. $a = 7, b = -3, a' = 0, b' = 1$
17. $a = 0, b = 1, a' = 0, b' = 2$
18. $a = 0, b = 1, a' = 0, b' = -1$
19. $a = 0, b = 1, a' = 1, b' = 0$
20. $a = 0, b = 100, a' = 1, b' = 0$

In Exercises 21–30, A, B, C, D, E, are points on a line l with coordinates 0, 1, 2, 3, 4, respectively, in a coordinate system on l. In each exercise simplify the given expression. Each expression names a number.

21. AB (in $\overline{AB}$ units) $= \boxed{?}$
22. AB (in $\overline{AC}$ units) $= \boxed{?}$
23. AC (in $\overline{AB}$ units) $= \boxed{?}$
24. AD (in $\overline{AB}$ units) $= \boxed{?}$
25. AD (in $\overline{AC}$ units) $= \boxed{?}$
26. AD (in $\overline{AE}$ units) $= \boxed{?}$
27. AE (in $\overline{AB}$ units) $= \boxed{?}$
28. AB (in $\overline{AE}$ units) $= \boxed{?}$
29. $\dfrac{AB \text{ (in } \overline{AE} \text{ units)}}{AD \text{ (in } \overline{AE} \text{ units)}} = \boxed{?}$
30. $\dfrac{AB \text{ (in } \overline{AC} \text{ units)}}{AD \text{ (in } \overline{AC} \text{ units)}} = \boxed{?}$

In Exercises 31–40, $\overrightarrow{AB}$ is a directed segment, P divides $\overrightarrow{AB}$ internally in the positive ratio $\frac{r}{s}$, Q divides $\overrightarrow{AB}$ externally in the ratio $\frac{r}{s}$, and a and b are the coordinates of A and B, respectively, in a coordinate system on $\overleftrightarrow{AB}$. In each exercise, draw an appropriate figure and find p and q, the coordinates of P and Q, respectively.

31. $a = 0, b = 1, r = 2, s = 1$
32. $a = 0, b = 1, r = 1, s = 2$
33. $a = 0, b = 1, r = 2, s = 3$
34. $a = 0, b = 1, r = 10, s = 5$
35. $a = 10, b = 5, \frac{r}{s} = \frac{1}{4}$
36. $a = 10, b = 5, \frac{r}{s} = \frac{4}{1}$
37. $a = -6, b = 9, \frac{r}{s} = \frac{1}{3}$
38. $a = -6, b = 9, \frac{r}{s} = 3$
39. $a = 1, b = 2, r = 5, s = 6$
40. $a = 1.3, b \doteq 7.2, \frac{r}{s} = \frac{1}{58}$

In Exercises 41–47, X is a point on a line l and x and k are its coordinates in two coordinate systems on l. The given equation tells how x and k are related for every X on l. In each exercise, copy and complete the given statement.

41. $x = 3k + 1$ — If $k = 0$, then $x = \boxed{?}$
42. $x = 3k + 1$ — If $k = 1$, then $x = \boxed{?}$
43. $x = 3k + 1$ — $0 \le k \le 1$ if and only if $\boxed{?}$ (a condition on x)
44. $x = -3k + 1$ — If $k = -3$, then $x = \boxed{?}$
45. $x = -3k + 1$ — If $k = -5$, then $x = \boxed{?}$
46. $x = -3k + 1$ — $k \le -3$ if and only if $\boxed{?}$ (a condition on x)
47. $x = -3k + 1$ — $k \ge -3$ if and only if $\boxed{?}$ (a condition on x)

Chapter 4

Fred Ward/Black Star

Angles, Ray-Coordinates, and Polygons

4.1 INTRODUCTION

In Chapter 3 we developed definitions and postulates for the concept of distance and for coordinate systems on a line. Betweenness for points is related to betweenness for real numbers through the idea of a coordinate system on a line. The length of a segment is related to the coordinates of its endpoints.

In this chapter we develop postulates and definitions for angular measure or, as we usually call it, angle measure, and for ray-coordinate systems in a plane. Betweenness for rays is related to betweenness of the coordinates matched with the rays. The measure of an angle is related to the coordinates of the rays that form the angle. Ray-coordinates are useful in developing the properties of angles.

Chapter 4 concludes with a discussion of polygons and dihedral angles. The idea of a polygon is a natural extension of the idea of a triangle, and the idea of a dihedral angle grows naturally from the idea of an angle.

4.2 ANGLE MEASURE AND CONGRUENCE

When a pie is cut into four quarters of equal size as indicated in Figure 4-1, the rim is also cut into quarters. This is true regardless of the size of the pie. If a pie is cut in the usual manner, then with each piece that is less than half a pie there is an associated angle, sometimes called the *associated central angle.* This angle is the union of the two rays which have the point at the center of the pie as their common endpoint and which contain the segments that are the cuts forming the piece of pie. For a quarter pie we might think of the size of this angle as $\frac{1}{4}$.

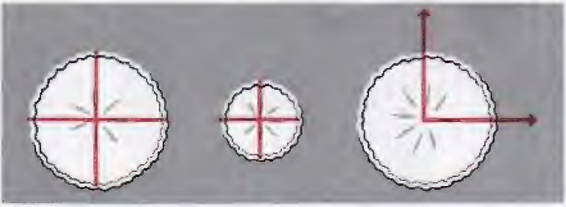


Figure 4-1

This corresponds to thinking of a revolution as the unit of measure. If we adopted this unit, then the measures of the angles in our geometry would be real numbers between 0 and $\frac{1}{2}$. We prefer to think of one revolution as equivalent to 360 degrees. Then the measures of angles in our formal geometry will be numbers between 0 and 180 as suggested in Figure 4-2.

Figure 4-2

It is said that the Babylonians originated the system of measurement that is based on what we now call the degree as the unit. To them the stars (except the sun) appeared to be fixed on a celestial sphere that rotated about an axis once each day. The sun appeared to com-

plete a circular path among the stars once each year (four successive seasons). They apparently knew that the length of the year was approximately 365 days, but, perhaps for convenience, took 360 days as their "calendar" year. Considering that the sun traveled over a circular path once each 360 days it was natural to divide that path into 360 equal parts and consider each part as corresponding to one day and 90 parts as corresponding to one season. In the early days of the Christian era, the Greek mathematicians of the School of Alexandria divided the circle into 360 equal parts and called each part a *moira*. This Greek word was translated into the Latin word *de-gradus*, meaning "a grade or step from." From this we get our word **degree**, meaning the first step down from a complete revolution, or $\frac{1}{360}$ of a revolution.

Of course, we could use other units of angular measure such as revolutions or right angles. Some units that you may not have heard of are mils, grads, and radians. There is no particular reason for using degrees for angle measures other than the fact that this is commonly done and has been done for a long time. To make our development simpler, we base our formal geometry of angle measure on just one unit of measure, the degree.

POSTULATE 20 (***Angle Measure Existence Postulate***) There exists a correspondence which associates with every angle in space a unique real number between 0 and 180.

> ***Definition 4.1*** The number which corresponds to an angle as in the Angle Measure Existence Postulate is called the **measure of the angle.**

Notation. The measure of $\angle ABC$ is denoted by $m\angle ABC$.

Note that if the number of degree units in the measure of $\angle ABC$ is 40, then $m\angle ABC = 40$ and *not* $m\angle ABC = 40°$. If an angle is marked 40° in a figure, it means that the measure of the angle is 40.

In Chapter 3 we agreed to call two segments congruent if they have the same length. We make a similar agreement for angles.

> ***Definition 4.2*** Two angles (whether distinct or not) are **congruent angles,** and each is said to be **congruent** to the other if they have the same measure.

Notation. $\angle ABC \cong \angle DEF$ denotes that $\angle ABC$ and $\angle DEF$ are congruent.

Although congruence and equality as applied to angles may seem alike, they are in reality different ideas. It is true that if $\angle A = \angle B$, then $\angle A \cong \angle B$, but the converse is not true. Many pairs of congruent angles are not pairs of equal angles. Remember that an angle is a set of points and that two sets of points are not equal unless they consist of the same points. For the angles suggested in Figure 4-3 we have

$$\angle ABC = \angle CBA = \angle ABK = \angle GBC = \angle GBK.$$

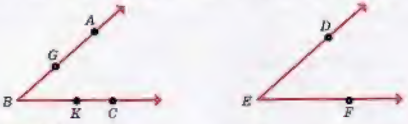


Figure 4-3

Also, $\angle ABC \neq \angle DEF$. But it is quite possible that $m\angle ABC = m\angle DEF$. If this is true, then $\angle ABC \cong \angle DEF$.

The most common device for measuring angles in informal geometry is a semicircular protractor with degree marks from 0 to 180 evenly spaced on the semicircular edge. To measure an angle such as in Figure 4-4a we can either place the protractor as indicated in (b) and read the measure 20 directly or we can place it as indicated in (c) and obtain the same measure by subtracting 65 from 85.

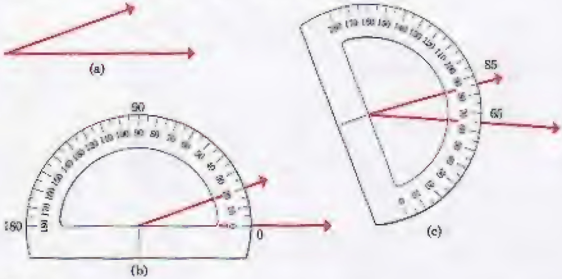


Figure 4-4

Another type of protractor is a circular one. (See Figure 4-5.) This 360-degree protractor has advantages in drawing certain figures. In using a 360-degree protractor, as in using a semicircular one, it is possible to obtain the measure of an angle from readings on its scale. This and other properties of the protractor suggest the concept of a ray-coordinate system which we define later.

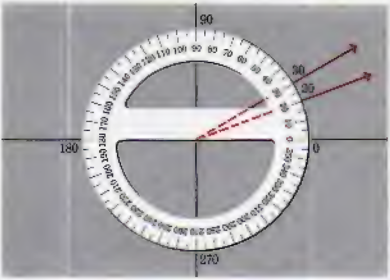


Figure 4-5

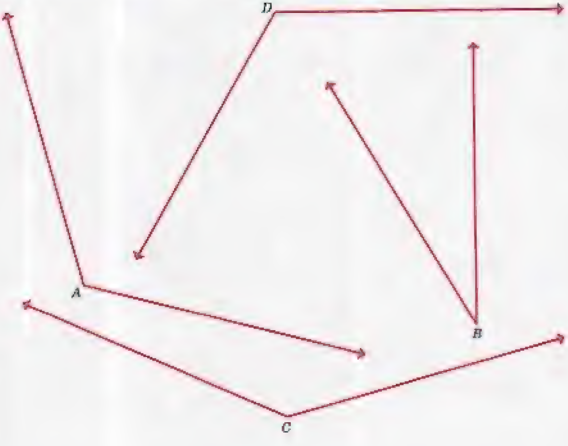
EXERCISES 4.2

1. Copy and complete the following definition of congruence of angles.

   If $\angle A \cong \angle B$, then $\boxed{?}$ and if $m\angle A = m\angle B$, then $\boxed{?}$.

2. (a) Use your protractor to construct three angles whose degree measures are 60, 90, and 135, respectively.
   (b) Why is it not possible, using our definition of angle measure, to construct an angle whose degree measure is 240?

3. Use your protractor to find the degree measure of the angles shown.



4. In Exercise 3, did you find two angles that are congruent? If so, name them and tell why they are congruent.
5. If Sandy Moser measures an angle and finds its degree measure to be 60 and Bob Blake measures the same angle with the same protractor and finds its degree measure to be 70, what can you conclude? What postulate are you using as a basis for your conclusion?

Exercises 6–10 refer to the angles in Figure 4-6.

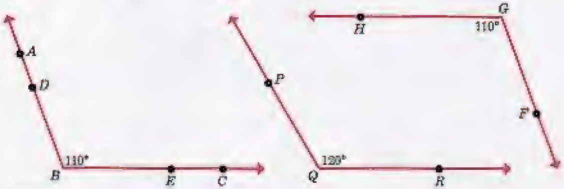
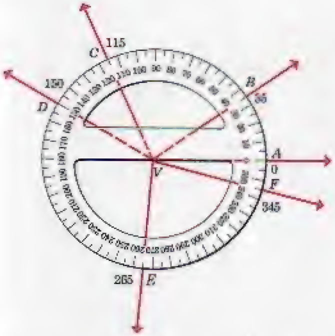


Figure 4-6

6. Using the notation of the figure, write two different names for equal angles, that is, for the same angle.
7. Name two angles that are congruent but not equal.
8. Write two names of angles such that the angles named are equal (and therefore congruent).
9. Name two angles that are not congruent.
10. Can you name two angles that are equal but not congruent?
11. Using a protractor, measure the angles of the two triangles in the following figure. List those pairs of angles that appear to be congruent.

12. In the figure, the readings for certain rays with endpoint V are shown in a circular protractor. Name four pairs of congruent angles, that is, name four pairs of angles such that the angles in each pair are congruent to each other.



13. Without using a protractor draw six angles whose degree measures you would estimate to be 30, 45, 60, 90, 120, and 150, respectively. After you have drawn each angle, measure it with your protractor and see how good your estimate was.
14. Draw a triangle ABC so that $m\angle A = 43$, $m\angle B = 57$, and $m\angle C = 80$.
15. Can you draw a triangle DEF so that $m\angle D = 54$, $m\angle E = 67$, and $m\angle F = 70$?
16. Draw a triangle DEF so that $m\angle D = 54$ and $m\angle E = 67$. Use your protractor to find $m\angle F$.

In Exercises 17 and 18, complete the proof of the following theorem.

THEOREM Congruence for angles is reflexive, symmetric, and transitive.

Proof: Let $\angle A$ be any angle; then $m\angle A$ is a number and $m\angle A = m\angle A$ by the reflexive property of equality. But if $m\angle A = m\angle A$, then $\angle A \cong \angle A$ by the definition of congruence for angles. Therefore congruence for angles is reflexive.

17. Prove that congruence for angles is symmetric.
18. Prove that congruence for angles is transitive.

19. (a) If $\angle A \cong \angle B$ and $\angle C \cong \angle B$, what can you conclude?
    (b) What properties justify your conclusion in (a)?

4.3 BETWEENNESS FOR RAYS

Recall that if A, B, C are three distinct points on a line, then B is between A and C if and only if

$$AB + BC = AC.$$

Also, B is between A and C if and only if B is between C and A. Betweenness for points is related to betweenness for numbers through the definition of a coordinate system on a line and the Ruler Postulate.

For three distinct coplanar rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ (with a common endpoint) we want to develop a concept of betweenness based on our intuitive notions of symmetry, our experiences with protractors, and our desire for additivity of angle measures in certain situations. Specifically, if $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, then we want $\overrightarrow{VB}$ to be between $\overrightarrow{VC}$ and $\overrightarrow{VA}$. Also, we want $\overrightarrow{VB}$ to be between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if

$$m\angle AVB + m\angle BVC = m\angle AVC.$$

These ideas suggest the following definition and postulate. Refer to Figure 4-7 as you read them.

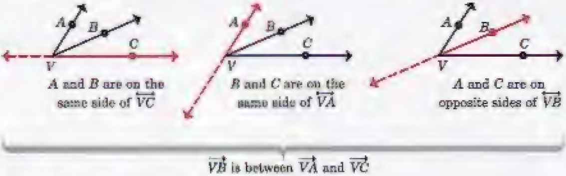


Figure 4-7

Definition 4.3 If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are rays, then $\overrightarrow{VB}$ is **between** $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if

1. A and B are in the same halfplane with edge $\overleftrightarrow{VC}$.
2. B and C are in the same halfplane with edge $\overleftrightarrow{VA}$.
3. A and C are in opposite halfplanes with edge $\overleftrightarrow{VB}$.

POSTULATE 21 (*Angle Measure Addition Postulate*) If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are distinct coplanar rays, then $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if

$$m\angle AVC = m\angle AVB + m\angle BVC.$$

We consider two important matters relating to Definition 4.3.

(A) Suppose that $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$. Is $\overrightarrow{VB}$ between $\overrightarrow{VC}$ and $\overrightarrow{VA}$?

(B) Suppose that $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, and that A', B', C' are any points, except V, on $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, respectively, as in Figure 4-8. Is $\overrightarrow{VB'}$ between $\overrightarrow{VA'}$ and $\overrightarrow{VC'}$?

In view of Definition 4.3, the questions raised in (A) and (B) amount to the following, expressed in terms of Figure 4-8. Suppose

(1) A and B are in the same halfplane with edge $\overleftrightarrow{VC}$,
(2) B and C are in the same halfplane with edge $\overleftrightarrow{VA}$,
(3) A and C are in opposite halfplanes with edge $\overleftrightarrow{VB}$.

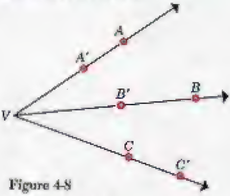


Figure 4-8

Does it follow that

(4) C and B are in the same halfplane with edge $\overleftrightarrow{VA}$,
(5) B and A are in the same halfplane with edge $\overleftrightarrow{VC}$,
(6) C and A are in opposite halfplanes with edge $\overleftrightarrow{VB}$?

Does it follow that

(7) A' and B' are in the same halfplane with edge $\overleftrightarrow{VC'}$,
(8) B' and C' are in the same halfplane with edge $\overleftrightarrow{VA'}$,
(9) A' and C' are in opposite halfplanes with edge $\overleftrightarrow{VB'}$?

Since (1) implies (5) and (7), (2) implies (4) and (8), and (3) implies (6) and (9), it follows that the answer to questions (A) and (B) is Yes.

Thus betweenness for rays is symmetric just as betweenness for points is symmetric. Indeed, $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if $\overrightarrow{VB}$ is between $\overrightarrow{VC}$ and $\overrightarrow{VA}$, and Q is between P and R if and only if Q is between R and P. Also, betweenness for rays depends on rays, not on the particular choice of points used in designating the rays.

Postulate 21 is consistent with these properties of betweenness. Thus, referring to Figure 4-8,

$$m\angle AVC = m\angle AVB + m\angle BVC \qquad \text{if and only if}$$
$$m\angle CVA = m\angle CVB + m\angle BVA \qquad \text{(Why?)}$$
and
$$m\angle AVC = m\angle AVB + m\angle BVC \qquad \text{if and only if}$$
$$m\angle A'VC' = m\angle A'VB' + m\angle B'VC' \qquad \text{(Why?)}$$

Example Consider the six coplanar and concurrent rays formed by the three intersecting lines in Figure 4-9. Then study the following two instances of betweenness in this figure and the four instances of "not betweenness."

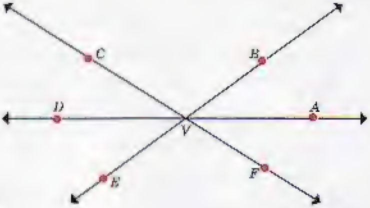


Figure 4-9

Betweenness

1. $\overrightarrow{VA}$ is between $\overrightarrow{VF}$ and $\overrightarrow{VB}$ since (a) B and F are on opposite sides of $\overleftrightarrow{VA}$, (b) A and F are on the same side of $\overleftrightarrow{VB}$, and (c) B and A are on the same side of $\overleftrightarrow{VF}$. From Postulate 21 it follows that

$$m\angle FVA + m\angle AVB = m\angle FVB.$$

2. $\overrightarrow{VA}$ is between $\overrightarrow{VB}$ and $\overrightarrow{VF}$. Indeed, the three statements to check are the same three statements that we just checked to verify that $\overrightarrow{VA}$ is between $\overrightarrow{VF}$ and $\overrightarrow{VB}$. From Postulate 21 it follows that

$$m\angle BVA + m\angle AVF = m\angle BVF,$$

which should not be surprising in view of the equation $m\angle FVA + m\angle AVB = m\angle FVB$ and the commutative property of addition.

Not Betweenness

1. $\overrightarrow{VA}$ is not between $\overrightarrow{VF}$ and $\overrightarrow{VC}$. Why? Let us check. Are F and C on opposite sides of $\overleftrightarrow{VA}$? Yes. Are A and F on the same side of $\overleftrightarrow{VC}$? No. Of course, one "No" in checking the three requirements is enough to establish "not betweenness."

An alternate method of checking betweenness in this instance involves using Postulate 21. According to this postulate, $\overrightarrow{VA}$ is between $\overrightarrow{VF}$ and $\overrightarrow{VC}$ if and only if

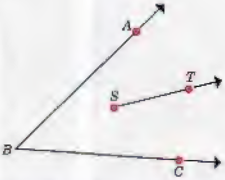
$$m\angle FVC = m\angle FVA + m\angle AVC.$$

But this equation is a false statement. Why? It is false because the right side of the equation is a number, whereas the left side is not. Indeed, there is no such angle as $\angle FVC$ and hence there is no number such as $m\angle FVC$. (See Section 4.2.)

2. $\overrightarrow{VA}$ is not between $\overrightarrow{VE}$ and $\overrightarrow{VB}$. Why not?
3. $\overrightarrow{VA}$ is not between $\overrightarrow{VE}$ and $\overrightarrow{VC}$. Why not?
4. $\overrightarrow{VA}$ is not between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Why not?

EXERCISES 4.3

1. In the figure below at the left, explain why $\overrightarrow{ST}$ is not between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.



2. In the figure above at the right, explain why $\overrightarrow{VC}$ is not between $\overrightarrow{VA}$ and $\overrightarrow{VB}$.
3. In the figure for Exercise 2, is $\overrightarrow{VB}$ between $\overrightarrow{VA}$ and $\overrightarrow{VC}$? Is $\overrightarrow{VA}$ between $\overrightarrow{VB}$ and $\overrightarrow{VC}$?
4. In the figure at the right, if A–B–C, then is $\overrightarrow{BD}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$? Why?

Exercises 5–9 refer to Figure 4-10. Assume that no two of the angles of the figure are congruent. In each exercise, name the missing angle.

5. $m\angle AEB + m\angle BEC = m\angle \boxed{?}$
6. $m\angle AEC + m\angle CED = m\angle \boxed{?}$
7. $m\angle ABC - m\angle ABE = m\angle \boxed{?}$
8. $m\angle BED - m\angle \boxed{?} = m\angle BEC$
9. $m\angle \boxed{?} - m\angle ECD = m\angle ECB$

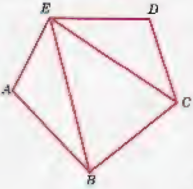


Figure 4-10

Exercises 10–13 refer to Figure 4-11. Name the missing angle or number.

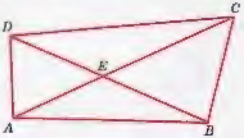
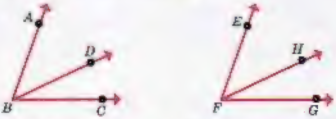


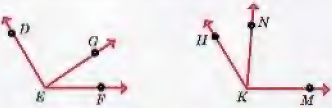
Figure 4-11

10. $m\angle ADE + m\angle EDC = m\angle \boxed{?}$
11. $m\angle DAB - m\angle DAC = m\angle \boxed{?}$
12. $DE + EB = \boxed{?}$
13. $AC - AE = \boxed{?}$

14. Given $m\angle AVB = 35$
$m\angle AVC = 115$
find $m\angle BVC$ if
(a) $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, and
(b) $\overrightarrow{VA}$ is between $\overrightarrow{VB}$ and $\overrightarrow{VC}$.
15. If, in a plane, $m\angle AVB = 70$ and $m\angle BVC = 44$, find $m\angle AVC$. Is there just one possible value for $m\angle AVC$? Illustrate with a figure.
16. In the figures below, $m\angle ABC = m\angle EFG = 70$ and $m\angle DBC = m\angle HFG = 25$. Prove that $\angle ABD \cong \angle EFH$.

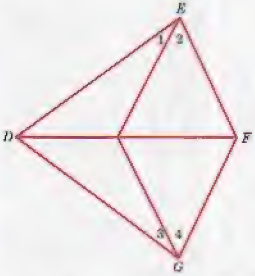


17. In the figures below, $m\angle DEF = m\angle HKM = 120$ and $m\angle GEF = m\angle HKN = 35$. Prove that $\angle DEG \cong \angle NKM$.



18. In the figures below, $m\angle PQS = m\angle DBC = 20$ and $m\angle SQR = m\angle ABD = 50$. Prove that $\angle PQR \cong \angle ABC$.

19. In the figure below, $m\angle 1 = m\angle 3$ and $m\angle 2 = m\angle 4$. Prove that $\angle DEF \cong \angle DGF$.



20. CHALLENGE PROBLEM. Let $\overrightarrow{VB}$ be between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ and let $\overrightarrow{RQ}$ be between $\overrightarrow{RP}$ and $\overrightarrow{RS}$ as shown in the figure below. It appears that all three of the following statements might be true: (a) $\angle CVB \cong \angle SRQ$, (b) $\angle BVA \cong \angle QRP$, and (c) $\angle CVA \cong \angle SRP$. Prove that if any two of these three statements are true, then the third one is also true.

Hint: There are three things to prove.

(A) If (a) and (b), then (c).
(B) If (a) and (c), then (b).
(C) If (b) and (c), then (a).

Proof of (A): There is a number r such that

$$m\angle CVB = m\angle SRQ = r. \qquad \text{Why?}$$

There is a number s such that

$$m\angle BVA = m\angle QRP = s. \qquad \text{Why?}$$

Then

$$m\angle CVA = r + s = m\angle SRP. \qquad \text{Why?}$$

Then

$$\angle CVA \cong \angle SRP. \qquad \text{Why?}$$

Now write proofs of (B) and (C).

4.4 RAY-COORDINATES AND THE PROTRACTOR POSTULATE

In the same way that the Ruler Postulate provides us with a mathematical, or abstract, ruler for assigning coordinates to points, we want a mathematical protractor for assigning ray-coordinates to rays. We first define what is meant by a *ray-coordinate system*. Then we adopt the Protractor Postulate which amounts to an agreement that ray-coordinate systems do exist and that they are unique if we pin them down in certain ways. The definition is based on experiences with protractors.

Definition 4.4 (See Figure 4-12.) Let V be a point in a plane α. A ray-coordinate system in α relative to V is a one-to-one correspondence between the set of all rays in α with endpoint V and the set of all real numbers x such that $0 \leq x < 360$ with the following property: If numbers r and s correspond to rays $\overrightarrow{VR}$ and $\overrightarrow{VS}$ in α, respectively, and if $r > s$, then

$m\angle RVS = r - s$	if $r - s < 180$
$m\angle RVS = 360 - (r - s)$	if $r - s > 180$
$\overrightarrow{VR}$ and $\overrightarrow{VS}$ are opposite rays	if $r - s = 180$.

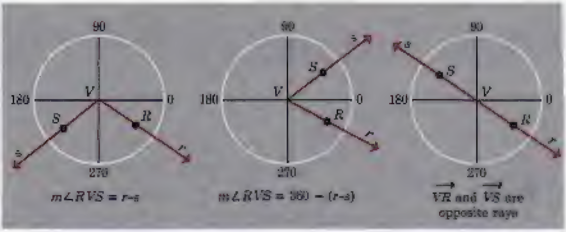


Figure 4-12

Definition 4.5 The number that corresponds to a ray in a given ray-coordinate system is called the ray-coordinate of that ray. The ray whose ray-coordinate is zero is called the zero-ray of that system.

Notation. We use $cd\ \overrightarrow{VX}$ as an abbreviation for ray-coordinate of $\overrightarrow{VX}$.

Example 1 Figure 4-13 suggests a ray-coordinate system where the ray-coordinates of several rays are given. For this example we have:

$$m\angle AVR = 70 - 0 = 70$$
$$m\angle AVR' = 360 - (250 - 0) = 110$$
$$m\angle PVR = 90 - 70 = 20$$
$$m\angle PVR' = 250 - 90 = 160$$
$$m\angle P'VA = 360 - (270 - 0) = 90$$
$$\overrightarrow{VR} = opp\ \overrightarrow{VR'} \text{ since } 250 - 70 = 180$$
$$\overrightarrow{VA} \text{ and } \overrightarrow{VA'} \text{ are opposite rays since } 180 - 0 = 180$$

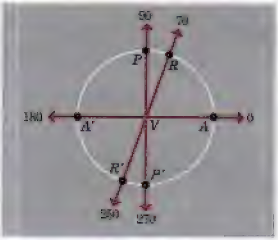


Figure 4-13

POSTULATE 22 (***Protractor Postulate***) If α is any plane and $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are noncollinear rays in α, then

(1) there is a unique ray-coordinate system $\mathcal{S}$ in α relative to V in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VB} = m\angle AVB$ and

(2) if X is any point on the B-side of $\overleftrightarrow{VA}$, then $cd\ \overrightarrow{VX}$ (in $\mathcal{S}$) $= m\angle AVX$.

Example 2 Figure 4-14 shows a line $\overleftrightarrow{VA}$ and two rays $\overrightarrow{VB}$ and $\overrightarrow{VX}$ with B and X on the same side of $\overleftrightarrow{VA}$.

Figure 4-14

If $m\angle AVB = 50$, there is a unique ray-coordinate system in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VB} = 50$. If $m\angle AVX = 105$ and X and B are on the same side of $\overleftrightarrow{VA}$, then $cd\ \overrightarrow{VX} = 105$.

We now proceed to prove several theorems using our postulates about angles.

THEOREM 4.1 If $\angle AVB$ is any angle in a plane α and if $\mathcal{S}$ is the ray-coordinate system in α relative to V in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VB} = m\angle AVB$, then the ray-coordinate of $\overrightarrow{VX}$ is

(1) 0 if $\overrightarrow{VX} = \overrightarrow{VA}$.
(2) 180 if $\overrightarrow{VX} = opp\ \overrightarrow{VA}$.
(3) between 0 and 180 if X is on the B-side of $\overleftrightarrow{VA}$.
(4) between 180 and 360 if X is on the not-B-side of $\overleftrightarrow{VA}$.

Proof: Figure 4-15 consists of four parts which correspond with those of the theorem.

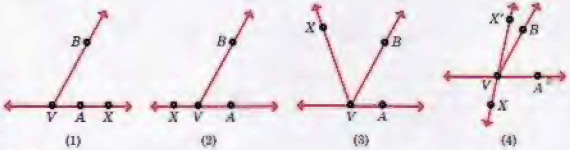


Figure 4-15

1. In a ray-coordinate system there is only one number matched with each ray. Since $\overrightarrow{VX} = \overrightarrow{VA}$ and since 0 is matched with $\overrightarrow{VA}$, then 0 is the number matched with $\overrightarrow{VX}$.
2. According to the definition of a ray-coordinate system two rays are opposite rays if and only if their ray-coordinates differ by 180. Since $cd\ \overrightarrow{VA} = 0$, since $\overrightarrow{VX} = opp\ \overrightarrow{VA}$, and since every ray-coordinate is either 0 or a positive number less than 360, it follows that $cd\ \overrightarrow{VX} = 180$.
3. Since X is on the B-side of $\overleftrightarrow{VA}$, then $cd\ \overrightarrow{VX} = m\angle AVX$. Since $m\angle AVX$ is a number between 0 and 180, it follows that $cd\ \overrightarrow{VX}$ is a number between 0 and 180.
4. Let X' be a point such that $\overrightarrow{VX}$ and $\overrightarrow{VX'}$ are opposite rays. Since X is on the not-B-side of $\overleftrightarrow{VA}$, it follows that X' is on the B-side of $\overleftrightarrow{VA}$. Then $cd\ \overrightarrow{VX'}$ is a number between 0 and 180 by part 3, and since $cd\ \overrightarrow{VX}$ is a number between 0 and 360 which differs from $cd\ \overrightarrow{VX'}$ by 180, it follows that $cd\ \overrightarrow{VX}$ is between 180 and 360.

The argument in part 4 expressed in symbols consists of the following steps:

$$0 < cd\ \overrightarrow{VX} < 360$$
$$0 < cd\ \overrightarrow{VX'} < 360$$
$$0 < cd\ \overrightarrow{VX'} < 180$$
$$cd\ \overrightarrow{VX} - cd\ \overrightarrow{VX'} = 180$$
$$cd\ \overrightarrow{VX} = cd\ \overrightarrow{VX'} + 180$$
$$180 < cd\ \overrightarrow{VX} < 360$$

In working with a protractor we know that if we are given any angle, say $\angle DEF$, and a ray $\overrightarrow{VA}$ on the edge of a halfplane $\mathcal{H}$, then we can draw a ray $\overrightarrow{VB}$ with B in $\mathcal{H}$ so that $\angle AVB \cong \angle DEF$. Our postulates permit us to do this in the abstract as the following theorem suggests.

THEOREM 4.2 (***Angle Construction Theorem***) If $\angle DEF$ is any angle, if $\overrightarrow{VA}$ is any ray, if $\mathcal{H}$ is any halfplane with edge $\overrightarrow{VA}$, then there is one and only one halfline $\overset{\circ\!\!\rightarrow}{VB}$ in $\mathcal{H}$ such that $\angle AVB \cong \angle DEF$.

Proof: (See Figure 4-16.) Let us suppose that $\angle DEF$, $\overrightarrow{VA}$, and $\mathcal{H}$ are given as in the statement of the theorem. Let α be the plane that contains $\mathcal{H}$ and let P be any point in $\mathcal{H}$. Let $\mathcal{S}$ be the unique ray-coordinate system in α relative to V in which

$$cd\ \overrightarrow{VA} = 0 \qquad \text{and} \qquad cd\ \overrightarrow{VP} = m\angle AVP.$$

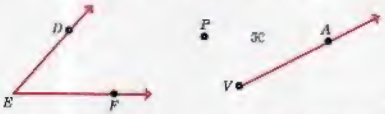


Figure 4-16

Let $m\angle DEF = b$. Then $0 < b < 180$ and there is exactly one ray $\overrightarrow{VB}$ with B in $\mathcal{H}$ such that $cd\ \overrightarrow{VB} = b$. Why? Then

$$m\angle AVB = b = m\angle DEF$$

and $\overset{\circ\!\!\rightarrow}{VB}$ is the unique halfline in $\mathcal{H}$ such that $\angle AVB \cong \angle DEF$.

The following theorem relates betweenness for rays and betweenness for coordinates.

THEOREM 4.3 If a ray-coordinate system in which $cd\,\overrightarrow{VA} = 0$, $cd\,\overrightarrow{VB} = b$, $cd\,\overrightarrow{VC} = c$ with $c < 180$ is given, then $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if b is between 0 and c.

Proof: Figure 4-17 suggests the "given" or hypothesis of our theorem. We prove two things.

1. If b is between 0 and c, then $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$.
2. If $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, then b is between 0 and c.

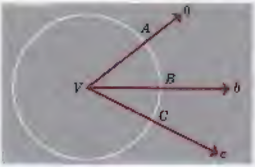


Figure 4-17

Proof of 1:

Statement	Reason
1. $c < 180$	1. Hypothesis
2. $c \geq 0$	2. All ray-coordinates are non-negative numbers.
3. $0 < b < c < 180$	3. Hypothesis and steps 1, 2
4. $m\angle AVB = b - 0$	4. Definition of ray-coordinate system
5. $m\angle BVC = c - b$	5. Definition of ray-coordinate system
6. $m\angle AVC = c - 0$	6. Definition of ray-coordinate system
7. $(b - 0) + (c - b) = c - 0$	7. Properties of real numbers
8. $m\angle AVB + m\angle BVC = m\angle AVC$	8. Substitution (steps 4, 5, 6, and 7)
9. $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$.	9. Angle Measure Addition Postulate

Proof of 2:

Statement	Reason
1. $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$.	1. Hypothesis
2. $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are distinct rays.	2. Definition of betweenness for rays
3. B is on the C-side of $\overleftrightarrow{VA}$.	3. Definition of betweenness for rays
4. 0, b, c are distinct numbers.	4. Definition of ray-coordinate system and step 2
5. $0 < c < 180$	5. Hypothesis

6. $0 < b < 180$	6. Steps 3, 4, 5 and definition of a ray-coordinate system
7. $m\angle AVB = b - 0$ $m\angle AVC = c - 0$	7. Definition of a ray-coordinate system
8. $m\angle AVB + m\angle BVC = m\angle AVC$	8. Angle Measure Addition Postulate
9. $b + m\angle BVC = c$	9. Substitution (steps 7 and 8)
10. $m\angle BVC > 0$	10. Angle Measure Existence Postulate
11. $b < c$	11. Steps 9 and 10
12. $0 < b < c$	12. Steps 6 and 11

THEOREM 4.4 (*Angle Measure Addition Theorem*) If distinct rays $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are between rays $\overrightarrow{VA}$ and $\overrightarrow{VD}$ and if $\angle AVB \cong \angle CVD$, then $\angle AVC \cong \angle BVD$.

Proof: Figure 4-18 suggests two possibilities. We prove both cases at the same time.

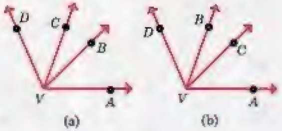


Figure 4-18

Suppose that $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are distinct rays as in Figure 4-18. From the Protractor Postulate and Theorem 4.3 it follows that there is a unique ray-coordinate system $\mathcal{S}$ such that $cd\ \overrightarrow{VA} = 0$, $0 < cd\ \overrightarrow{VD} < 180$, $0 < cd\ \overrightarrow{VB} < cd\ \overrightarrow{VD}$, and $0 < cd\ \overrightarrow{VC} < cd\ \overrightarrow{VD}$. Let $cd\ \overrightarrow{VB} = b$, $cd\ \overrightarrow{VC} = c$, $cd\ \overrightarrow{VD} = d$.

We must show that if $\angle AVB \cong \angle CVD$, then $\angle AVC \cong \angle BVD$. Since

$$m\angle AVB = b - 0 = b$$
$$m\angle CVD = d - c$$
$$m\angle AVC = c$$
$$m\angle BVD = d - b$$

our problem amounts to proving that if $b = d - c$, then $c = d - b$. Suppose then that $b = d - c$. Using the Addition Property of Equality, we may add $c - b$ to both sides of this equation. The result is

$$b + (c - b) = (d - c) + (c - b),$$

which simplifies to $c = d - b$, the desired conclusion.

Notice the similarity of Theorem 4.4 to Theorem 3.4.

COROLLARY 4.4.1 If distinct rays $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are between rays $\overrightarrow{VA}$ and $\overrightarrow{VD}$ and if $\angle AVC \cong \angle BVD$, then $\angle AVB \cong \angle CVD$.

Proof: If $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are between $\overrightarrow{VA}$ and $\overrightarrow{VD}$, then $\overrightarrow{VC}$ and $\overrightarrow{VB}$ are between $\overrightarrow{VA}$ and $\overrightarrow{VD}$. The corollary follows immediately from Theorem 4.4 by interchanging $\overrightarrow{VB}$ and $\overrightarrow{VC}$, that is, by renaming ray $\overrightarrow{VB}$ as ray $\overrightarrow{VC}$ and renaming ray $\overrightarrow{VC}$ as ray $\overrightarrow{VB}$.

COROLLARY 4.4.2 If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$ are distinct coplanar rays such that A–V–D, B and C are on the same side of $\overleftrightarrow{AD}$, and $\angle AVB \cong \angle CVD$, then $\angle AVC \cong \angle BVD$.

Proof: There is a unique ray-coordinate system in which $cd\,\overrightarrow{VA} = 0$, $cd\ \overrightarrow{VB} = b$, $cd\ \overrightarrow{VC} = c$, $cd\ \overrightarrow{VD} = 180$, $b \neq c$, $0 < b < 180$, and $0 < c < 180$. Then

$$\begin{aligned} m\angle AVB &= b - 0 = b, \\ m\angle AVC &= c - 0 = c, \\ m\angle BVD &= 180 - b, \\ m\angle CVD &= 180 - c. \end{aligned}$$

Since $\angle AVB \cong \angle CVD$, then $b = 180 - c$. It follows that $b + c = 180$, $c = 180 - b$, $m\angle AVC = m\angle BVD$, and $\angle AVC \cong \angle BVD$.

EXERCISES 4.4

Exercises 1–10 refer to a ray-coordinate system in which the numbers assigned to $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$, $\overrightarrow{VE}$, and $\overrightarrow{VF}$ are 0, 28, 47, 139, 263, and 319, respectively. In Exercises 2–9, compute the angle measures using the given ray-coordinates.

1. Draw a figure to illustrate the given situation.
2. $m\angle BVC$
3. $m\angle BVF$
4. $m\angle CVE$
5. $m\angle DVE$
6. $m\angle FVE$
7. $m\angle AVE$
8. $m\angle BVD$
9. $m\angle CVF$
10. What can you say about rays $\overrightarrow{VD}$ and $\overrightarrow{VF}$?

11. Let three distinct concurrent rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VA'}$ such that $\overrightarrow{VA} = opp\ \overrightarrow{VA'}$ be given. Prove that

$$m\angle AVB + m\angle BVA' = 180.$$

12. In the situation of Exercise 11, explain why it is incorrect in our formal geometry to say that

$$m\angle AVB + m\angle BVA' = m\angle AVA'.$$

Exercises 13–19 refer to a ray-coordinate system in which the numbers assigned to $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, and $\overrightarrow{VD}$ are 0, b, c, and d, respectively, where $0 < b < c < d < 180$. In Exercises 14–19, express the angle measures using the given ray-coordinates.

13. Draw a figure to illustrate the given situation.
14. $m\angle AVC$
15. $m\angle BVD$
16. $m\angle BVC$
17. $m\angle CVD$
18. $m\angle AVD$
19. $m\angle AVB$

In Exercises 20–26, the same situation as in Exercises 13–19 is given except that $0 < b < c < 90$ and $270 < d < 360$. In Exercises 21–26, express the angle measures using the given ray-coordinates.

20. Draw a figure to illustrate the given situation.
21. $m\angle AVC$
22. $m\angle BVD$
23. $m\angle BVC$
24. $m\angle CVD$
25. $m\angle AVD$
26. $m\angle AVB$

Exercises 27–38 refer to Figure 4-19. If the ray-coordinates of $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$, $\overrightarrow{VE}$, and $\overrightarrow{VF}$ are as shown in the figure, name the theorem, definition, or postulate that justifies the statement in the exercise.

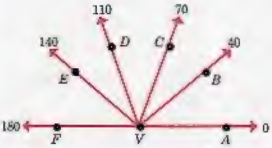


Figure 4-19

27. $\overrightarrow{VD}$ is between $\overrightarrow{VC}$ and $\overrightarrow{VE}$.
28. $m\angle CVE = m\angle CVD + m\angle DVE$
29. $\overrightarrow{VC}$ is between $\overrightarrow{VF}$ and $\overrightarrow{VB}$.
30. $m\angle BVF = m\angle BVC + m\angle CVF$
31. $m\angle BVC = 70 - 40 = 30$
32. $m\angle DVE = 140 - 110 = 30$
33. $\angle BVC \cong \angle DVE$
34. $\angle BVD \cong \angle CVE$
35. $m\angle AVE = 140 - 0 = 140$
36. $m\angle BVF = 180 - 40 = 140$
37. $\angle AVE \cong \angle BVF$
38. $\angle AVB \cong \angle EVF$

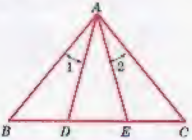
Exercises 39 and 40 refer to Figure 4-20. Use Corollary 4.4.2.

Figure 4-20

39. If $\angle MVN \cong \angle RVS$, name an angle congruent to $\angle MVR$.
40. If $\angle SVN \cong \angle RVM$, name an angle congruent to $\angle SVR$.

41. In the figure below, $\angle DBA \cong \angle CBE$. What can you conclude about $\angle 1$ and $\angle 2$? What are you assuming from the figure about the points A, B, C, D, E? About the points A, B, C in particular? About D and E in relation to AC?

42. In the figure below, $\angle 1 \cong \angle 2$. What can you conclude about $\angle BAE$ and $\angle CAD$? Name the theorem you are using. What assumption are you making about the figure?



4.5 SOME PROPERTIES OF ANGLES

Two distinct intersecting lines form four angles as suggested by Figure 4-21. Two angles such as a and c or b and d, which appear "opposite" each other in the figure, are called *vertical angles*. Two angles such as a and b or b and c are called a *linear pair of angles*. We state these ideas more precisely in the following definitions.

Figure 4-21

Definition 4.6 Two angles are called vertical angles if and only if their sides form two pairs of opposite rays.

Definition 4.7 Two angles are called a linear pair of angles if and only if they have one side in common and the other sides are opposite rays.

Note that two angles are vertical angles if their union is the union of two distinct intersecting lines. Note also that two angles are a linear pair of angles if their union is the union of a line and a ray whose endpoint lies on that line.

THEOREM 4.5 Vertical angles are congruent.

Proof: Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at V to form two vertical angles $\angle AVC$ and $\angle BVD$ as in Figure 4-22. Let α be the plane determined by A, V, and C. Let $\mathcal{S}$ be the unique ray-coordinate system in α relative to V in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VC} = m\angle AVC$. (Which postulate tells us there is one and only one ray-coordinate system with these properties?) For convenience, let $cd\ \overrightarrow{VB} = b$, $cd\ \overrightarrow{VC} = c$, $cd\ \overrightarrow{VD} = d$.

Figure 4-22

The rest of the proof follows from the definition of a ray-coordinate system.

$b = 180$ and $d = c + 180$ Why?

$m\angle BVD = d - b = (c + 180) - 180 = c = m\angle AVC$

$\angle AVC \cong \angle BVD$

Definition 4.8 Two angles (distinct or not) are complementary, and each is called a complement of the other if the sum of their measures is 90. Two angles (distinct or not) are supplementary, and each is called a supplement of the other if the sum of their measures is 180.

THEOREM 4.6 If two angles form a linear pair of angles, then they are supplementary angles.

Proof: Let a linear pair of angles, $\angle AVB$ and $\angle BVC$, in plane α as indicated in Figure 4-23 be given. Let $\mathcal{S}$ be the unique ray-coordinate system relative to V in α such that $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VB} = m\angle AVB$.

Figure 4-23

Then $\qquad cd\ \overrightarrow{VB} < 180 \qquad$ and $\qquad cd\ \overrightarrow{VC} = 180$

$$m\angle AVB + m\angle BVC = (cd\ VB - 0) + (180 - cd\ VB) = 180.$$

Therefore $\angle AVB$ and $\angle BVC$ are supplementary.

Notice that Exercise 11 of Exercises 4.4 is Theorem 4.6.

THEOREM 4.7 Complements of congruent angles are congruent.

Proof: Let $\angle A$ and $\angle B$ be two congruent angles, let $\angle C$ be a complement of $\angle A$, and let $\angle D$ be a complement of $\angle B$. Then

$$m\angle A + m\angle C = 90$$
$$m\angle B + m\angle D = 90$$
$$m\angle A = m\angle B$$
$$m\angle A + m\angle C = m\angle B + m\angle D$$
$$m\angle C = m\angle D$$
$$\angle C \cong \angle D$$

THEOREM 4.8 Supplements of congruent angles are congruent.

Proof: Assigned as an exercise.

A special ray associated with an angle is its *midray*. Here is a definition for a midray.

Definition 4.9 A ray is a midray of an angle if it is between the sides of the angle and forms with them two congruent angles. A midray of an angle is said to bisect the angle; it is sometimes called the bisector of the angle or, briefly, the angle bisector.

It appears from Figure 4-24 that an angle should have one and only one midray. This suggests our next theorem.

Figure 4-24

THEOREM 4.9 Every angle has a unique midray.

Proof: Let $\angle AVC$ as in Figure 4-25 be given. Let $\mathcal{S}$ be the unique ray-coordinate system in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VC} = c = m\angle AVC$. Then there is a unique ray $\overrightarrow{VB}$ such that $cd\ \overrightarrow{VB} = b = \frac{c}{2}$. Why?

$m\angle AVB + m\angle BVC = (b - 0) + (c - b)$

$= c = m\angle AVC$

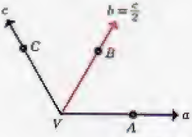


Figure 4-25

$\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$.

$m\angle AVB = b - 0 = b$

$m\angle BVC = c - b = 2b - b = b$

$m\angle AVB = m\angle BVC$

Therefore $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, and $\overrightarrow{VB}$ forms with $\overrightarrow{VA}$ and $\overrightarrow{VC}$ two congruent angles. Therefore $\overrightarrow{VB}$ is a midray of $\angle AVC$.

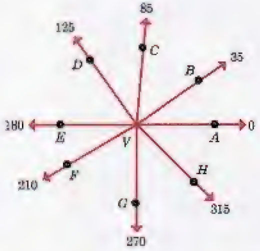
We prove next that $\angle AVC$ has only one midray. Suppose that $\overrightarrow{VD}$ is a midray of $\angle AVC$. Let $cd\ \overrightarrow{VD} = d$.

Then $\overrightarrow{VD}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$.	Why?
$0 < d < c$	by Theorem 4.3
$m\angle AVD = d - 0 = d$	
$m\angle DVC = c - d$	
$c - d = d, \quad c = 2d, \quad d = \frac{c}{2} = b$	
$\overrightarrow{VD} = \overrightarrow{VB}$	Why?

Therefore the midray is unique.

EXERCISES 4.5

1. In the figure several rays and their ray-coordinates are marked. Compute the measures of the following angles.
   (a) $\angle AVB$
   (b) $\angle AVC$
   (c) $\angle AVH$
   (d) $\angle AVG$
   (e) $\angle AVF$
   (f) $\angle DVC$
   (g) $\angle DVB$
   (h) $\angle DVA$
   (i) $\angle DVH$
   (j) $\angle DVG$



2. In a ray-coordinate system the numbers assigned to $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$ are 0, 30, 130, 180, respectively.
   (a) Draw a figure to illustrate the given situation.
   (b) Compute several angle measures and use them to prove that $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$.
   (c) Is $\overrightarrow{VB}$ between $\overrightarrow{VA}$ and $\overrightarrow{VD}$? Justify your answer.
   (d) Is $\overrightarrow{VA}$ between $\overrightarrow{VB}$ and $\overrightarrow{VC}$? Justify your answer.
   (e) Is $\overrightarrow{VC}$ between $\overrightarrow{VB}$ and $\overrightarrow{VD}$? Justify your answer.
3. In a ray-coordinate system the numbers assigned to $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are 0 and 100, respectively. All of the rays and angles in this exercise are in the plane determined by the points A, V, and B.
   (a) If $m\angle AVC = 50$ and if B and C are on the same side of $\overleftrightarrow{VA}$, find $cd\ \overrightarrow{VC}$.
   (b) If $m\angle AVC = 50$ and if B and C are on opposite sides of $\overleftrightarrow{VA}$, find $cd\ \overrightarrow{VC}$.
   (c) If $m\angle BVD = 150$ and if A and D are on the same side of $\overleftrightarrow{VB}$, find $cd\ \overrightarrow{VD}$.
   (d) If $m\angle BVD = 150$ and if A and D are on opposite sides of $\overleftrightarrow{VB}$, find $cd\ \overrightarrow{VD}$.
   (e) If $cd\ \overrightarrow{VE} = 200$, is $\overrightarrow{VB}$ between $\overrightarrow{VA}$ and $\overrightarrow{VE}$?
   (f) If $cd\ \overrightarrow{VE} = 200$, is $\overrightarrow{VA}$ between $\overrightarrow{VB}$ and $\overrightarrow{VE}$?

(g) If $cd\ \overrightarrow{VE} = 200$, is $\overrightarrow{VE}$ between $\overrightarrow{VB}$ and $\overrightarrow{VA}$?

(h) If $cd\ \overrightarrow{VF} = 281$, is $\overrightarrow{VB}$ between $\overrightarrow{VA}$ and $\overrightarrow{VF}$?

(i) If $cd\ \overrightarrow{VF} = 281$, is $\overrightarrow{VA}$ between $\overrightarrow{VB}$ and $\overrightarrow{VF}$?

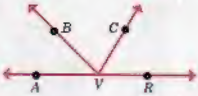
(j) If $cd\ \overrightarrow{VF} = 281$, is $\overrightarrow{VF}$ between $\overrightarrow{VB}$ and $\overrightarrow{VA}$?

(k) Find $cd\ \overrightarrow{VK}$ if $\overrightarrow{VK}$ is the midray of $\angle AVB$.

4. Given a ray-coordinate system in which the numbers assigned to $\overrightarrow{VA}$, $\overrightarrow{VN}$, $\overrightarrow{VD}$, $\overrightarrow{VY}$ are a, b, c, d, respectively, let $\overrightarrow{VA'}$, $\overrightarrow{VN'}$, $\overrightarrow{VD'}$, $\overrightarrow{VY'}$, be the rays opposite to $\overrightarrow{VA}$, $\overrightarrow{VN}$, $\overrightarrow{VD}$, $\overrightarrow{VY}$, respectively. Assume that $0 < a < b < c < d < 180$. Derive formulas in terms of a, b, c, d for the following measures.

(a) $m\angle AVN$
(b) $m\angle AVY$
(c) $m\angle AVN'$
(d) $m\angle AVD'$
(e) $m\angle DVA$
(f) $m\angle DVY$
(g) $m\angle DVN$
(h) $m\angle DVN'$

5. Referring to the figure, describe in your own words the following sets.



(a) The union of $\overrightarrow{VA}$ and $\overrightarrow{VR}$.
(b) The union of $\overrightarrow{VA}$ and $\overrightarrow{VB}$.
(c) The union of $\angle AVB$ and $\angle BVC$.
(d) The union of $\angle AVB$ and $\angle RVC$.

6. Referring to the figure, describe in your own words the following sets.

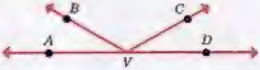
(a) The intersection of $\overrightarrow{VR}$ and $\overrightarrow{VP}$.
(b) The intersection of $\angle RVP$ and $\angle PVX$.
(c) The intersection of $\angle RVP$ and $\angle XVY$.
(d) The intersection of $\angle PVX$ and $\angle XVP$.

7. Let $\overleftrightarrow{AB}$ be the edge of a halfplane $\mathcal{H}$ that contains points C and D. Assume that A, C, D are noncollinear and that B, C, D are noncollinear. Describe the following sets.

(a) The intersection of $\mathcal{H}$ and $\overleftrightarrow{AB}$.
(b) The intersection of $\angle CAD$ and $\mathcal{H}$.
(c) The intersection of $\angle CAD$ and $\overleftrightarrow{AB}$.
(d) The intersection of $\angle CAD$ and $\angle CBD$.
(e) The union of $\mathcal{H}$ and $\overleftrightarrow{AB}$.
(f) The union of $\mathcal{H}$ and $\angle CAD$.
(g) The union of $\mathcal{H}$ and $\angle ACB$.
(h) The intersection of $\mathcal{H}$ and $\angle ACB$.

8. Let four coplanar rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$ be given such that $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, $\overrightarrow{VC}$ is between $\overrightarrow{VB}$ and $\overrightarrow{VD}$, and $\overrightarrow{VD} = opp\ \overrightarrow{VA}$ as indicated in the figure. Complete the following proof that

$$m\angle AVB + m\angle BVC + m\angle CVD = 180.$$

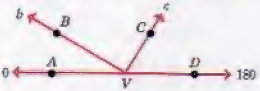


Proof: $m\angle BVD = m\angle BVC + m\angle CVD$ Why?
$m\angle AVB + m\angle BVD = 180$ Why?

Therefore $\boxed{?}$.

9. Let four coplanar rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$ be given such that $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, $\overrightarrow{VD} = opp\ \overrightarrow{VA}$, and $m\angle AVC > m\angle AVB$ as indicated in the figure. Justify each step in the following proof that

$$m\angle AVB + m\angle BVC + m\angle CVD = 180.$$



Proof: Let $\mathcal{S}$ be the unique ray-coordinate system in which $cd\ \overrightarrow{VA} = 0$ and

$$cd\ \overrightarrow{VB} = m\angle AVB = b < 180.$$

Then B and C are on the same side of $\overleftrightarrow{AV}$. Let $cd\ \overrightarrow{VC} = c$. Then $0 < c < 180$, where $c = m\angle AVC$. Then $0 < b < c < 180$. Therefore

$$m\angle AVB + m\angle BVC + m\angle CVD$$
$$= (b - 0) + (c - b) + (180 - c) = 180.$$

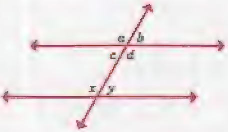
10. In the figure $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O forming four angles. If $m\angle AOD = 133$, find
(a) $m\angle COA$
(b) $m\angle BOC$
(c) $m\angle BOD$

11. If $\angle A$ and $\angle B$ are supplementary angles and x is a number such that $m\angle A = 3x + 6$ and $m\angle B = 2x + 12$, find the measures of $\angle A$ and $\angle B$. Check your results by finding the sum of these measures.

12. If $\angle P$ and $\angle Q$ are complementary angles and y is a number such that $m\angle P = y + 30$ and $m\angle Q = y - 30$, find the measures of $\angle P$ and $\angle Q$. Check your result by addition.

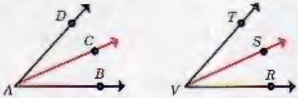
13. Twice the measure of an angle is 24 more than five times the measure of its supplement. Find the measure of the angle and check your result.

14. Find the measure of an angle if its measure is twice the measure of its complement.
15. Find the measure of an angle if its measure is one-half the measure of its complement.
16. If $\angle A$ and $\angle B$ are both congruent and supplementary, find the measure of each.
17. If $\angle A$ and $\angle B$ are both congruent and complementary, find the measure of each.
18. The figure shows three coplanar lines and six angles marked a, b, c, d, x, and y. Complete the following statements. (There are several correct responses for some items.)



(a) a and b are a $\boxed{?}$ pair of angles.
(b) a and b are $\boxed{?}$ angles.
(c) b and c are $\boxed{?}$ angles.
(d) If $a \cong x$, then $b \cong \boxed{?}$.

19. Given an angle $\angle AVB$, let $\mathcal{H}$ be the set of all points that are on the same side of $\overleftrightarrow{VB}$ as A, that is, $\mathcal{H}$ is the A-side of $\overleftrightarrow{VB}$. Let $\mathcal{K}$ be the B-side of $\overleftrightarrow{VA}$. Make a sketch showing $\mathcal{H}$ by shading with vertical halflines and $\mathcal{K}$ by shading with horizontal halflines. How is the interior of $\angle AVB$ marked in your sketch? Is the interior of $\angle AVB$ the intersection of $\mathcal{H}$ and $\mathcal{K}$, or the union of $\mathcal{H}$ and $\mathcal{K}$, or some other set related to $\mathcal{H}$ and $\mathcal{K}$?
20. In the proof of Theorem 4.7, justify each of the five equations and the congruence.
21. Prove Theorem 4.8.
22. In the figure below, $\overrightarrow{AC}$ is the midray of $\angle BAD$ and $\overrightarrow{VS}$ is the midray of $\angle RVT$. If $\angle BAD \cong \angle RVT$, prove that $\angle BAC \cong \angle RVS$.

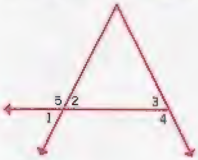


23. Speaking informally, does the result of Exercise 22 prove that "halves" of congruent angles are congruent?
24. Using Theorem 4.8 and the reflexive property of congruence for angles, write an alternate proof of Theorem 4.5.
25. In the figure at the right,

$$\angle 2 \cong \angle 3.$$

Prove that

$$\angle 1 \cong \angle 3 \qquad \text{and} \qquad \angle 4 \cong \angle 5.$$



4.6 INTERIORS OF ANGLES

In Chapter 2 we defined the interior of $\angle ABC$ as the intersection of two halfplanes, the C-side of $\overleftrightarrow{AB}$ and the A-side of $\overleftrightarrow{BC}$.

Another way to think of the interior of an angle is in terms of the rays between the sides of the angle. From our definition of betweenness for rays in Section 4.3, we know that if $\overrightarrow{VD}$ is a ray between the sides of $\angle AVC$, then every point of $\overrightarrow{VD}$ except V (that is, the halfline $\overset{\circ}{\overrightarrow{VD}}$) lies on the A-side of $\overleftrightarrow{VC}$ and on the C-side of $\overleftrightarrow{AV}$. Hence $\overset{\circ}{\overrightarrow{VD}}$ is contained in the interior of the angle. What we have shown for $\overrightarrow{VD}$ is true for every ray between $\overrightarrow{VA}$ and $\overrightarrow{VC}$. For convenience let $\mathcal{R}$ denote the union of the interiors of all rays between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ and let $\mathcal{I}$ denote the interior of $\angle AVC$. We have shown that $\mathcal{R}$ lies in $\mathcal{I}$, that is, that

$$\mathcal{R} \subset \mathcal{I}.$$

In Lemma 4.10.1 we shall show that if a point is in $\mathcal{I}$, it is also in $\mathcal{R}$. Then we shall have $\mathcal{R} \subset \mathcal{I}$ and $\mathcal{I} \subset \mathcal{R}$, and hence $\mathcal{I} = \mathcal{R}$, which we state formally in Theorem 4.10.

LEMMA 4.10.1 If a point is in the interior of an angle, then it is an interior point of a ray between the sides of that angle.

Proof: Let D be a point in the interior of $\angle AVC$, and let α be the plane containing the points A, V, C, D. Then it follows from the definition of the interior of an angle that (1) A and D are on the same side of $\overleftrightarrow{VC}$ and (2) D and C are on the same side of $\overleftrightarrow{VA}$. We shall show that $\overrightarrow{VD}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, and hence that D is an interior point of a ray between the sides of $\angle AVC$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be ray-coordinate systems in α relative to V with coordinates as indicated in the table and such that $0 < c < 180$, $0 < a' < 180$. Since $0 < c < 180$ and since C and D are on the same side of $\overleftrightarrow{VA}$, it follows that $0 < d < 180$. Since $0 < a' < 180$ and since A and D are on the same side of $\overleftrightarrow{VC}$, it follows that $0 < d' < 180$.

	$\mathcal{S}$	$\mathcal{S}'$
$cd\ \overrightarrow{VA}$	0	a'
$cd\ \overrightarrow{VD}$	d	d'
$cd\ \overrightarrow{VC}$	c	0

Suppose that $c = d$. Then $d' = 0$ (Why?), D lies on $\overrightarrow{VC}$, and D is not in the interior of $\angle AVC$. Since this contradicts the hypothesis of the theorem, it follows that $c \neq d$.

Then $0 < d < c < 180$ or $0 < c < d < 180$. Suppose that

$$0 < c < d < 180.$$

Then it follows from Theorem 4.3 that $\overrightarrow{VC}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VD}$ and from Definition 4.3 that A and D are on opposite sides of $\overleftrightarrow{VC}$. However, A and D are on the same side of $\overleftrightarrow{VC}$. (See (1) in proof.) Since this contradicts the Plane Separation Postulate, it follows that

$$0 < d < c < 180.$$

Using Theorem 4.3 and Definition 4.3 again, we deduce that $\overrightarrow{VD}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ and hence that D is an interior point of a ray between the sides of $\angle AVC$.

We have proved the following theorem.

THEOREM 4.10 The interior of an angle is the union of the interiors of all rays between the sides of the angle.

Another way to consider the interior of an angle is in terms of the segments whose endpoints lie on the sides of the angle. (See Figure 4-26.)

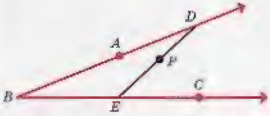


Figure 4-26

Given $\angle ABC$, let D be any point on $\overrightarrow{BA}$ except B and let E be any point on $\overrightarrow{BC}$ except B. Let P be any interior point of the segment $\overline{DE}$. In other words, P is any point between D and E. Since all of $\overline{DE}$ except E lies on the A-side of $\overleftrightarrow{BC}$ and all of $\overline{DE}$ except D lies on the C-side of $\overleftrightarrow{AB}$, it follows that P lies in the interior of $\angle ABC$. Thus we have the following theorem.

THEOREM 4.11 If $\overline{AB}$ is a segment joining an interior point of one side of an angle to an interior point of the other side, then the interior of $\overline{AB}$ is contained in the interior of the angle.

4.7 ADJACENT ANGLES AND PERPENDICULARITY

A linear pair of angles is a special case of a pair of coplanar angles having one side in common and interiors which do not intersect. (See Figure 4-27.) It is convenient to introduce a special word for angles with this property.

Figure 4-27

Definition 4.10 Two coplanar angles are adjacent angles if they have one side in common and the intersection of their interiors is empty.

Figure 4-28

In Figure 4-28, $\angle AVB$ and $\angle BVC$ are adjacent angles and $\angle AVB$ and $\angle AVC$ are adjacent angles. Although $\angle DWE$ and $\angle EWF$ are adjacent angles, note that $\angle DWE$ and $\angle DWF$ are *not* adjacent angles.

All of you have a background of experience with right angles, perpendicular lines, acute angles, and obtuse angles. Following are the formal definitions for these terms.

Definition 4.11 An angle whose measure is 90 is a right angle. An angle whose measure is less than 90 is an acute angle. An angle whose measure is greater than 90 is an obtuse angle. (See Figure 4-29.)

Figure 4-29

THEOREM 4.12 If the two angles in a linear pair are congruent, they are right angles.

Proof: (See Figure 4-30.) Let the measure of each angle in the linear pair be r. It follows from Theorem 4.6 that $r + r = 180$. Therefore $r = 90$ and each of the angles is a right angle.

Figure 4-30

THEOREM 4.13 Any two right angles are congruent.

Proof: Every right angle has a measure of 90. Hence all right angles have the same measure and hence all right angles are congruent to each other.

Definition 4.12 If the union of two intersecting lines contains a right angle, then the lines are **perpendicular**.

If $\angle AVB$ is a right angle, then $\overleftrightarrow{AV}$ and $\overleftrightarrow{VB}$ are perpendicular lines. (See Figure 4-31.)

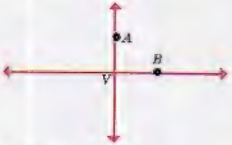


Figure 4-31

Figure 4-32

It is easy to show that two perpendicular lines form four right angles. Let $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ be perpendicular lines which intersect at V as indicated in Figure 4-32 and let $\angle AVB$ be a right angle. Then $\angle A'VB'$ is a right angle since $\angle A'VB'$ and $\angle AVB$ are vertical angles and vertical angles are congruent. Also, $\angle AVB'$ is a right angle since $\angle AVB'$ and $\angle AVB$ form a linear pair. Then $m\angle AVB = 90$, $m\angle AVB + m\angle AVB' = 180$, and therefore $m\angle AVB' = 90$. Why is $\angle A'VB$ a right angle?

When are two rays perpendicular? It seems natural enough to say that two rays are perpendicular if the lines which contain them are perpendicular. We extend this idea to any combination of segments, rays, and lines in the following definition.

Definition 4.13 Two sets, each of which is a segment, a ray, or a line, and which determine two perpendicular lines are called perpendicular sets, and each is said to be perpendicular to the other.

Notation. We write $\mathcal{A} \perp \mathcal{B}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ are perpendicular sets. We may read $\mathcal{A} \perp \mathcal{B}$ as "$\mathcal{A}$ is perpendicular to $\mathcal{B}$." Note that if $\mathcal{A} \perp \mathcal{B}$, then $\mathcal{B} \perp \mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are perpendicular sets, then $\mathcal{A}$ is contained in some line l, $\mathcal{B}$ is contained in some line m, and $l \perp m$. Since perpendicular lines are distinct intersecting lines, it follows that if $\mathcal{A} \perp \mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ have at most one point in common. As indicated in Figure 4-33, the intersection of two perpendicular sets is a set consisting of at most one point.

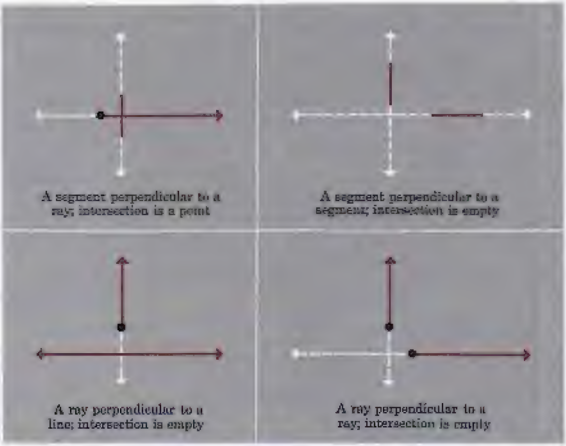


Figure 4-33

In a figure two perpendicular sets may be marked with a little "square corner" as illustrated in Figure 4-34.

Figure 4-34

A typical exercise in informal geometry involves constructing a perpendicular to a line at a point on it. Our formal geometry is sufficiently developed now so that we can prove that such perpendiculars exist.

THEOREM 4.14 For each point on a line in a plane, there is one and only one line which lies in the given plane, contains the given point, and is perpendicular to the given line.

Proof: See Figure 4-35. Let P be a point on line l in plane α. Let $\mathcal{H}$ be one of the two halfplanes in α with edge l. There exists a point A different from P on l. Why?

Figure 4-35

Let R be any point in $\mathcal{H}$. Let $\mathcal{S}$ be the unique ray-coordinate system in α relative to P in which $cd\ \overrightarrow{PA} = 0$ and $cd\ \overrightarrow{PR} = m\angle APR$.

Let $\overrightarrow{PB}$ be the unique ray with $cd\ \overrightarrow{PB} = 90$. $\angle APB$ is a right angle and $\overleftrightarrow{PB} \perp \overleftrightarrow{PA}$. Why?

Therefore there is at least one line in α through P perpendicular to l.

Let m be any line in α through P and perpendicular to l. Then $m \cup l$ contains four right angles. One of these right angles is the union of $\overrightarrow{PA}$ and a ray, say $\overrightarrow{PC}$, such that $\overset{\circ}{\overrightarrow{PC}} \subset \mathcal{H}$.

$$m\angle APC = 90$$
$$0 < cd\ \overrightarrow{PC} < 180$$
$$cd\ \overrightarrow{PC} = 90$$
$$\overrightarrow{PC} = \overrightarrow{PB} \qquad \text{Why?}$$
$$m = \overleftrightarrow{PB} \qquad \text{Why?}$$

Hence there is only one line in α through P and perpendicular to l.

In connection with Theorem 4.14 there is an interesting question to consider. If l is a line and P is a point not on l, is there one and only one line through P that is perpendicular to l? (See Figure 4-36.) Maybe there is no line m such that $m \perp l$. Maybe there is just one such line m. Maybe there are several. We shall prove later that in our formal geometry there is one and only one line m through P and perpendicular to l. The point in which m intersects l is called the foot of this perpendicular from P to l.

Figure 4-36

It is important to emphasize that all lines considered in Theorem 4.14 lie in one plane. If we remove this restriction, it seems reasonable that there are many lines perpendicular to a given line at a given point on it, as suggested in Figure 4-37. This idea comes up again in a later chapter.

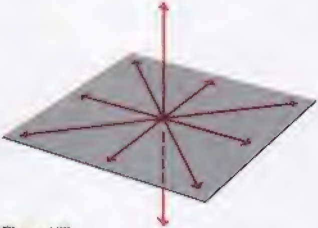


Figure 4-37

EXERCISES 4.7

Let a ray-coordinate system $\mathcal{S}$ in a plane α relative to a point V be such that the ray-coordinates of $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are a, b, c, respectively. In Exercises 1–5, given the values of a, b, c, determine whether or not B is in the interior of $\angle AVC$.

1. $a = 0, b = 90, c = 175$
2. $a = 0, b = 90, c = 185$
3. $a = 0, b = 270, c = 185$
4. $a = 0, b = 270, c = 175$
5. $a = 90, b = 260, c = 0$

In Exercises 6–16, A, B, C are three noncollinear points and D is an interior point of $\overline{BC}$.

6. Is D an element of the interior of $\angle BAC$?
7. Is D an element of the interior of $\angle BCA$?
8. Is D an element of the interior of $\angle CAB$?
9. Is B an element of the interior of $\angle CAD$?
10. Is C an element of the interior of $\angle DAB$?
11. Is $\overleftrightarrow{AB}$ a subset of $\angle ABC$?
12. Is $\overline{AB}$ a subset of $\angle ABC$?
13. Is the interior of $\overline{AB}$ a subset of $\angle ABC$?
14. Is $\overline{BC}$ a subset of $\angle ABC$?
15. Is $\overrightarrow{AD}$ a subset of the union of $\angle BAC$ and its interior?
16. Answer each question under "reason" with a definition, a postulate, or a theorem in the following proof that $m\angle DAC + m\angle DAB = m\angle BAC$.

Statement	Reason
(a) D is an interior point of $\overline{BC}$.	(a) Given
(b) D is in the interior of $\angle BAC$.	(b) Why?
(c) $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.	(c) Why?
(d) $m\angle DAC + m\angle DAB = m\angle BAC$	(d) Why?

In Exercises 17–22, state which of the following descriptions, (a) to (e), correctly identifies the perpendicular sets in the given diagrams.

(a) Two perpendicular rays.
(b) Two perpendicular segments.
(c) A line and a ray perpendicular to each other.
(d) A ray and a segment perpendicular to each other.
(e) A line and a segment perpendicular to each other.

17.


18.


19.


20.


21.


22.


In Exercises 23–27, supply the missing word so that each sentence is true.

23. If the angles in a linear pair of angles are congruent to each other, then each of them is a [?] angle.
24. If the measure of an angle is less than 90, then it is an [?] angle.
25. If the measure of an angle is greater than 90, then it is an [?] angle.
26. If one angle of a linear pair of angles is acute, then the other one is [?].
27. If $\angle ABC$ is a right angle, then $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ are [?] lines.

Let a set of coplanar rays be given as in Figure 4-38 such that $m\angle DVM = 130$, $m\angle MVR = 65$, $m\angle SVM = 30$, $\overrightarrow{VT}$ is the midray of $\angle RVD$, $\overrightarrow{VS}$ is between $\overrightarrow{VM}$ and $\overrightarrow{VR}$, $\overrightarrow{VR}$ is between $\overrightarrow{VS}$ and $\overrightarrow{VD}$. Use this information to find the measure of the given angle in Exercises 28–32.

28. $\angle RVS$
29. $\angle DVR$
30. $\angle DVT$
31. $\angle DVS$
32. $\angle TVM$

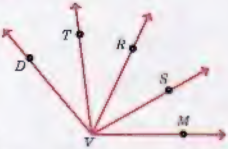


Figure 4-38

In Figure 4-39, $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ are coplanar lines that intersect at V and $m\angle AVC = 32.3$ and $m\angle AVF = 151.7$. Use this information to find the measure of the given angle in Exercises 33–37.

33. $\angle CVF$
34. $\angle FVB$
35. $\angle BVE$
36. $\angle EVA$
37. $\angle EVC$

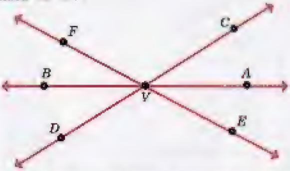


Figure 4-39

In Exercises 38–42, $\angle AVB$ in plane α and a ray-coordinate system in α such that $cd\,\overrightarrow{VA} = 270$ are given. In each exercise, supply the two numbers that are as close together as possible and that make the resulting statement true.

38. If $\angle AVB$ is obtuse and $cd\,\overrightarrow{VB} < 90$, then $cd\,\overrightarrow{VB}$ is a number between [?] and [?].
39. If $\angle AVB$ is acute and $cd\,\overrightarrow{VB} < 270$, then $cd\,\overrightarrow{VB}$ is a number between [?] and [?].
40. If $\angle AVB$ is obtuse and $cd\,\overrightarrow{VB} > 90$, then $cd\,\overrightarrow{VB}$ is a number between [?] and [?].
41. If $\angle AVB$ is acute and $cd\,\overrightarrow{VB} > 270$, then $cd\,\overrightarrow{VB}$ is a number between [?] and [?].
42. If $\angle AVB$ is a right angle, then $cd\,\overrightarrow{VB}$ is [?] or [?].

43. Copy and complete the proof of the following theorem. Draw a figure to help you understand the result.

THEOREM If $\angle AVB$, $\angle BVC$, and $\angle CVA$ are three coplanar angles such that the interiors of no two of them intersect, then the sum of the measures of these three angles is 360.

Proof: In plane AVB there is a unique ray-coordinate system $\mathcal{S}$ relative to V in which $cd\,\overrightarrow{VA} = 0$ and $cd\,\overrightarrow{VB} = m\angle AVB$. From the $\boxed{?}$ Postulate it follows that $m\angle AVB$ is less than $\boxed{?}$, hence that $cd\,\overrightarrow{VB}$ is less than $\boxed{?}$. By $\boxed{?}$ the interiors of $\angle AVB$ and $\angle AVC$ do not intersect. Therefore C and B lie on $\boxed{?}$ (the same side, opposite sides) of $\overleftrightarrow{AV}$. Therefore the ray-coordinate of $\overrightarrow{VC}$, call it c, is $\boxed{?}$ than 180. Since the interiors of $\angle BVC$ and $\angle AVC$ do not intersect, it follows that c is less than $b + \boxed{?}$. Then

$$m\angle AVB = b - 0 = b$$
$$m\angle BVC = \boxed{?}$$
$$m\angle CVA = \boxed{?}$$
$$m\angle AVB + m\angle BVC + m\angle CVA = b + (c - b) + \boxed{?} = 360$$

44. In the proof of Theorem 4.14 there is a statement that $m \cup l$ is the union of four right angles. One of these angles is described in terms of $\overrightarrow{PA}$ and a ray $\overrightarrow{PC}$ such that $\overset{\circ}{\overrightarrow{PC}} \subset \mathcal{H}$. Describe the other three angles in a similar way.

4.8 POLYGONS

As stated before, triangles are three-sided polygons and quadrilaterals are four-sided polygons. A polygon is a plane figure with n vertices and n sides, where n is an integer greater than or equal to 3. The figures shown in Figure 4-40 are polygons, but the figures shown in Figure 4-41 are not.

Figure 4-40

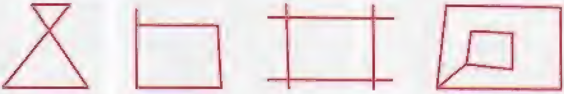


Figure 4-41

Our formal definition is as follows.

Definition 4.14 Let n be any integer greater than or equal to 3. Let $P_1, P_2, \ldots, P_{n-1}, P_n$ be n distinct coplanar points such that the n segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}, \overline{P_nP_1}$ have the following properties:

1. No two of these segments intersect except at their endpoints.
2. No two of these segments with a common endpoint are collinear.

Then the union of these n segments is a **polygon.** Each of the n given points is a **vertex** of the polygon. Each of the n segments is a **side** of the polygon.

If $n = 3$, the definition of a polygon yields a triangle; if $n = 4$, it yields a quadrilateral. Sometimes a polygon with n sides is called an n-gon. For example, a polygon with 13 sides is a 13-gon. The following list gives the names commonly used for the polygons having the number of sides indicated. You should learn these names.

Number of Sides	Name of Polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
10	Decagon
12	Dodecagon

Notation. The polygon whose vertices are $P_1, P_2, \ldots, P_n$ and whose sides are $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}, \overline{P_nP_1}$ is called the polygon $P_1P_2 \ldots P_n$.

Definition 4.15 Two vertices of a polygon that are endpoints of the same side are called **consecutive** vertices. Two sides of a polygon that have a common endpoint are called **consecutive** sides. A **diagonal** of a polygon is a segment whose endpoints are vertices, but not consecutive vertices, of the polygon.

Most of our work with polygons is restricted to polygons like those in Figure 4-42.

Figure 4-42

We usually exclude polygons like those in Figure 4-43 because they are not convex polygons.

Figure 4-43

The polygons that interest us are the *convex polygons*. The following is our formal definition.

> ***Definition 4.16*** A polygon is a convex polygon if and only if each of its sides lies on the edge of a halfplane which contains all of the polygon except that one side.

You must be careful not to confuse the idea of a convex polygon with that of a convex *set* as defined in Section 2.5. A convex polygon is *not* a convex set of points, although the union of a convex polygon and its interior is (see Definition 4.17).

Figure 4-44 illustrates Definition 4.16. It shows the halfplanes associated with the sides as required in the definition.

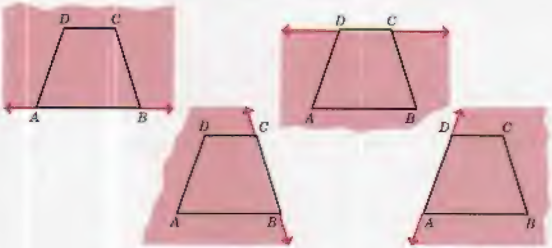


Figure 4-44

Figure 4-45 shows a quadrilateral $ABCD$ that is not convex. Neither the D-side of $\overleftrightarrow{BC}$ nor the A-side of $\overleftrightarrow{BC}$ contains all points of the quadrilateral not on $\overline{BC}$. Draw a figure to convince yourself that every triangle is a convex polygon.

Figure 4-45

Definition 4.17 The interior of a convex polygon is the intersection of all of the halfplanes, each of which has a side of the polygon on its edge and each of which contains all of the polygon except that side.

Figure 4-46 illustrates this definition for a convex quadrilateral $ABCD$.

$$\mathcal{H} = C\text{-side of } \overleftrightarrow{AB}$$
$$\mathcal{S} = D\text{-side of } \overleftrightarrow{BC}$$
$$\mathcal{J} = A\text{-side of } \overleftrightarrow{CD}$$
$$\mathcal{K} = B\text{-side of } \overleftrightarrow{DA}$$

Interior of $ABCD = \mathcal{H} \cap \mathcal{S} \cap \mathcal{J} \cap \mathcal{K}$

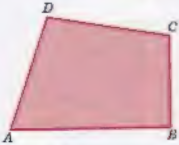


Figure 4-46

Definition 4.18 An angle determined by two consecutive sides of a convex polygon is called an angle of the polygon. Two angles of a polygon are called consecutive angles of the polygon if their vertices are consecutive vertices of the polygon.

For a triangle, any two of its vertices are consecutive vertices, any two of its sides are consecutive sides, and any two of its angles are consecutive angles. For a quadrilateral we use the word *opposite* when consecutive is not applicable as in the following definition.

Definition 4.19 If two sides (or vertices, or angles) of a quadrilateral are not consecutive sides (or vertices, or angles), then they are opposite sides (or vertices, or angles) and each is said to be opposite the other.

Figure 4-47 illustrates this definition. For this example we have

A and C are opposite vertices.
B and D are opposite vertices.
$\overline{AB}$ and $\overline{CD}$ are opposite sides.
$\overline{BC}$ and $\overline{DA}$ are opposite sides.
$\angle A$ and $\angle C$ are opposite angles.
$\angle B$ and $\angle D$ are opposite angles.

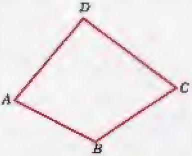


Figure 4-47

4.9 DIHEDRAL ANGLES

Angles are plane figures. Every angle is a subset of a plane. Closely related to the idea of an angle is the idea of a *dihedral angle.* Sometimes we say plane angle when we want to emphasize that an angle is not a dihedral angle. A plane angle is the union of two noncollinear rays having the same endpoint. A dihedral angle is formed by two halfplanes and a line. Here is the formal definition.

> ***Definition 4.20*** If two noncoplanar halfplanes have the same edge, then the union of these halfplanes and the line which is their common edge is a **dihedral angle.** The union of this common edge and either one of these two halfplanes is a **face** of the dihedral angle. The common edge is the **edge** of the dihedral angle.

A dihedral angle is suggested by Figure 4-48. In this diagram, B and C are points on the edge, A is a point in one face but not on the edge, D is a point in the other face but not on the edge. A suitable name for this dihedral angle is A–BC–D or $\angle A$–BC–D. The letters at both ends of this symbol are names of two points, one in each face but not on the edge; the two letters in the middle are names of distinct points on the edge.

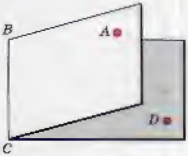


Figure 4-48

The same way two intersecting lines form four angles, two intersecting planes form four dihedral angles as indicated in Figure 4-49.

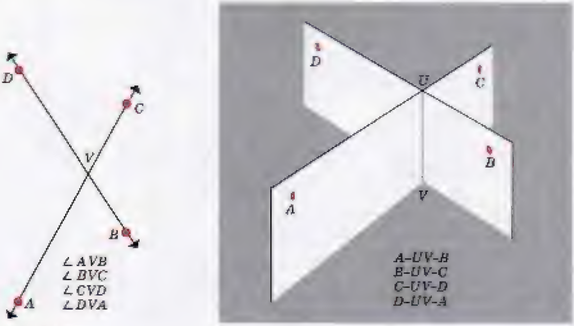


Figure 4-49

Definition 4.21 Two dihedral angles, such as A–UV–B and D–UV–C in Figure 4-49, which have a common edge and whose union is the union of the two intersecting planes are **vertical dihedral angles.**

EXERCISES 4.9

1. Copy and complete the following definition of a pentagon:
   Let P, Q, R, S, T be five distinct coplanar points such that the five segments $\boxed{?}$, $\boxed{?}$, $\boxed{?}$, $\boxed{?}$, $\boxed{?}$ have the following properties.
   (a) $\boxed{?}$
   (b) $\boxed{?}$
   Then the $\boxed{?}$ of these five segments is a pentagon.

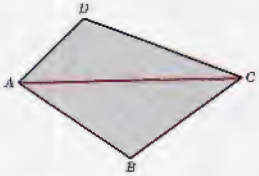
In Exercises 2–7, one of the sides of hexagon $ABCDEF$ is given. In each exercise, name two sides which are consecutive with the given side.

2. $\overline{AB}$
3. $\overline{BC}$
4. $\overline{CD}$
5. $\overline{DE}$
6. $\overline{EF}$
7. $\overline{FA}$

8. Copy and complete the following proof that if $ABCD$ is a convex quadrilateral, then the interior of $\overline{AC}$ is contained in the interior of the quadrilateral.

Proof: Let $\mathcal{I}$ denote the interior of $\overline{AC}$ and let $\mathcal{J}$ denote the interior of the quadrilateral.

$\mathcal{I} \subset$ C-side of $\overleftrightarrow{AB}$
$\mathcal{I} \subset$ C-side of $\overleftrightarrow{AD}$
$\mathcal{I} \subset$ A-side of $\overleftrightarrow{BC}$
$\mathcal{I} \subset$ A-side of $\overleftrightarrow{CD}$



But $\mathcal{J}$ is the intersection of $\boxed{?}$. Therefore $\mathcal{I}$ $\boxed{?}$.

9. (a) Draw a convex pentagon and all of its diagonals.
   (b) How many diagonals does a convex pentagon have?
   (c) Does the interior of each diagonal of a convex pentagon lie in the interior of the pentagon?
10. Draw a pentagon which is not convex. Label it and explain why it is not convex.
11. Draw all of the diagonals of the pentagon you drew in Exercise 10. How many diagonals are there?

■ Figure 4-50 shows a labeled cube. In Exercises 12–16, a quadrilateral whose vertices are vertices of this cube is given. In each exercise, write a suitable name, using four of the letters from the figure, for a dihedral angle containing the given quadrilateral.

12. $ABCD$
13. $ABFE$
14. $BCGF$
15. $CGHD$
16. $EFGH$

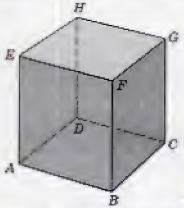


Figure 4-50

■ Exercises 17–21 also refer to Figure 4-50. Base your answers on examining the figure. In some cases the formal geometry that we have developed up to this point is inadequate to prove the correctness of these answers.

17. Is A–BF–H a dihedral angle?
18. Is A–BF–C a dihedral angle?
19. Is A–BF–E a dihedral angle?
20. Is A–BF–D a dihedral angle?
21. Is G–HE–C a dihedral angle?

In Exercises 22 and 23, a dihedral angle $A\text{-}BC\text{-}D$, a line l in plane ABC through A and not intersecting $\overleftrightarrow{BC}$, and a line m in plane DBC through D and not intersecting $\overleftrightarrow{BC}$ are given.

22. Draw a figure to illustrate this situation.
23. Explain why l and m do not intersect.

24. Figure 4-51 shows two planes, α and β, intersecting in line $\overleftrightarrow{AB}$. How many dihedral angles are formed by these two intersecting planes? Name them.
25. Define the interior and exterior of a dihedral angle.
26. Define adjacent dihedral angles.
27. Name a pair of adjacent dihedral angles in Figure 4-51.
28. Name a pair of vertical dihedral angles in Figure 4-51.

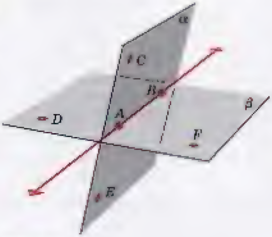


Figure 4-51

CHAPTER SUMMARY

The central theme of this chapter is properties of angles. We introduced three new postulates: The ANGLE MEASURE EXISTENCE POSTULATE, the ANGLE MEASURE ADDITION POSTULATE, and the PROTRACTOR POSTULATE. The definitions include the following.

CONGRUENT ANGLES
MEASURE OF AN ANGLE
BETWEENNESS FOR RAYS
RAY-COORDINATE SYSTEM
VERTICAL ANGLES
LINEAR PAIR OF ANGLES
RIGHT ANGLE
OBTUSE ANGLE
ACUTE ANGLE
COMPLEMENTARY ANGLES
SUPPLEMENTARY ANGLES
MIDRAY OF AN ANGLE
ANGLE BISECTOR
ADJACENT ANGLES
PERPENDICULAR SEGMENTS, RAYS, AND LINES
POLYGON
CONVEX POLYGON
DIHEDRAL ANGLE

This list of terms should remind you of many of the ideas and theorems of this chapter. Ray-coordinates have been used extensively in developing the ideas and in proving the theorems that are included. Theorems 4.2 (ANGLE CONSTRUCTION THEOREM), 4.4 (ANGLE MEASURE ADDITION THEOREM), 4.5, 4.6, 4.7, and 4.8 are important for the work that you will do in Chapter 5. Be sure that you know the statements of these theorems.

REVIEW EXERCISES

In Exercises 1–10, write definitions for the following terms.

1. Congruent angles
2. Vertical angles
3. Adjacent angles
4. Linear pair of angles
5. Complementary angles
6. Supplementary angles
7. Right angle
8. Acute angle
9. Obtuse angle
10. Dihedral angle

In Exercises 11–20, a, b, c are the ray-coordinates of $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, respectively. In each exercise, determine if one of the three rays is between the other two. If one is, name it. If none is, write "none."

11. $a = 0$, $b = 10$, $c = 170$
12. $a = 0$, $b = 10$, $c = 180$
13. $a = 0$, $b = 10$, $c = 190$
14. $a = 0$, $b = 10$, $c = 200$
15. $a = 350$, $b = 50$, $c = 110$
16. $a = 270$, $b = 180$, $c = 135$
17. $a = 270$, $b = 0$, $c = 135$
18. $a = 180$, $b = 90$, $c = 20$
19. $a = 135$, $b = 315$, $c = 0$
20. $a = 0$, $b = 300$, $c = 100$

In Exercises 21–30, a and b are the ray-coordinates of $\overrightarrow{VA}$ and $\overrightarrow{VB}$, respectively. In each exercise, compute the measure of $\angle AVB$.

21. $a = 38$, $b = 106$
22. $a = 300$, $b = 150$
23. $a = 300$, $b = 100$
24. $a = 359$, $b = 1$
25. $a = 270$, $b = 100$
26. $a = 38$, $b = 50$
27. $a = 198$, $\mathrm{b} = 0$
28. $a = 15$, $b = 300$
29. $a = 6$, $b = 40$
30. $a = 315$, $b = 345$

In Exercises 31–40,

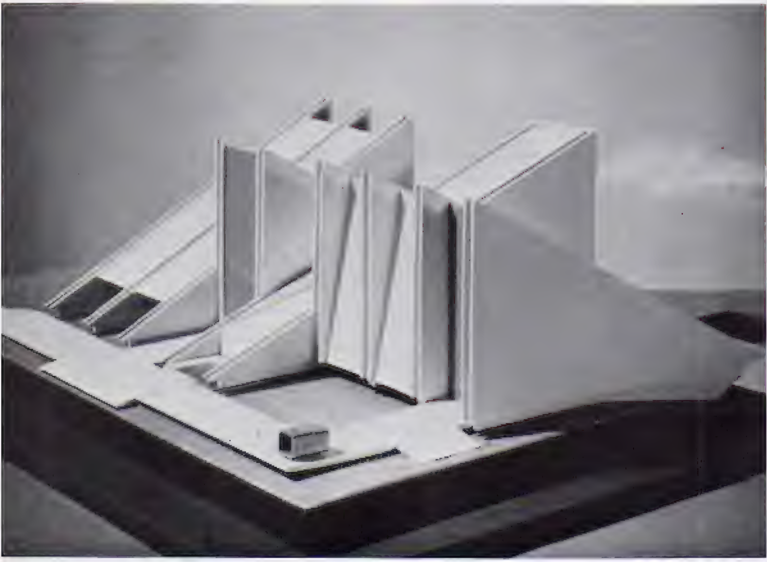
$cd\ \overrightarrow{VA} = 0$	$cd\ \overrightarrow{VJ} = 150$
$cd\ \overrightarrow{VB} = 10$	$cd\ \overrightarrow{VK} = 200$
$cd\ \overrightarrow{VC} = 20$	$cd\ \overrightarrow{VL} = 250$
$cd\ \overrightarrow{VD} = 30$	$cd\ \overrightarrow{VM} = 300$
$cd\ \overrightarrow{VE} = 40$	$cd\ \overrightarrow{VN} = 305$
$cd\ \overrightarrow{VF} = 50$	$cd\ \overrightarrow{VP} = 310$
$cd\ \overrightarrow{VG} = 75$	$cd\ \overrightarrow{VQ} = 315$
$cd\ \overrightarrow{VH} = 90$	$cd\ \overrightarrow{VR} = 325$
$cd\ \overrightarrow{VI} = 100$	$cd\ \overrightarrow{VS} = 330$

In each exercise, determine whether or not the given statement is true.

31. $\angle BVH \cong \angle JVL$
32. $\angle IVH \cong \angle DVE$
33. $\angle SVB \cong \angle BVF$
34. $\angle BVE$ and $\angle DVH$ are supplementary angles.
35. $\angle CVE$ and $\angle SVI$ are supplementary angles.
36. $\angle KVL$ and $\angle AVE$ are complementary angles.
37. $\overrightarrow{VH}$ is the midray of $\angle DVJ$.
38. $\overrightarrow{VN}$ is the midray of $\angle MVP$.
39. $\overrightarrow{VM}$ is the midray of $\angle KVE$.
40. $\angle SVA$ is an acute angle.

41. Explain how betweenness for rays is related to betweenness for ray-coordinates.

42. Explain how betweenness for rays is related to the addition of angle measures.
43. If $\angle A$ and $\angle B$ are complementary angles and if $m\angle A = x$ and $m\angle B = 3x + 10$, find x.
44. If $\angle T$ and $\angle W$ are supplementary angles and if $m\angle T = x$ and $m\angle W = 3x + 10$, find x.
45. Use ray-coordinates to prove that vertical angles are congruent.
46. Use ray-coordinates to prove that the angles of a linear pair are supplementary.
47. If $A\text{-}V\text{-}A'$, $B\text{-}V\text{-}B'$, $cd\ \overrightarrow{VA} = 143$, $cd\ \overrightarrow{VB} < cd\ \overrightarrow{VA}$, and $\overleftrightarrow{AA'} \perp \overleftrightarrow{BB'}$, find $cd\ \overrightarrow{VB}$, $cd\ \overrightarrow{VA'}$, $cd\ \overrightarrow{VB'}$.
48. If $ABCD$ is a convex quadrilateral and E is a point not in the plane of the quadrilateral, which of the following are names of dihedral angles: $A\text{-}BC\text{-}D$, $A\text{-}BC\text{-}E$, $D\text{-}BC\text{-}E$, $E\text{-}AD\text{-}C$?
49. Explain why a convex polygon is not a convex set.
50. Explain why a dihedral angle is not a convex set.



Chapter 5

Courtesy of Quebec Government Tourist Office

Congruence of Triangles

5.1 INTRODUCTION

Suppose you placed a sheet of carbon paper between two sheets of paper and drew a picture of a triangle on the top sheet. Will the carbon copy of the triangle that appears on the second sheet be the same size and shape as the one you have drawn on the top sheet? The idea of two physical objects being carbon copies of one another is what we have in mind when we say that the two objects have the same size and shape. Obviously, we cannot take a physical object such as a house and, using carbon paper, make a carbon copy of it. We can, however, take the same set of blueprints that were used in constructing one house and construct another house that is exactly like it, that is, having the same size and shape as the first one.

In this chapter we are concerned with the "size and shape" of geometrical objects such as segments, angles, and triangles. These geometrical objects are not the physical objects that we draw on our paper. They are the mathematical objects which exist in our minds and whose properties have been described in our postulates, definitions, and theorems. The mathematical concept corresponding to "same size and shape" is *congruence*. We have already defined congruence of segments in Chapter 3 and congruence of angles in Chapter 4. Since all segments have the same shape, we say that two segments are congruent

if they have the same size, that is, if the measures of their lengths are equal. Similarly, two angles have the same size and shape or are congruent if their measures are equal.

5.2 CONGRUENCE OF TRIANGLES

In order to arrive at a definition of congruence for geometrical triangles, let us consider again two physical triangles, one of them a carbon copy of the other. Consider first a triangle that has sides of three different lengths, a scalene triangle. The definition we will obtain applies, however, to all triangles regardless of their shapes. Figure 5-1 shows two scalene triangles, $\triangle ABC$ and its carbon copy, $\triangle A'B'C'$.

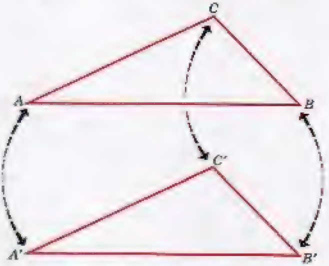


Figure 5-1

If you are to make $\triangle ABC$ "fit" on $\triangle A'B'C'$, you must match up the vertices of the two triangles according to the scheme in Figure 5-1:

$$A \longleftrightarrow A',$$
$$B \longleftrightarrow B',$$
$$C \longleftrightarrow C'.$$

This matching is called a one-to-one correspondence between the two sets of vertices. The correspondence between the vertices of the two triangles also gives a correspondence between the sides of the triangle.

$$\overline{AB} \longleftrightarrow \overline{A'B'},$$
$$\overline{BC} \longleftrightarrow \overline{B'C'},$$
$$\overline{AC} \longleftrightarrow \overline{A'C'}.$$

If a correspondence between the vertices of two triangles is such that the corresponding angles and corresponding sides of the two triangles are congruent, then the correspondence is called a congruence between the two triangles. The correspondence we have been discussing is a congruence between $\triangle ABC$ and $\triangle A'B'C'$. On the other hand,

it is possible to write a correspondence between the vertices of the two triangles that is not a congruence. For example, the correspondence,

$$A \longleftrightarrow B',$$
$$B \longleftrightarrow C',$$
$$C \longleftrightarrow A',$$

is not a congruence between the two triangles shown in Figure 5-1 because, by this matching of vertices, it is not possible to make $\triangle ABC$ coincide with $\triangle A'B'C'$. Write four more correspondences between the vertices of $\triangle ABC$ and $\triangle A'B'C'$ such that no two are the same and such that none is a congruence between the two triangles.

It is convenient to write a correspondence, such as

$$A \longleftrightarrow A',$$
$$B \longleftrightarrow B',$$
$$C \longleftrightarrow C',$$

on one line as $ABC \longleftrightarrow A'B'C'$. When this notation is used, it is understood that the first letter on the left corresponds to the first letter on the right of the double arrow, the second corresponds to the second, and the third corresponds to the third as shown below.

$$ABC \longleftrightarrow A'B'C'$$

Note that there are several ways of writing this same correspondence. For example, both $BAC \longleftrightarrow B'A'C'$ and $CBA \longleftrightarrow C'B'A'$ name the same correspondence as $ABC \longleftrightarrow A'B'C'$.

The foregoing discussion about corresponding vertices of physical triangles can be made to apply equally as well to abstract geometrical triangles. Thus, if $ABC \longleftrightarrow DEF$ is a correspondence between the vertices of any two geometric triangles, this correspondence provides us with three pairs of points. We are interested primarily in these points as vertices of the angles of the triangles and as endpoints of the sides of the triangles. In connection with this correspondence we speak of six pairs of corresponding parts. Three of these six pairs are pairs of angles:

$$\angle A \longleftrightarrow \angle D,$$
$$\angle B \longleftrightarrow \angle E,$$
$$\angle C \longleftrightarrow \angle F.$$

The other three pairs are pairs of sides:

$$\overline{AB} \longleftrightarrow \overline{DE},$$
$$\overline{BC} \longleftrightarrow \overline{EF},$$
$$\overline{AC} \longleftrightarrow \overline{DF}.$$

We are now ready to state the definition of a congruence between two triangles.

> ***Definition 5.1*** Two triangles (not necessarily distinct) are **congruent** if and only if there exists a one-to-one correspondence between their vertices in which the corresponding parts are congruent. Such a one-to-one correspondence between the vertices of two congruent triangles is called a **congruence**.

If $ABC \longleftrightarrow DEF$ is a congruence between $\triangle ABC$ and $\triangle DEF$, then we write $\triangle ABC \cong \triangle DEF$ and note that the following six statements are true:

$$\begin{aligned} \angle A &\cong \angle D, \qquad & \overline{AB} &\cong \overline{DE}, \\ \angle B &\cong \angle E, & \overline{BC} &\cong \overline{EF}, \\ \angle C &\cong \angle F, & \overline{AC} &\cong \overline{DF}. \end{aligned}$$

We can also say that, for $\triangle ABC$ and $\triangle DEF$, if all of these six statements are true, then $ABC \longleftrightarrow DEF$ is a congruence, or what means the same thing, $\triangle ABC \cong \triangle DEF$. In view of the definition of congruent triangles, we sometimes say that "corresponding parts of congruent triangles are congruent."

If $\triangle ABC \cong \triangle DEF$, explain why each of the following six equations is true:

$$\begin{aligned} m\angle A &= m\angle D, \qquad & AB &= DE, \\ m\angle B &= m\angle E, & BC &= EF, \\ m\angle C &= m\angle F, & AC &= DF. \end{aligned}$$

Note that if $\triangle ABC$ and $\triangle DEF$ are distinct triangles and if $ABC \longleftrightarrow DEF$ is a congruence, then it is correct to write $\triangle ABC \cong \triangle DEF$, but that it is incorrect to write $\triangle ABC = \triangle DEF$. The statement "$\triangle ABC \cong \triangle DEF$" is a short way of saying "$ABC \longleftrightarrow DEF$ is a congruence;" it is a statement about a one-to-one correspondence between the vertices of two triangles. The statement "$\triangle ABC = \triangle DEF$" is a statement that two sets are equal; it means that $\triangle ABC$ and $\triangle DEF$ are names for the same triangle.

Note also that

$$\text{if} \qquad \triangle ABC \cong \triangle DEF, \qquad \text{then} \qquad \triangle BAC \cong \triangle EDF.$$

For $\triangle ABC \cong \triangle DEF$ means that $ABC \longleftrightarrow DEF$ is a congruence; $\triangle BAC \cong \triangle EDF$ means that $BAC \longleftrightarrow EDF$ is a congruence; and $ABC \longleftrightarrow DEF$ and $BAC \longleftrightarrow EDF$ are two ways of describing the same one-to-one correspondence between vertices. On the other hand, if $ABC \longleftrightarrow DEF$ is a congruence but $ABC \longleftrightarrow FED$ is *not* a con-

gruence between the two triangles, then it is incorrect to write $\triangle ABC \cong \triangle FED$.

In drawing figures it is convenient to label congruent angles and congruent sides of triangles with the same number of marks as shown in Figure 5-2.

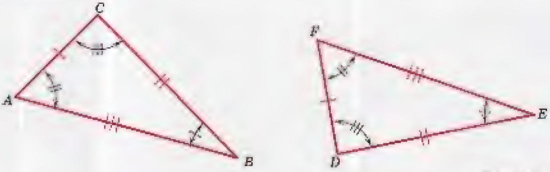


Figure 5-2

Three of the six congruences indicated in the figure are

$$\angle A \cong \angle F, \quad \angle B \cong \angle E, \quad \angle C \cong \angle D.$$

Name the three pairs of congruent sides indicated in the figure. The six congruences tell us $ABC \longleftrightarrow FED$ and $\triangle ABC \cong \triangle FED$.

It is also helpful to label the side that is opposite a given angle in a triangle with the same number of marks used in labeling the angle, as has been done in Figure 5-2. For example, in $\triangle ABC$, $\angle C$ and side $\overline{AB}$ (which is opposite $\angle C$) both have the same number of marks.

EXERCISES 5.2

Exercises 1–6 refer to Figure 5-3.

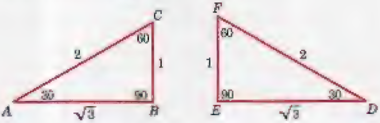


Figure 5-3

1. Is $\triangle ABC \cong \triangle DEF$? Why?
2. Explain why the correspondence $ABC \longleftrightarrow EDF$ is not a congruence.
3. Is it correct to write $\triangle ABC \cong \triangle EDF$?
4. Is it correct to write $\triangle BCA \cong \triangle EFD$? Why?
5. Write four more statements of congruence between the two triangles, each of which follows immediately from the fact that $ABC \longleftrightarrow DEF$ is a congruence.
6. Write three correspondences between the vertices of the two triangles such that each is not a congruence.

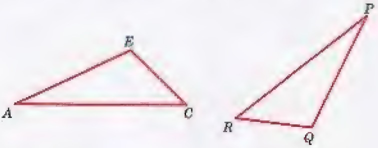
7. The following figure shows two scalene triangles with corresponding congruent sides and angles marked alike.

Copy and complete the following correspondences so that the resulting statements are true.

$LKP \longleftrightarrow \boxed{?}$ is a congruence.
$LPK \longleftrightarrow \boxed{?}$ is a congruence.
$KLP \longleftrightarrow \boxed{?}$ is a congruence.
$KPL \longleftrightarrow \boxed{?}$ is a congruence.
$PKL \longleftrightarrow \boxed{?}$ is a congruence.
$PLK \longleftrightarrow \boxed{?}$ is a congruence.

8. In Exercise 7, why is the correspondence $LKP \longleftrightarrow RST$ not a congruence?

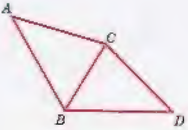
9. In the figure, $\triangle AEC \cong \triangle PQR$. Copy and complete the following statements by supplying the missing symbols.



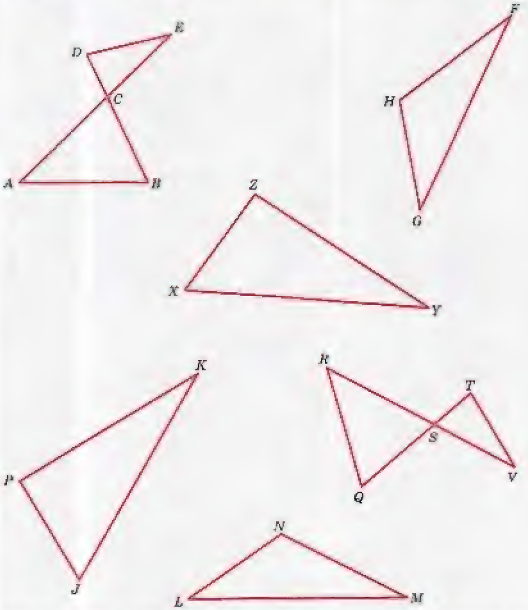
The correspondence $A\boxed{?} \longleftrightarrow \boxed{?}QR$ is a congruence.

$\angle A \cong \angle P$	$\overline{AE} \cong \boxed{?}$
$\angle E \cong \boxed{?}$	$\overline{EC} \cong \boxed{?}$
$\angle C \cong \boxed{?}$	$\overline{CA} \cong \boxed{?}$

10. In the figure, $\triangle ABC \cong \triangle DBC$. List the six pairs of corresponding, congruent parts of these two triangles.



11. In Exercise 10, if $\angle ABC$ and $\angle DBC$ are distinct coplanar acute angles, does ray $\overrightarrow{BC}$ bisect $\angle ABD$? Why?
12. In Exercise 10 (with the figure appropriately modified), if $\angle ABC$ and $\angle DBC$ are coplanar obtuse angles, does ray $\overrightarrow{BC}$ bisect $\angle ABD$? Why?
13. The figure below shows eight triangles. If two triangles look congruent, assume that they really are congruent. Write congruences between such congruent pairs of triangles.



14. Without using a figure, list the six pairs of corresponding, congruent parts for the triangle congruence

$$\triangle EFK \cong \triangle ABT.$$

15. Without using a figure, list the six pairs of corresponding, congruent parts for the triangle congruence

$$\triangle RPS \cong \triangle LSP.$$

16. For $\triangle ABC$ and $\triangle GRS$, it is true that $\overline{AB} \cong \overline{SG}$, $\overline{AC} \cong \overline{SR}$, $\overline{BC} \cong \overline{GR}$, $\angle A \cong \angle S$, $\angle C \cong \angle R$, and $\angle B \cong \angle G$. Write a statement of congruence between these two triangles.
17. If $\triangle DEF$ is a scalene triangle, prove that the statement
$$\triangle DEF \cong \triangle DFE$$
is false.
18. If $\triangle RST \cong \triangle STR$, what special property does $\triangle RST$ have? Draw a suitable figure for $\triangle RST$.
19. If $\triangle LMN \cong \triangle MLN$, what special property does $\triangle LMN$ have? Draw a suitable figure for $\triangle LMN$.

In Exercises 20 and 21, complete the proof of the following theorem.

THEOREM Congruence for triangles is reflexive, symmetric, and transitive.

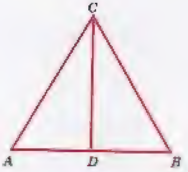
Proof: Let $\triangle ABC$ be any triangle. Then $\angle A \cong \angle A$, $\angle B \cong \angle B$, and $\angle C \cong \angle C$ by the reflexive property of congruence for angles. Also
$$\overline{AB} \cong \overline{AB}, \quad \overline{BC} \cong \overline{BC}, \quad \text{and} \quad \overline{AC} \cong \overline{AC}$$
by the reflexive property of congruence for segments. Therefore $\triangle ABC \cong \triangle ABC$ by the definition of congruent triangles. Therefore congruence for triangles is reflexive.

20. Prove that congruence for triangles is symmetric.
21. Prove that congruence for triangles is transitive.

22. In the figure at the right,

$\overline{CD} \perp \overline{AB}$,
$\overline{AC} \cong \overline{BC}$,
$AD = BD$,
$\angle ACD \cong \angle BCD$,
$m\angle A = 60$,
$m\angle B = 60$.

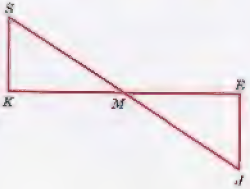
Prove that $\triangle ACD \cong \triangle BCD$.



23. In the figure at the right,

M is the midpoint of $\overline{SJ}$,
M is the midpoint of $\overline{KE}$,
$\overline{KS} \cong \overline{EJ}$,
$m\angle S = m\angle J$,
$\overline{SK} \perp \overline{KE}$,
$\overline{JE} \perp \overline{KE}$.

Prove that $\triangle SKM \cong \triangle JEM$.



5.3 "IF-THEN" STATEMENTS AND THEIR CONVERSES

Many of our definitions, postulates, and theorems have been stated in the "if-then" form. They have been statements of the type "If p, then q," where p and q are statements. (Remember that a statement is a sentence which is either true or false.) In other instances we have used the phrase "if and only if" in the statement of some of our definitions, postulates, and theorems.

As an example of the use of the phrase "if and only if" consider the definition of congruent segments given in Chapter 3: Two segments are congruent if and only if they have the same length. This statement is a conjunction of the following two statements:

1. Two segments are congruent if they have the same length.
2. Two segments are congruent only if they have the same length.

Statements 1 and 2 can be restated in the "if-then" form as follows:

3. If two segments have the same length, then they are congruent.
4. If two segments are congruent, then they have the same length.

We note two important things about these last two statements.

(a) Each is written in the "if-then" form. The if-clause of each statement is the then-clause of the other. (This also means that the then-clause of each is the if-clause of the other.)
(b) Both statements may be used in proofs. For example, if we know $\overline{AB} \cong \overline{CD}$ and want to establish $AB = CD$, we can use statement 4 to justify our writing $AB = CD$. On the other hand, if we know $AB = CD$ and wish to establish $\overline{AB} \cong \overline{CD}$, we can use statement 3 for justification.

The two statements, 1 and 2, or alternatively 3 and 4, are called **converses** of each other. That is, the statement "If p, then q" is called the *converse* of the statement "If q, then p," and "If q, then p" is the *converse* of "If p, then q." The converse of a statement in the "if-then" form can be obtained by interchanging the if- and then-clauses.

If p, then q.

If q, then p.

For example, the converse of the statement:

"If I live in Seattle, then I live in the state of Washington"

is the statement

"If I live in the state of Washington, then I live in Seattle."

It is evident from this example that a statement and its converse need not both be true.

When a definition is given in the "if-then" form, it is understood that the statements of the definition and its converse are both true. As an example, consider Definition 1.1 of Chapter 1:

Space is the set of all points.

This definition can be restated in the "if-then" form as follows:

1. If S is space, then S is the set of all points.

The converse of (1) is

2. If S is the set of all points, then S is space.

Thus the complete definition is: If S is space, then S is the set of all points; and if S is the set of all points, then S is space. The "if and only if" form (which is logically equivalent to the complete definition) is: S is space if and only if S is the set of all points.

Although a definition in "if-then" form always implies the converse statement, this is certainly not true of all postulates and theorems. We discuss the "if-then" form of a theorem more fully in Section 5.4.

As an example of a postulate whose converse is not true, consider Postulate 2 (The Line-Point Postulate).

"Every line is a set of points and contains at least two points."

We can restate this postulate in the "if-then" form as follows:

"If l is a line, then l is a set of points and contains at least two points."

The converse of this statement is

"If l is a set of points and contains at least two points, then l is a line."

Clearly, this last statement is false. There are many sets of points such as planes that contain at least two points and that are not lines.

Consider the following theorem proved in Chapter 4.

"Vertical angles are congruent."

In the "if-then" form, this theorem can be stated as follows:

"If $\angle A$ and $\angle B$ are vertical angles, then $\angle A \cong \angle B$."

Surely the converse of this theorem, stated as follows, is false:

"If $\angle A \cong \angle B$, then $\angle A$ and $\angle B$ are vertical angles."

We conclude this section with some remarks about definitions and truth. A definition in our formal geometry is accepted as a true statement. Why is "Space is the set of all points" a true statement? It is true "by definition." Definitions help us communicate. It is helpful to have one word that means the same thing as "the set of all points." It is helpful to have one word to describe several points that all lie on the same line. Why do collinear points all lie on the same line? They do, by definition.

EXERCISES 5.3

In Exercises 1–5, a definition in "if-then" form is given. Write its converse.

1. If the points of a set are collinear, then there is a line which contains all of them.
2. If there is a plane which contains all the points of a set, then those points are coplanar.
3. If point A is between points B and C, then rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.
4. If an angle is a right angle, then its measure is 90.
5. If $\overrightarrow{BP}$ is the midray of $\angle ABC$, then $\overrightarrow{BP}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ and $\angle ABP \cong \angle PBC$.

In Exercises 6–8, write the definition of the given phrase using the "if and only if" form.

6. Acute angle
7. Linear pair
8. Vertical angles

In Exercises 9–16, is the statement true or false? Write the converse of the statement. Is the converse true or false?

9. If two sets are convex, then their intersection is convex.
10. If two angles are right angles, then they are congruent.
11. If two angles are complements of congruent angles, then they are congruent.
12. If S is the interior of an angle, then S is a convex set.
13. If two angles are congruent, then they are supplements of congruent angles.
14. If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
15. If $\triangle ABC \cong \triangle KLM$, then $\triangle KLM \cong \triangle ABC$.
16. If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle RST$, then $\triangle ABC \cong \triangle RST$.

5.4 THE USE OF CONDITIONAL STATEMENTS IN PROOFS

As you have seen, many of the theorems have the form "If p, then q," where p and q are statements. Not all theorems are stated in this way, however, because it is sometimes easier to state them otherwise, but every theorem can be restated in the "If p, then q" form.

A statement of the form "If p, then q" is called a **conditional**. The if-clause (the p statement) is called the **hypothesis** and the then-clause (the q statement) is called the **conclusion**. In order to understand mathematical proof better, we examine how such statements are used in proofs.

Let us consider the following statements concerning two numbers x and y about which we know nothing except what we are told in the statement (A).

(A) If $x = y$, then $x^2 = y^2$.

We see that (A) involves two statements.

(B) $x = y$ (the hypothesis)
(C) $x^2 = y^2$ (the conclusion)

Even though we do not know the replacements for x and y, can we say anything about the truth of statements (A), (B), and (C)? Do we know that (B) is true? Do we know that (C) is true? What about statement (A)? Your experience in working with numbers should help you to see that even though (B) need not be true and that (C) need not be true, (A) *is* true. Thus a conditional may be true even though its hypothesis and conclusion are not. Replace x and y with several pairs of numbers, some of which are equal and some of which are not. You should find that in those cases where you chose unequal numbers for x and y, both (B) and (C) are false and in those cases where you chose equal numbers for x and y, both (B) and (C) are true. But, *in every case*, (A) is true.

Let us examine one such case where the replacements for x and y are unequal numbers. Suppose we replace x with 2 and y with 1. Then statement (B) becomes $2 = 1$ and statement (C) becomes $2^2 = 1^2$, or $4 = 1$. Of course, both statements (B) and (C) are false. However, if we accept the hypothesis that $2 = 1$, use the multiplication property of equality to write $3 \cdot 2 = 3 \cdot 1$, or $6 = 3$, and use the subtraction property of equality to write $6 - 2 = 3 - 2$, or $4 = 1$, we have shown that the statement "If $2 = 1$, then $2^2 = 1^2$" is true.

Of course, a general statement in the form of a conditional is of no value in a specific situation if the hypothesis of the conditional is false

in that situation. The truth of "If p, then q" does not by itself guarantee the truth of either p or q. But the truth of the conditional and of the hypothesis is a different story, as we shall see.

Let us go back, then, to the three statements (A), (B), and (C) involving x and y. Suppose that (A) and (B) are both true. That is, suppose that the conditional and the hypothesis are both true. Then it follows logically that the conclusion (C) is true. Check this with our example. This is a most important concept in mathematical proofs. It means that we can assert (C) after we have proved or know that both (A) and (B) have been established. On the other hand, it does not mean that (B) follows from (A) and (C). (In our example, $x = y$ does not follow from $x^2 = y^2$, since we could also have $x = -y$.) In general,

> if a conditional and its hypothesis are both known to be true, then the conclusion of the conditional is also true.

More concisely, if we know that the following two statements have been established:

1. if p, then q
2. p

then we may conclude that q has been established. This means, in our example, that if we know the following two statements are both true:

(A) if $x = y$, then $x^2 = y^2$
(B) $x = y$

then we can conclude $x^2 = y^2$.

How are you to know when a conditional is true? In our example, the conditional

$$\text{if} \quad x = y, \quad \text{then} \quad x^2 = y^2$$

is a theorem that can be proved using the properties of numbers. In our geometry, a conditional is accepted as being true if it is a postulate, a previously proved theorem, or part of a definition.

Let us look at two more examples.

Example 1 We know that the conditional

$$\text{if} \quad \overline{AB} \cong \overline{CD}, \quad \text{then} \quad AB = CD$$

is true. Why? Suppose that we also know or have been able to establish that $\overline{AB} \cong \overline{CD}$. What can we conclude?

Example 2 We know that the conditional

if two angles are right angles, then they are congruent

is true. Why? Suppose that you are able to establish that $\angle A$ and $\angle B$ are right angles. What can you conclude? If, in connection with the same conditional, you are able to establish that $\angle A \cong \angle B$, can you then conclude that $\angle A$ and $\angle B$ are right angles? Why?

EXERCISES 5.4

Write the theorems in Exercises 1–6 in "if-then" form. State the hypothesis and the conclusion of each.

1. Supplements of congruent angles are congruent.
2. Right angles are congruent.
3. Vertical angles are congruent.
4. Two angles of a linear pair are supplementary.
5. The intersection of two convex sets of points is a convex set.
6. The interior of a triangle is a convex set.

In Exercises 7–10, a statement p and a statement q are given. In each exercise, write the truth value (that is, true or false) (a) for the statement "If p, then q" and (b) for the statement "If q, then p." The answers to Exercise 7 have been given as a sample.

7. p: $m\angle A = 90$ and $m\angle B = 90$
   q: $\angle A \cong \angle B$
   (a) T (b) F
8. p: $\overrightarrow{VB}$ is the midray of $\angle AVC$.
   q: $\angle AVB \cong \angle BVC$
9. p: A–M–B and $\overline{AM} \cong \overline{MB}$
   q: M is the midpoint of $\overline{AB}$.
10. p: $\angle A$ and $\angle B$ are supplementary angles.
    q: $\angle A$ and $\angle B$ are a linear pair of angles.

In Exercises 11–20, certain given statements are to be accepted as true. Then a conclusion is stated. In each exercise, state whether the conclusion is true, false, or inconclusive (that is, not enough information is given to decide whether the conclusion is true or false).

11. *Given:* $x + y = 16$, $x - y = 12$.
    Conclusion: $x = 15$.
12. *Given:* If there is not a cloud in the sky, then it is not raining. There is not a cloud in the sky.
    Conclusion: It is not raining.

13. *Given:* If there is not a cloud in the sky, then it is not raining. It is not raining.
Conclusion: There is not a cloud in the sky.

14. *Given:* You are a member of the team only if you obey the training rules. Ken is a member of the team.
Conclusion: Ken obeys the training rules.

15. *Given:* You are a member of the team only if you obey the training rules. Bill obeys the training rules.
Conclusion: Bill is a member of the team.

16. *Given:* $2x + 3y = 12, \quad x - y = 4$
Conclusion: $3x + 2y = 16$

17. *Given:* The intersection of two convex sets of points is a convex set. S and T are convex sets of points. $S \cap T = R$.
Conclusion: R is a convex set.

18. *Given:* If a and b are numbers and if $ab = 0$, then $a = 0$ or $b = 0$. a and b are numbers, $ab = 0$, and $a \neq 0$.
Conclusion: $b = 0$

19. *Given:* If $m\angle A = 30$ and $m\angle B = 60$, then $\angle A$ and $\angle B$ are complementary angles. $\angle A$ and $\angle B$ are complementary angles.
Conclusion: $m\angle A = 30$ and $m\angle B = 60$

20. *Given:* Linda will marry Joe only if he will buy her a new house. Joe will buy Linda a new house.
Conclusion: Linda will marry Joe.

5.5 PROOFS IN TWO-COLUMN FORM

Students often ask, "What is a correct proof?" Unfortunately, there is no simple answer to the question. Making correct proofs is something that each of us learns by experience. A proof that may seem convincing to you may not be at all convincing to another person with much less experience in geometry than you. A deductive proof of a theorem is a set of statements, one or several or many, that shows how the conclusion follows logically from the hypothesis. To make a good proof it is important to think clearly about what is given and what is to be proved, and to consider various possibilities of statements which will lead from what is given to what is to be proved. It will help you considerably if you have a firm understanding of the postulates, definitions, and theorems already stated or proved.

The proof of a theorem is often given in paragraph form. We have used this form of proof for most of the theorems proved thus far. For illustrative purposes look at the proof for Theorem 4.13 in Chapter 4.

THEOREM 4.13 Any two right angles are congruent.

Proof: Every right angle has a measure of 90. Hence all right angles have the same measure, and hence they are congruent to each other.

This proof consists of two sentences. Since the second sentence consists of two parts, there are really three steps in the proof. These three steps, or links, form a chain of reasoning that shows how the conclusion,

they are congruent,

follows logically from the hypothesis,

two angles are right angles.

In writing this proof we did not provide reasons to support these three steps because we felt that the proof as given could be understood by someone with your background in geometry. If we were trying to convince someone with less practice, it would be necessary to give additional statements for justifying each of the statements in the proof.

One of the advantages of the paragraph type of proof is that it is not always necessary to give the reasons for all the statements when those reasons are obvious. However, when you give a proof in this form, you should be prepared to fill in the reasons.

A two-column proof of a theorem consists of a chain of statements written in one column with a supporting reason for each statement written in a second column. When a statement in this chain is established because it is part of the hypothesis of the theorem, we simply write "hypothesis" or "given" as the reason. Otherwise, a statement may have as its supporting reason a combination of a conditional and its hypothesis. As stated before, a conditional is acceptable if it is a postulate, a part of a definition, or a previously proved theorem. The hypothesis of this conditional should have appeared as an earlier statement in the proof. The conclusion of the conditional should apply to the statement that is being supported.

A two-column proof of Theorem 4.13 follows. Note that in reasons which are conditionals, we write the numbers of the statements in which we have established the hypothesis of the conditional.

THEOREM 4.13 Any two right angles are congruent.

RESTATEMENT: If $\angle A$ and $\angle B$ are any two right angles, then $\angle A \cong \angle B$.

Proof:

Hypothesis: $\angle A$ and $\angle B$ are right angles.

Conclusion: $\angle A \cong \angle B$

Statement	Reason
1. $\angle A$ and $\angle B$ are right angles.	1. Hypothesis
2. $m\angle A = 90$, $m\angle B = 90$	2. If an angle is a right angle (1), then its measure is 90.
3. $m\angle A = m\angle B$	3. Substitution property of equality (2)
4. $\angle A \cong \angle B$	4. If two angles have equal measures (3), then they are congruent.

We note several important points about this proof.

1. The proof is not complete until the last statement in the left-hand column is the same as the conclusion.
2. When a statement is part of the hypothesis, we write "hypothesis" or "given" as its reason.
3. When a reason is in the "if-then" form, its hypothesis refers to an earlier statement or statements for support. For example, the if-clause of reason 2 refers to statement 1. However, the then-clause of reason 2 refers to statement 2.
4. When a reason is not in the "if-then" form, and it can be written in that form, then it must satisfy the requirements stated in (3). For example, reason 3 simply states: "Substitution property of equality (2)." We could also have stated reason 3 as follows: "If a, b, and c are numbers and if $a = c$ and $b = c$ (2), then $a = b$."
5. In proving the theorem we have not proved statement 4 considered by itself as an isolated statement. Rather we have proved the following conditional: *If statement 1, then statement 4.*

Your teacher may permit you to list your reasons simply by identifying the postulate, definition, theorem, or property of equality which supports each statement. If this is the case, the proof of Theorem 4.13 might read as follows:

Statement	Reason
1. $\angle A$ and $\angle B$ are right angles.	1. Given
2. $m\angle A = 90$, $m\angle B = 90$	2. Statement (1) and the definition of right angle
3. $m\angle A = m\angle B$	3. Statement (2) and the substitution property of equality
4. $\angle A \cong \angle B$	4. Statement (3) and the definition of congruent angles

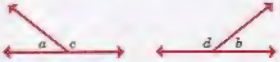
We have shown three examples of proofs of the same theorem. Which is the best proof? The answer is that they are all good. The two-column proof reminds us that we must be able to give a reason for every statement we make, and it also makes it easier to see which hypothesis we accept to begin our proof. Which proof you choose may depend on whom you are trying to convince. Usually in writing a proof your objective will be to convince your teacher that you understand the proof! Your teacher may want you to have experience in writing both the paragraph type of proof and the two-column type.

EXERCISES 5.5

1. In the following two-column proof several reasons are given in the "if-then" form. For each such reason indicate to which statement the if-clause refers and to which statement the then-clause refers.

THEOREM Supplements of congruent angles are congruent.

RESTATEMENT: If $\angle c$ is a supplement of $\angle a$ and if $\angle d$ is a supplement of $\angle b$ and if $\angle a \cong \angle b$, then $\angle c \cong \angle d$.



Proof:

Hypothesis: $\angle c$ is a supplement of $\angle a$.
$\angle d$ is a supplement of $\angle b$.
$\angle a \cong \angle b$

Conclusion: $\angle c \cong \angle d$

Statement	Reason
1. $\angle c$ is a supplement of $\angle a$. $\angle d$ is a supplement of $\angle b$.	1. Hypothesis
2. $m\angle c + m\angle a = 180$ $m\angle d + m\angle b = 180$	2. If two angles are supplementary, then the sum of their measures is 180.
3. $\angle a \cong \angle b$	3. Hypothesis
4. $m\angle a = m\angle b$	4. If two angles are congruent, then they have the same measure.
5. $m\angle c + m\angle a = m\angle d + m\angle b$	5. If x, y, and z are numbers and if $x = y$ and $z = y$, then $x = z$.
6. $m\angle c = m\angle d$	6. If a, b, x, y are numbers and if $a = b$ and $x = y$, then $x - a = y - b$.
7. $\angle c \cong \angle d$	7. If two angles have the same measure, then they are congruent.

2. Identify each of reasons 2, 4, 5, 6, and 7 in Exercise 1 as a postulate, definition, theorem, or property of equality.
3. Write a two-column proof for these theorems from Chapter 4.
   (a) Complements of congruent angles are congruent.
   (b) Vertical angles are congruent.
4. Write a proof for the following theorem in (a) paragraph form and (b) two-column form.

   THEOREM If two angles are both congruent and supplementary, then each is a right angle.

5. Write a two-column proof of the theorem that vertical angles are congruent using (a) the definition of vertical angles, (b) the definition of a linear pair, (c) the theorem that if two angles form a linear pair, then they are supplementary, and (d) the theorem that supplements of congruent angles are congruent. (*Hint:* In the figure, let $\angle a$ and $\angle c$ be a pair of vertical angles. Prove $\angle a \cong \angle c$.)

In Exercises 6–11, you are to perform some experiments with physical triangles. You will need a ruler, a protractor, and a compass. You are to use these experiments as a basis for formulating the next three postulates in Section 5.6. Do the required constructions and measurements carefully and answer all the questions before proceeding to the next section.

6. Construct a triangle, $\triangle ABC$, in which $AB = 2\frac{1}{2}$ in., $AC = 1\frac{3}{4}$ in., and $m\angle A = 50$. Measure the remaining three parts ($\overline{BC}$, $\angle B$, $\angle C$) of your constructed triangle and compare your measurements with those of two or three of your classmates. Are they nearly the same?
7. Construct a triangle, $\triangle RST$, in which $RS = 2$ in., $m\angle R = 40$, and $m\angle S = 60$. Measure the remaining three parts ($\angle T$, $\overline{RT}$, $\overline{ST}$) of your constructed triangle and compare your measurements with those of some of your classmates. Are they nearly the same?
8. Construct a triangle, $\triangle PQR$, in which $PQ = 5$ cm., $PR = 4.5$ cm., and $RQ = 6$ cm. (You may need to use a compass for this construction.) Measure the three angles of your constructed triangle and compare your measurements with those of some of your classmates. Are they nearly the same?
9. Construct a triangle, $\triangle DEF$, in which $m\angle D = 50$, $m\angle E = 60$, and $m\angle F = 70$. Measure the three sides of your constructed triangle in centimeters. Compare your measurements with those of some of your classmates. Are they nearly the same?

10. Construct a triangle, $\triangle LMN$, in which $m\angle L = 40$, $LM = 5$ cm., and $MN = 3.5$ cm. (You may need to use your compass again for this construction.) Measure $\angle M$, $\angle N$, and $\overline{LN}$ of your constructed triangle and compare your measurements with those of some of your classmates. Are they nearly the same?
11. Using only your ruler, construct a triangle, $\triangle ADE$, which has no two of its sides congruent. Now construct $\triangle A'D'E'$ (distinct from $\triangle ADE$) such that $\triangle ADE \cong A'D'E'$. (You need not restrict yourself to using only your ruler for this construction.) How many of the six parts of $\triangle ADE$ did you use in obtaining $\triangle A'D'E'$? Could you have obtained $\triangle A'D'E'$ by using a set of parts different from those that you actually used? What is the least number of congruent parts of $\triangle ADE$ needed to be sure that $\triangle A'D'E'$ is congruent to $\triangle ADE$?

5.6 THE CONGRUENCE POSTULATES FOR TRIANGLES

We asked you to perform certain constructions in Exercises 5.5. We now wish to examine these constructions in more detail, but first we need some definitions. In $\triangle ABC$ (Figure 5-4) we say that $\angle A$ *is included* by sides $\overline{AC}$ and $\overline{AB}$. Similarly, we say that side $\overline{BC}$ is *included* by angles $\angle B$ and $\angle C$.

Figure 5-4

> ***Definition 5.2*** An angle of a triangle is said to be included by two sides of that triangle if the angle contains those sides. A side of a triangle is said to be included by two angles of that triangle if the endpoints of the side are the vertices of those angles.

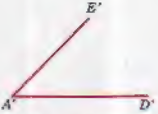
In $\triangle ABC$, shown in Figure 5-4, which angle is included by sides $\overline{BC}$ and $\overline{BA}$? Which side is included by $\angle A$ and $\angle C$? Were you able to answer these last two questions without looking at the picture of the triangle? Without looking at a picure of $\triangle RST$, state which angle is included by sides $\overline{ST}$ and $\overline{RT}$. Which side is included by $\angle R$ and $\angle S$?

In Exercise 6 of Exercises 5.5, you should have concluded that all triangles having the given parts are congruent; similarly for Exercises 7 and 8. When this is true, we say that the three given parts *determine* a triangle. In Exercise 6, the three given parts of the triangle were two sides and the included angle. In Exercise 7, the three given parts were two angles and the included side, and in Exercise 8 the three given parts were the three sides. In Exercises 9 and 10, however, you should have found that not all triangles having the three given parts are con-

gruent. How many triangles of different size can be constructed if only the measures of the three angles are given (assuming there is at least one triangle with angles having these measures)? How many triangles of different sizes can be constructed using the data of Exercise 10?

How did you construct $\triangle A'D'E'$ in Exercise 11? Make a list of the steps you used. Perhaps you have one of the following lists.

List 1	List 2	List 3
1. Draw $\overline{A'D'} \cong \overline{AD}$.	1. Draw $\angle A' \cong \angle A$.	1. Draw $\overline{A'D'} \cong \overline{AD}$.
2. Draw $\angle A' \cong \angle A$.	2. Draw $\overline{A'D'} \cong \overline{AD}$.	2. Draw an arc with A' as center and $A'E'$ as radius.
3. Draw $\overline{A'E'} \cong \overline{AE}$.	3. Draw $\angle D' \cong \angle D$.	3. Draw an arc with D' as center and $D'E'$ as radius.
4. Complete the construction by connecting E' and D' with $\overline{D'E'}$.	4. Complete the construction by drawing the sides of $\angle A'$ and $\angle D'$ long enough.	4. Complete the construction by connecting the intersection of the arcs to A' and D'.



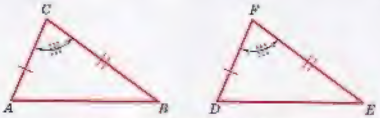
For each list, the figure at the bottom shows what the construction looks like just before it is completed.

Look at the first list. How many side measures are used? How many angle measures? Is the angle between the two sides? This combination of *two sides and the included angle* is abbreviated by the symbol S.A.S.; the correspondence $ADE \longleftrightarrow A'D'E'$ is referred to as an **S.A.S. congruence** because we feel that this much information about congruent pairs is enough to guarantee that all matched pairs of parts are congruent. We make this conclusion formal in Postulate 23.

Look at the second list. What combination of measures is used? This combination of *two angles and the included side* is abbreviated A.S.A.; the correspondence $ADE \longleftrightarrow A'D'E'$ is referred to as an **A.S.A. congruence.** We make this conclusion formal in Postulate 24.

Look at the third list. This combination of *three sides* is abbreviated S.S.S., the correspondence $ADE \longleftrightarrow A'D'E'$ is referred to as an **S.S.S. congruence.** We make this conclusion formal in Postulate 25.

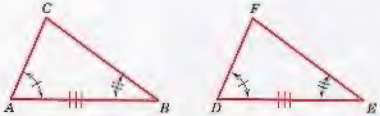
POSTULATE 23 (*The S.A.S. Postulate*) Let a one-to-one correspondence between the vertices of two triangles (not necessarily distinct) be given. If *two sides and the included angle* of the first triangle are congruent, respectively, to the corresponding parts of the second triangle, then the correspondence is a congruence. (See Figure 5-5.)



$\triangle ABC \cong \triangle DEF$ by the S.A.S. Postulate

Figure 5-5

POSTULATE 24 (*The A.S.A. Postulate*) Let a one-to-one correspondence between the vertices of two triangles (not necessarily distinct) be given. If *two angles and the included side* of the first triangle are congruent, respectively, to the corresponding parts of the second triangle, then the correspondence is a congruence. (See Figure 5-6.)



$\triangle ABC \cong \triangle DEF$ by the A.S.A. Postulate

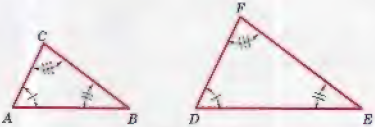
Figure 5-6

POSTULATE 25 (*The S.S.S. Postulate*) Let a one-to-one correspondence between the vertices of two triangles (not necessarily distinct) be given. If the *three sides* of the first triangle are congruent, respectively, to the corresponding sides of the second triangle, then the correspondence is a congruence. (See Figure 5-7.)

$\triangle ABC \cong \triangle DEF$ by the S.S.S. Postulate

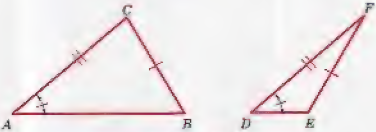
Figure 5-7

Note that there is no A.A.A. Postulate (Figure 5-8) and no S.S.A. Postulate (Figure 5-9). Also note how the results of Exercises 9 and 10 of Exercises 5.5 are related to this statement.



$\triangle ABC \not\cong \triangle DEF$

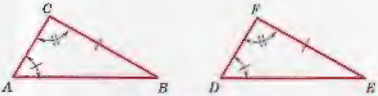
Figure 5-8



$\triangle ABC \not\cong \triangle DEF$

Figure 5-9

Our experience with physical triangles suggests that it would be proper to include an S.A.A. Postulate (Figure 5-10). Actually, we do not need such a postulate. The statement that you might expect as a postulate is, in fact, given as a theorem in a later chapter. We defer the proof because we need not only the congruence postulates but also a postulate about parallel lines, which appears later, before we can prove the S.A.A. statement. Since it is easy to prove later, we have decided not to adopt it formally as a postulate.



$\triangle ABC \cong \triangle DEF$ by S.A.A.

Figure 5-10

The A.S.A. and S.S.S. Postulates can be proved as theorems once the S.A.S. Postulate is assumed. The proofs are difficult, however, and so we have adopted these statements as postulates in order to make simpler the development of our geometry.

EXERCISES 5.6

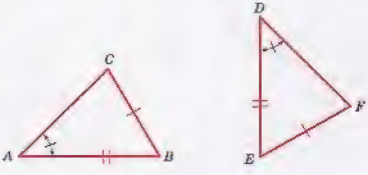
In Exercises 1–16, like markings on the triangles indicate congruent parts. In each exercise, determine if a pair of triangles can be proved congruent. If a congruence can be proved, write a triangle congruence (in the form $\triangle BAC \cong \triangle FDE$) and the postulate (S.A.S., A.S.A., S.S.S.) you would use to prove it. Exercise 1 has been worked as a sample.

1.

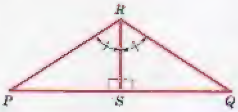
$\triangle BAC \cong \triangle FDE$
by S.A.S.

2.

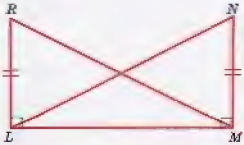
3.



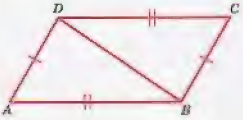
4.



6.

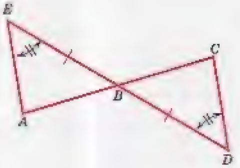


5.

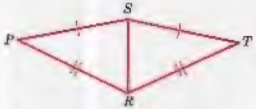


7.

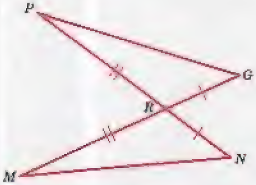
8.



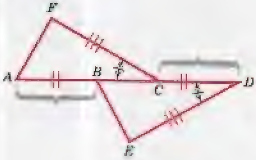
9.



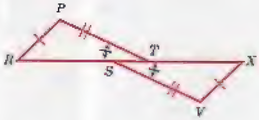
10.



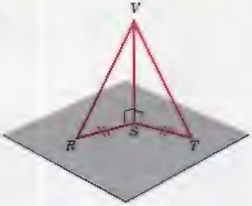
11.



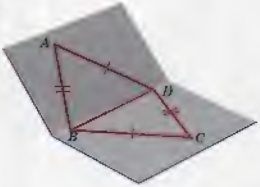
12.



13.

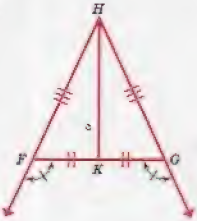


14.



15.

16.



17. For $\triangle ABC$ and $\triangle DEF$,

$$\overline{AB} \cong \overline{EF},$$
$$\angle A \cong \angle E,$$
$$\overline{AC} \cong \overline{ED}.$$

Write a congruence between the two triangles. What postulate are you using?

18. For $\triangle RSU$ and $\triangle GKL$,

$$\angle R \cong \angle L \qquad \text{and} \qquad \angle U \cong \angle K.$$

Which two sides must be proved congruent if

$$\triangle RUS \cong \triangle LKG$$

by the A.S.A. Postulate?

19. For $\triangle PQR$ and $\triangle QSR$,

$$\overline{PR} \cong \overline{SR} \qquad \text{and} \qquad \overline{PQ} \cong \overline{SQ}.$$

Are the triangles necessarily congruent? Why? Draw a figure.

20. For $\triangle ABC$ and $\triangle BCD$,

$$\angle ABC \cong \angle DBC \qquad \text{and} \qquad \angle ACB \cong \angle DCB.$$

Are the triangles necessarily congruent? Why? Draw a figure.

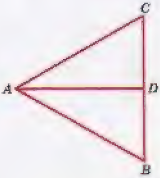
In Exercises 21–30, two triangles appear to be congruent in the given figure. In each exercise, certain information is given about the figure. Assume that all points are coplanar and have the relative positions shown.

(a) Copy and mark each figure, as was done in Exercises 1–16, to show the given information.

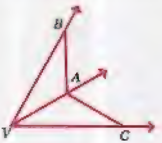
(b) If the given information is sufficient to prove the triangles congruent, state a congruence between the triangles and the postulate you would use (S.A.S., A.S.A., S.S.S.) to prove it. If the given information is insufficient, write "Insufficient."

21. M is the midpoint of $\overline{PQ}$.
$\overline{PR} \cong \overline{QR}$

22. $\overline{AD} \perp \overline{BC}$ at D.
D is the midpoint of $\overline{BC}$.

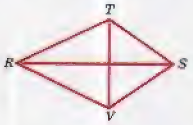


23. $\overrightarrow{VA}$ is the midray of $\angle BVC$.
$m\angle VAC = m\angle VAB$

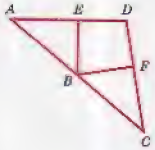


24. Consider only $\triangle PQR$ and $\triangle SRQ$.
$\overline{PQ} \perp \overline{QR}$, $\overline{SR} \perp \overline{QR}$, and
$\overline{PR} \cong \overline{QS}$

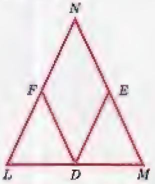
25. Consider only $\triangle RTS$ and $\triangle RVS$.
$\overline{RT} \cong \overline{RV}$
$\overline{ST} \cong \overline{SV}$
$m\angle STV = m\angle SVT$
$m\angle RTV = m\angle RVT$



26. A–E–D, D–F–C
B is the midpoint of $\overline{AC}$.
$\overline{BE} \perp \overline{AD}$
$\overline{BF} \perp \overline{CD}$
$AE = CF$

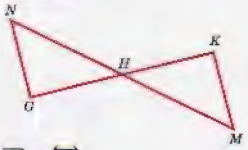


27. L–F–N, N–E–M
D is the midpoint of $\overline{LM}$.
$LF = ME$
$DF = DE$

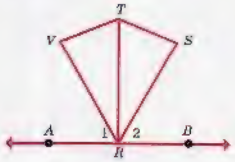


28. In Exercise 27, if, in addition to the information given for the figure, we also have $LN = MN$, is $\triangle FND \cong \triangle END$? Why? (*Hint:* Use $\overline{ND}$ in your proof. This segment exists even though it is not shown in the figure.)

29. N–H–M
H is the midpoint of $\overline{GK}$.
$\overline{NG} \perp \overline{GK}$
$\overline{MK} \perp \overline{GK}$



30. $\overline{TR} \perp \overleftrightarrow{AB}$
$\angle 1 \cong \angle 2$
$\overline{VR} \cong \overline{SR}$
A–R–B



5.7 USING THE S.A.S., A.S.A., AND S.S.S. POSTULATES IN WRITING PROOFS

In this section and the following one, you will be asked to write your own proofs. In writing these proofs you will usually need to prove one or more pairs of triangles congruent by using the S.A.S., A.S.A., and S.S.S. Postulates. Therefore, in planning your proof, you should look for the opportunity to apply one of these postulates to some pair of triangles. We illustrate with some examples.

Example 1 If M is the midpoint of $\overline{AB}$ and $\overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

In starting to construct a proof for this statement, we first draw a figure which seems to fit the hypothesis.

Figure 5-11

Figure 5-11 shows M to be the midpoint of segments $\overline{AB}$ and $\overline{CD}$. (Can you draw a different figure which shows the same information?) We have marked $\overline{AM}$ and $\overline{MB}$ with the same number of marks since, by the definition of a midpoint, we know that $\overline{AM} \cong \overline{MB}$. $\overline{CM}$ and $\overline{MD}$ have been marked alike for the same reason.

Before attempting to construct a proof of a theorem, it is helpful to have a definite plan in mind. In this example, we want to prove that two segments are congruent. We know that two segments are congruent if they are corresponding sides of congruent triangles. Our plan is to prove $\triangle AMC \cong \triangle BMD$ in Figure 5-11. For completeness the figure is included as a part of the proof, as it should be.

Proof:

Hypothesis: M is the midpoint of $\overline{AB}$ and $\overline{CD}$.
Conclusion: $\overline{AC} \cong \overline{BD}$
(*Plan:* Prove $\triangle AMC \cong \triangle BMD$.)

Statement	Reason
1. M is the midpoint of $\overline{AB}$ and $\overline{CD}$.	1. Hypothesis
2. $\overline{AM} \cong \overline{MB}$. $\overline{CM} \cong \overline{MD}$	2. Definition of midpoint (1)
3. $\angle AMC \cong \angle BMD$	3. Vertical angles are congruent.
4. $\triangle AMC \cong \triangle BMD$	4. S.A.S. Postulate (2, 3)
5. $\overline{AC} \cong \overline{BD}$	5. If two triangles are congruent (4), then their corresponding parts are congruent.

Note that there are essentially the following five steps to writing a geometric proof in two-column form.

1. Draw a figure which seems to fit the hypothesis and, where possible, mark on the figure the information given in the hypothesis.
2. State what is given (the hypothesis) expressed in terms of the figure.
3. State what is to be proved (the conclusion) also expressed in terms of the figure.
4. State a plan for the proof. The plan need not be expressed in written form, but should be carefully thought through before attempting to write the proof.
5. Write out the proof in two-column form.

Now let us examine the proof for Example 1. You probably noticed in the statement of Step 3 that we relied strongly on the figure for giving us the information that $\angle AMC$ and $\angle BMD$ are a pair of vertical angles. If our figures are carefully drawn, we can rely on them to give us correct information. However, we must always be prepared to be able to justify any information that we take from a figure by postulates, definitions, and theorems. In the figure for the example, we know that points A, M, B are collinear and that points C, M, D are collinear (Why?). We also know that when two lines intersect, vertical angles are formed. Since the figure clearly shows all of this information, we did not bother to establish in our proof that $\angle AMC$ and $\angle BMD$ are vertical angles.

Since this may be your first experience in writing geometric proofs in two-column form, your teacher may want you to write a more complete proof than the one given. In other words, your teacher may not want you to assume any information from a figure without establishing this information in your proof. If this is the case, we give a second, more complete proof of the theorem in Example 1 for your consideration.

Hypothesis: M is the midpoint of $\overline{AB}$ and $\overline{CD}$.
Conclusion: $\overline{AC} \cong \overline{BD}$

Statement	Reason
1. M is the midpoint of $\overline{AB}$ and $\overline{CD}$.	1. Hypothesis
2. M is between A and B, and M is between C and D.	2. Definition of midpoint (1)
3. $\overrightarrow{MA}$, $\overrightarrow{MB}$ and $\overrightarrow{MC}$, $\overrightarrow{MD}$ are two pairs of opposite rays.	3. Definition of opposite rays (2)
4. $\angle AMC$ and $\angle BMD$ are vertical angles.	4. Definition of vertical angles (3)
5. $\angle AMC \cong \angle BMD$	5. If two angles are vertical (4), then they are congruent.
6. $\overline{AM} \cong \overline{MB}$. $\overline{CM} \cong \overline{MD}$	6. Definition of midpoint (1)
7. $\triangle AMC \cong \triangle BMD$	7. S.A.S. Postulate (5, 6)
8. $\overline{AC} \cong \overline{BD}$	8. If two triangles are congruent (7), then their corresponding parts are congruent.

Note that some of the reasons are written in abbreviated form. For example, reason 4 is "Definition of vertical angles" rather than the complete statement of this definition. Be quite certain that you know the complete statement before using the abbreviated form in your proofs.

Example 2 If, in quadrilateral $ABCD$, $\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{CD}$, then $\angle A \cong \angle C$.

Hypothesis: $\overline{AD} \cong \overline{BC}$
$\overline{AB} \cong \overline{CD}$

Conclusion: $\angle A \cong \angle C$

(*Plan:* We want to prove $\angle A \cong \angle C$ by showing they are corresponding angles of congruent triangles. But there are no triangles in our figure. We therefore draw $\overline{DB}$ to show which triangles we shall use.)

Note that the segment $\overline{DB}$ in Figure 5-12 is dashed to distinguish it from the parts of the figure given in the hypothesis.

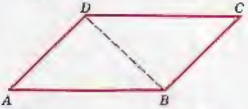


Figure 5-12

We call a segment such as $\overline{DB}$ an **auxiliary segment.** Thus an auxiliary segment is a segment that is not a part of the figure given in the hypothesis, but does exist by the definition of a segment and the Point-Line Postulate. Such segments should be drawn into your figure when it is convenient to use them in the proof. We now continue with our plan.

Since two sides of $\triangle ABD$ are congruent to the corresponding sides of $\triangle CDB$, we would expect to prove these triangles congruent by either the S.A.S. Postulate or the S.S.S. Postulate. If we were to use S.A.S., we would need $\angle A \cong \angle C$. But since this is what we are trying to prove, we cannot use it as part of our proof. That leaves the S.S.S. Postulate. In $\triangle ABD$, the third side is $\overline{BD}$ and in $\triangle CDB$, the third side is $\overline{DB}$. But $\overline{BD}$ and $\overline{DB}$ are the same segment and are congruent by the reflexive property of congruence. We can now write the proof.

Statement	Reason
1. $\overline{AB} \cong \overline{CD}$	1. Hypothesis
2. $\overline{AD} \cong \overline{CB}$	2. Hypothesis
3. Segment $\overline{BD}$ exists.	3. Point-Line Postulate and the definition of segment
4. $\overline{BD} \cong \overline{DB}$	4. The reflexive property of congruence for segments.
5. $\triangle ABD \cong \triangle CDB$	5. S.S.S. Postulate (1, 2, 4)
6. $\angle A \cong \angle C$	6. If two triangles are congruent (5), then their corresponding parts are congruent.

Note that when we wrote $\overline{AD} \cong \overline{CB}$ in statement 2 and $\overline{BD} \cong \overline{DB}$ in statement 4, we were adhering to the correspondence $ABD \longleftrightarrow CDB$ for the two triangles. Of course, it would be correct to write $\overline{AD} \cong \overline{BC}$ in statement 2 and $\overline{BD} \cong \overline{BD}$ in statement 4. But it helps to keep things clear for both the writer of the proof and someone reading it if it is written the way we wrote it, keeping the order of the letters consistent with their order in the congruence we wish to prove.

Example 3 If points are as shown in Figure 5-13 and if $\overline{AB} \cong \overline{CB}$, $\overline{FE} \cong \overline{DF}$, and $\overline{BE} \cong \overline{BD}$, prove $\overline{AF} \cong \overline{FC}$.

How do we formulate a plan for this proof? A plan for a proof can often be formed by "working backwards." That is, we begin with the conclusion and try to work our way back to the hypothesis. *If* these steps can then be reversed, we have a plan for our proof. This method consists of writing (or thinking of) information in two columns:

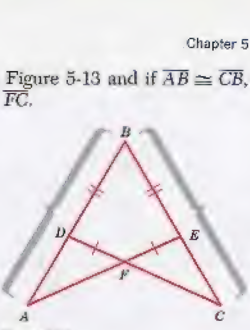


Figure 5-13

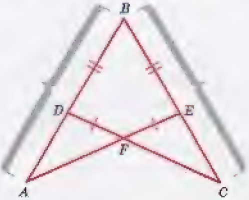
I Can Prove	If I Can Prove
1. $\overline{AF} \cong \overline{FC}$	1. (a) $\overline{AE} \cong \overline{CD}$?
	(b) $\overline{FE} \cong \overline{DF}\surd$
2. $\overline{AE} \cong \overline{CD}$	2. $\triangle ABE \cong \triangle CBD$?
3. $\triangle ABE \cong \triangle CBD$	3. (a) $\overline{AB} \cong \overline{CB}$ (S)$\surd$
	(b) $\angle B \cong \angle B$ (A)$\surd$
	(c) $\overline{BE} \cong \overline{BD}$ (S)$\surd$

We read line 1 as follows: I can prove $\overline{AF} \cong \overline{FC}$ if I can prove (a) $\overline{AE} \cong \overline{CD}$ and (b) $\overline{FE} \cong \overline{DF}$. (Note that if we know statements (a) and (b) of line 1 in the second column, then we can deduce statement 1 in the first column from Corollary 3.4.3.) Because (b) is given in the hypothesis, it is checked. Statement (a) is then to be considered. It is brought down to line 2. We read line 2 as follows: "I can prove $\overline{AE} \cong \overline{CD}$ if I can prove $\triangle ABC \cong \triangle CBD$." Since this triangle congruence is not given, we bring it down to line 3. Line 3 is read as follows: "I can prove $\triangle ABE \cong \triangle CBD$ if I can prove (a) $\overline{AB} \cong \overline{CB}$, (b) $\angle B \cong \angle B$, and (c) $\overline{BE} \cong \overline{BD}$." Since all three of these statements are given or can easily be proved, they are checked. We can now write the proof by reversing the order of the statements.

Proof:

Hypothesis: $\overline{AB} \cong \overline{CB}$
$\overline{FE} \cong \overline{DF}$
$\overline{BE} \cong \overline{BD}$

Conclusion: $\overline{AF} \cong \overline{FC}$



Statement	Reason
1. $\overline{BE} \cong \overline{BD}$	1. Hypothesis
2. $\angle B \cong \angle B$	2. Reflexive property of congruence for angles
3. $\overline{AB} \cong \overline{CB}$	3. Hypothesis
4. $\triangle ABE \cong \triangle CBD$	4. S.A.S. Postulate (1, 2, 3)
5. $\overline{AE} \cong \overline{CD}$	5. If two triangles are congruent (4), then their corresponding parts are congruent.
6. $\overline{FE} \cong \overline{DF}$	6. Hypothesis
7. $\overline{AF} \cong \overline{FC}$	7. Corollary 3.4.3 (5, 6)

Example 4 In Figure 5-14, all points are coplanar.

Hypothesis:

C is in the interior of $\angle ADE$.
E is in the interior of $\angle CDB$.
$\overline{AD} \perp \overline{DC}$
$\overline{BD} \perp \overline{DE}$
$\overline{AD} \cong \overline{CD}$
$\overline{DE} \cong \overline{DB}$

Copy and complete the proof that $\overline{AE} \cong \overline{CB}$.

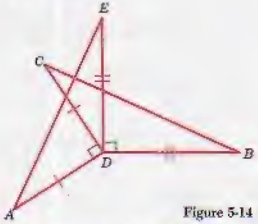


Figure 5-14

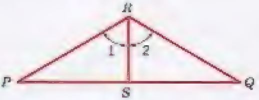
(*Plan:* We plan to use $ADE \longleftrightarrow CDB$. We can show $\overline{AD} \cong \overline{CD}$, $\angle ADE \cong \angle CDB$ by the Angle Measure Addition Postulate, and $\overline{DE} \cong \overline{DB}$. We can then use S.A.S.)

Statement	Reason
1. $\overline{AD} \perp \overline{DC}$, $\overline{BD} \perp \overline{DE}$	1. Hypothesis
2. $m\angle ADC = 90$, $m\angle BDE = 90$	2. $\boxed{?}$ (1)
3. C is in the interior of $\angle ADE$. E is in the interior of $\angle CDB$.	3. $\boxed{?}$
4. $m\angle ADE = 90 + m\angle CDE$ $m\angle CDB = 90 + m\angle CDE$	4. Angle Measure Addition Postulate (2, 3)
5. $m\angle ADE = m\angle CDB$	5. Substitution property of equality (4)
6. $\angle ADE \cong \angle CDB$	6. $\boxed{?}$ ($\boxed{?}$)
7. $\overline{AD} \cong \overline{CD}$, $\overline{DE} \cong \overline{DB}$	7. $\boxed{?}$
8. $\triangle ADE \cong \triangle CDB$	8. $\boxed{?}$ (6, 7)
9. $\overline{AE} \cong \overline{CB}$	9. $\boxed{?}$ ($\boxed{?}$). $\boxed{?}$

EXERCISES 5.7

In Exercises 1–22, write two-column proofs. In each exercise, copy the figure and mark on it the given congruences.

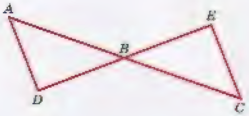
1. In the figure, S is between P and Q, $\overline{RS} \perp \overline{PQ}$, and $\angle 1 \cong \angle 2$. Prove that S is the midpoint of $\overline{PQ}$.



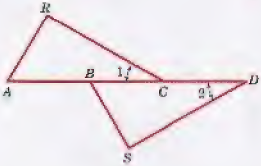
Copy the following outline and supply the missing reasons, including the numbers of supporting statements.

Statement	Reason
1. $\angle 1 \cong \angle 2$	1. Hypothesis
2. $\overline{RS} \cong \overline{RS}$	2. $\boxed{?}$
3. $\overline{RS} \perp \overline{PQ}$	3. $\boxed{?}$
4. $\angle RSP$ and $\angle RSQ$ are right angles.	4. If lines determined by two segments are perpendicular ($\boxed{?}$), then $\boxed{?}$.
5. $\angle RSP \cong \angle RSQ$	5. Any two right angles are $\boxed{?}$ ($\boxed{?}$).
6. $\triangle PSR \cong \triangle QSR$	6. $\boxed{?}$ ($\boxed{?}$, $\boxed{?}$, $\boxed{?}$)
7. $\overline{PS} \cong \overline{QS}$	7. If $\boxed{?}$ ($\boxed{?}$), then $\boxed{?}$.
8. S is between P and Q.	8. $\boxed{?}$
9. S is the midpoint of $\overline{PQ}$.	9. $\boxed{?}$ ($\boxed{?}$, $\boxed{?}$)

2. In the figure, $\angle ABD$ and $\angle CBE$ are vertical angles, B is the midpoint of $\overline{DE}$, and $\angle D \cong \angle E$. Prove that B is the midpoint of $\overline{AC}$.



3. In the figure, point B is between points A and C and point C is between points B and D. Given $\overline{CR} \cong \overline{DS}$, $\angle 1 \cong \angle 2$, and $\overline{AB} \cong \overline{CD}$, copy and complete the proof that $\angle R \cong \angle S$.



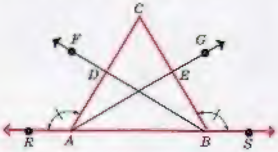
Statement	Reason
1. B is between A and C, and C is between B and D.	1. Hypothesis
2. $\overline{AB} \cong \overline{CD}$	2. ?
3. $\overline{BC} \cong \overline{BC}$	3. ?
4. ? $\cong$?	4. Length-Addition Theorem for Segments (1, 2, 3)
5. $\angle 1 \cong \angle 2$	5. ?
6. ? $\cong$?	6. Hypothesis
7. $\triangle ACR \cong \triangle BDS$	7. ? (?, ?, ?)
8. ? $\cong$?	8. Corresponding parts of congruent triangles are congruent (?).

4. In the figure, A is between C and D, B is between C and E, K is the midpoint of $\overline{AB}$. $\angle 1 \cong \angle 2$, and $AC = BC$. Copy and complete the proof that $\angle ACK \cong \angle BCK$.

Statement	Reason
1. $\angle 1 \cong \angle 2$	1. ?
2. D–A–C and E–B–C	2. Hypothesis
3. $\angle 1$ and $\angle CAK$ form a linear pair, and $\angle 2$ and $\angle CBK$ form a linear pair.	3. Definition of linear pair (2)
4. $\angle CAK$ is a supplement of $\angle 1$, and $\angle CBK$ is a supplement of $\angle 2$.	4. If two angles form a linear pair (3), then they are supplementary.
5. $\angle CAK \cong \angle CBK$	5. The supplements of congruent angles are ? (1, 4).
6. K is the midpoint of $\overline{AB}$.	6. ?
7. ? $\cong$?	7. Definition of midpoint (6)
8. ? = ?	8. Hypothesis
9. $\overline{AC} \cong \overline{BC}$	9. Definition of congruent segments (8)
10. $\triangle CAK \cong \triangle CBK$	10. ? (?, ?, ?)
11. $\angle ACK \cong \angle BCK$	11. ? (?)

5. In the figure, $LR = RN$ and $LM = MN$. Prove that $m\angle L = m\angle N$.

6. *Hypothesis:* For $\triangle ABC$, $\overrightarrow{BF}$ is the midray of $\angle ABC$ intersecting $\overline{AC}$ in D. $\overrightarrow{AG}$ is the midray of $\angle BAC$ intersecting $\overline{BC}$ in E. Points R, A, B, S are collinear in that order. $\angle RAD \cong \angle SBE$.



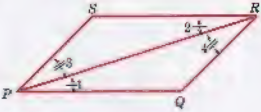
Conclusion: $\overline{AE} \cong \overline{BD}$

(*Plan:* Show that $\overline{AB} \cong \overline{BA}$, that $\angle ABE \cong \angle BAD$ (supplements of congruent angles), and that $m\angle BAE = m\angle ABD$ (halves of equals). Then use the A.S.A. Postulate to prove $\triangle ABE \cong \triangle BAD$.)

7. *Given:* $\overline{DA} \cong \overline{CB}$
$\overline{DA} \perp \overline{AB}$
$\overline{CB} \perp \overline{AB}$
To Prove: $\angle D \cong \angle C$

8. Does your proof for Exercise 7 depend on points A, B, C, and D being coplanar?

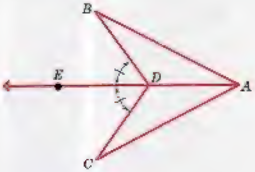
9. *Given:* $\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
To Prove: $PS = RQ$ and $m\angle S = m\angle Q$



10. *Given:* D and E are between A and B as shown in the figure.
$\angle ACE \cong \angle BCD$
$\overline{AC} \cong \overline{BC}$
$\overline{DC} \cong \overline{EC}$
To Prove: $\triangle ACE \cong \triangle BCD$
$\overline{AE} \cong \overline{BD}$

11. Use the same hypothesis and figure given in Exercise 10 and prove
(a) $\triangle ACD \cong \triangle BCE$. (b) $\overline{AD} \cong \overline{BE}$.

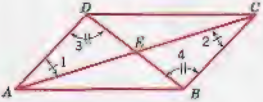
12. In the figure, E is in the interior of $\angle BAC$, $A\text{–}D\text{–}E$, $\angle EDC \cong \angle EDB$, and $\overline{BD} \cong \overline{CD}$. Prove that $\overrightarrow{AE}$ is the midray of $\angle BAC$.



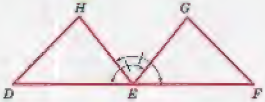
13. *Given:* $AB = CB$
$AD = CD$
To Prove: $\angle 1 \cong \angle 2$
(*Hint:* Draw $\overline{BD}$.)

14. In the figure, R, S, T, and N are collinear in that order. $\angle RSV \cong \angle PTN$, $\overline{RS} \cong \overline{TN}$, $\overline{PT} \cong \overline{VS}$.
Prove: $\angle P \cong \angle V$

15. In the figure, $A\text{–}E\text{–}C$, $D\text{–}E\text{–}B$, $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, and $\overline{AD} \cong \overline{BC}$.
Prove that $\overline{AC}$ and $\overline{BD}$ bisect each other at E.

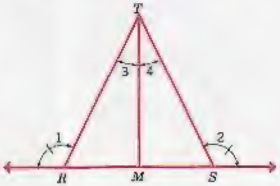


16. In the figure, E is the midpoint of $\overline{DF}$, H is in the interior of $\angle DEG$, G is in the interior of $\angle FEH$, $\angle DEG \cong \angle FEH$, and $\overline{HE} \cong \overline{GE}$. Prove that $\angle H \cong \angle G$.

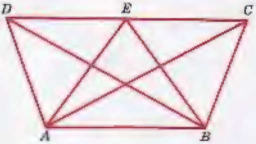


17. Given the situation of Exercise 16, prove in two different ways that $\angle DEH \cong \angle FEG$.

18. *Given:* $\angle 1 \cong \angle 2$
M is the midpoint of $\overline{RS}$
$\overline{RT} \cong \overline{ST}$
Prove: $\angle 3 \cong \angle 4$



19. In the figure, $AD = BC$, E is the midpoint of $\overline{CD}$, and $AE = BE$. Prove that $\overline{AC} \cong \overline{BD}$.

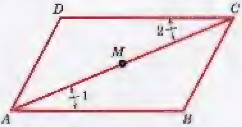


Copy and complete the statements in the following plan.

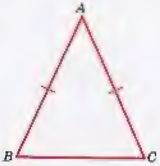
I Can Prove	If I Can Prove
1. $\overline{AC} \cong \overline{BD}$	1. $\triangle ADC \cong \triangle BCD$?
2. $\triangle ADC \cong \triangle BCD$	2. (a) $\boxed{?} \cong \boxed{?}$ (S)√
	(b) $\boxed{?} \cong \boxed{?}$ (A)?
	(c) $\overline{DC} \cong \overline{CD}$ (S)√
3. $\angle ADC \cong \angle BCD$	3. $\triangle EDA \cong \triangle ECB$?
4. $\triangle\boxed{?} \cong \triangle\boxed{?}$	4. (a) $\overline{ED} \cong \boxed{?}$ (S)√
	(b) $\overline{DA} \cong \boxed{?}$ (S)√
	(c) $\boxed{?} \cong \boxed{?}$ (S)√

Now reverse the steps in the second column and complete the proof.

20. In quadrilateral $ABCD$, M is the midpoint of $\overline{AC}$ and $\angle 1 \cong \angle 2$. Prove that any segment which contains M and has its endpoints in the interiors of $\overline{CD}$ and $\overline{AB}$ is bisected at M. (*Hint:* Draw $\overline{EF}$ such that E–M–F, A–E–B, and D–F–C. Now prove that $\triangle EMA \cong \triangle FMC$ and hence that $\overline{EM} \cong \overline{FM}$.)



21. CHALLENGE PROBLEM. In $\triangle ABC$, $\overline{AC} \cong \overline{AB}$. Prove that the correspondence $CAB \longleftrightarrow BAC$ is a congruence using the S.A.S. Postulate. Hence conclude that $\triangle CAB \cong \triangle BAC$. Does this prove that $\angle B \cong \angle C$? Why? Does this prove that if a triangle has two congruent sides, then the angles opposite those sides are congruent?



22. CHALLENGE PROBLEM. In $\triangle RST$, $\angle S \cong \angle T$. Prove that the correspondence $STR \longleftrightarrow TSR$ is a congruence using the A.S.A. Postulate. Hence conclude that $\triangle STR \cong \triangle TSR$. Does this prove that $\overline{RS} \cong \overline{RT}$? Why? Does this prove that if a triangle has two congruent angles, then the sides opposite those angles are congruent?

5.8 ISOSCELES TRIANGLES

Each triangle shown in Figure 5-15 has at least two congruent sides. Such triangles are called *isosceles* (from the Greek words *iso* and *skelos* meaning "equivalent" and "legs," respectively).

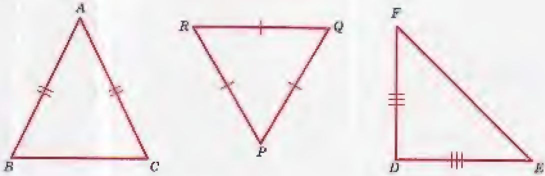


Figure 5-15

Definition 5.3 An isosceles triangle is a triangle with (at least) two congruent sides. If two sides are congruent, then the remaining side is called the base. The angle opposite the base is called the vertex angle. The two angles that are opposite the congruent sides are called the base angles.

In $\triangle ABC$ of Figure 5-15, $\overline{AB}$ and $\overline{AC}$ are congruent sides. The base is $\overline{BC}$, the vertex angle is $\angle A$, and the base angles are $\angle B$ and $\angle C$. Name the base, vertex angle, and base angles of $\triangle DEF$ in Figure 5-15.

If a triangle has three congruent sides as does $\triangle PQR$ in Figure 5-15, then any side may be considered as a base of the triangle. The angle opposite a base is considered the vertex angle corresponding to that base, and the angles that include a base are called the base angles corresponding to that base.

Definition 5.4 A triangle with three congruent sides is called an **equilateral** triangle. A triangle with three congruent angles is called an **equiangular** triangle.

If you worked Exercises 21 and 22 of Exercises 5.7, you already proved the following two theorems.

THEOREM 5.1 (***The Isosceles Triangle Theorem***) The base angles of an isosceles triangle are congruent.

RESTATEMENT: In $\triangle ABC$ (Figure 5-16), $\overline{AC} \cong \overline{AB}$. Prove that $\angle B \cong \angle C$.

Figure 5-16

(*Plan:* We will show that, for $\triangle ABC$, the correspondence $BAC \longleftrightarrow CAB$ is a congruence using S.A.S., and hence $\angle B \cong \angle C$ by the definition of congruent triangles.)

Proof:

Statement	Reason
1. $\overline{AC} \cong \overline{AB}$	1. Hypothesis
2. $\angle A \cong \angle A$	2. Reflexive property of congruence for angles
3. $\overline{AB} \cong \overline{AC}$	3. Symmetric property of congruence for segments (1)
4. $\triangle BAC \cong \triangle CAB$	4. S.A.S. Postulate (1, 2, 3)
5. $\angle B \cong \angle C$	5. Corresponding parts of congruent triangles (4) are congruent.

Theorem 5.1 implies the result which follows. Its proof is left as an exercise.

COROLLARY 5.1.1 If a triangle is equilateral, then it is equiangular.

We know from Theorem 5.1 that if a triangle has a pair of congruent sides, then the angles opposite these sides are congruent. The converse of Theorem 5.1 is also true, and we state it as our next theorem.

THEOREM 5.2 (*Converse of the Isosceles Triangle Theorem*) If a triangle has two congruent angles, then the sides opposite these angles are congruent and the triangle is isosceles.

RESTATEMENT: In $\triangle DEF$ (Figure 5-17), $\angle E \cong \angle F$. Prove that $\overline{DF} \cong \overline{DE}$ and hence that $\triangle DEF$ is isosceles.

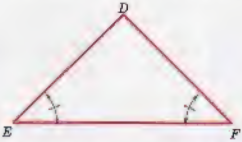


Figure 5-17

(*Plan:* We will show that, for $\triangle DEF$, the correspondence $EFD \longleftrightarrow FED$ is a congruence using A.S.A. and hence $\overline{DF} \cong \overline{DE}$ by the definition of congruent triangles.)

Proof:

Statement	Reason
1. $\angle E \cong \angle F$	1. Hypothesis
2. $\overline{EF} \cong \overline{FE}$	2. Reflexive property of congruence for segments
3. $\angle F \cong \angle E$	3. Symmetric property of congruence for angles (1)
4. $\triangle EFD \cong \triangle FED$	4. A.S.A. Postulate (1, 2, 3)
5. $\overline{DF} \cong \overline{DE}$	5. Corresponding parts of congruent triangles (4) are congruent.
6. $\triangle DEF$ is isosceles.	6. Definition of isosceles triangle (5)

The proof of the following corollary of Theorem 5.2 is left as an exercise.

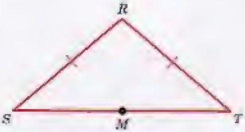
COROLLARY 5.2.1 If a triangle is equiangular, then it is equilateral.

EXERCISES 5.8

1. Is every equilateral triangle isosceles? Is every isosceles triangle equilateral?

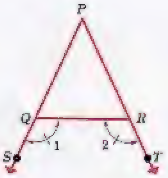
2. In the figure, P is in the interior of $\triangle ABC$. $\overline{AB} \cong \overline{AC}$ and $\overline{PB} \cong \overline{PC}$. Prove, without using congruent triangles, that $\angle ABP \cong \angle ACP$.

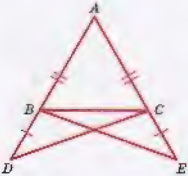
3. Write a two-column proof of Corollary 5.1.1.
4. Write a proof, in paragraph form, of Corollary 5.2.1.
5. Prove Theorem 5.1 using the S.S.S. Postulate.
6. In the figure, $\triangle RST$ is an isosceles triangle with vertex angle at R. Give a proof different from that given for Theorem 5.1 that base angles of an isosceles triangle are congruent. (*Hint:* Let M be the midpoint of $\overline{ST}$ and prove $\triangle SRM \cong \triangle TRM$.)



7. *Given:* P–Q–S, P–R–T, and $\angle 1 \cong \angle 2$
   Prove: $\triangle QPR$ is isosceles.

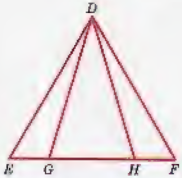
8. *Given:* A–B–D, A–C–E, $\overline{AB} \cong \overline{AC}$, and $\overline{BD} \cong \overline{CE}$
   Prove: $\triangle BDC \cong \triangle CEB$



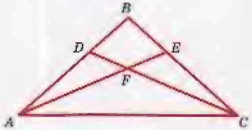


9. CHALLENGE PROBLEM. Did you use the Isosceles Triangle Theorem in your proof of Exercise 8? If not, write a different proof, using the Isosceles Triangle Theorem. If you did, write a different proof in which the Isosceles Triangle Theorem is not used.

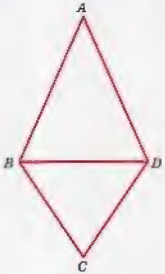
10. *Given:* $\triangle DEF$ is isosceles with vertex angle at D, $\overline{EG} \cong \overline{HF}$
Prove: (a) $\triangle GDH$ is isosceles.
(b) $\angle DGH \cong \angle DHG$



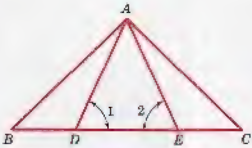
11. If points A, B, C, D, E, F have the betweenness relations shown in the figure, if $\triangle ABC$ is isosceles with vertex angle at B, and if $\triangle AFC$ is isosceles with vertex angle at F, prove $\overline{AD} \cong \overline{EC}$. (*Plan:* Use the Isosceles Triangle Theorem to prove $\triangle ADC \cong \triangle CEA$ by A.S.A.)



12. In convex quadrilateral $ABCD$, $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$. (a) Prove $\angle ABC \cong \angle ADC$ without proving any triangles congruent. (b) Draw $\overline{AC}$ intersecting $\overline{BD}$ at E. Prove $\overline{BE} \cong \overline{ED}$ and $\overline{AC} \perp \overline{BD}$.

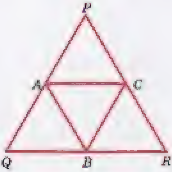


13. In the figure, $\angle 1 \cong \angle 2$, points B, D, E, C are collinear, and $BD = EC$. Prove that $\triangle ABC$ is isosceles.



14. *Given:* $\triangle RST$ is equilateral.
Prove: $\triangle RST$ is equiangular. (*Plan:* Show that the correspondence $RST \longleftrightarrow TRS$ is a congruence by S.S.S. and that $\angle R \cong \angle T \cong \angle S$ by the definition of congruent triangles.)

15. *Given:* $\triangle PQR$ is equilateral, with A, B, and C the midpoints of $\overline{PQ}$, $\overline{QR}$, and $\overline{PR}$, respectively.
Prove: $\triangle ABC$ is equiangular.



16. CHALLENGE PROBLEM. We can prove the A.S.A. Postulate as a theorem once we have assumed the S.A.S. Postulate. Complete the proof of the following statement.

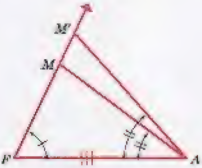
Given an A.S.A. correspondence

$$FAM \longleftrightarrow RSP$$

as indicated in the figures, if

$$\angle F \cong \angle R,$$
$$\overline{FA} \cong \overline{RS},$$
$$\angle FAM \cong \angle S,$$

prove that $\triangle FAM \cong \triangle RSP$.



Proof: There is a point M' on $\overrightarrow{FM}$ such that $\overline{FM'} \cong \overline{RP}$. Why? (Note that our figure shows M' and M to be different points. We will prove that they are the same point.) Therefore $\triangle FAM' \cong \triangle RSP$ by S.A.S. (Show this.) $\angle FAM' \cong \angle RSP$. Why? Therefore $\overrightarrow{AM'} = \overrightarrow{AM}$ by the Angle Construction Theorem. It follows that $M' = M$ (Why?) and $\triangle FAM \cong \triangle RSP$.

17. CHALLENGE PROBLEM. We can prove the S.S.S. Postulate as a theorem once we have assumed the S.A.S. Postulate and proved the Isosceles Triangle Theorem. Complete the proof of the following statement.

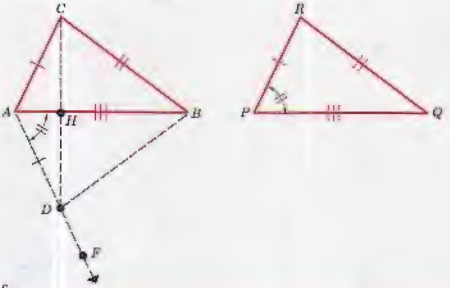
Given an S.S.S. correspondence

$$ABC \longleftrightarrow PQR$$

as indicated in the figure, if

$$\overline{AB} \cong \overline{PQ},$$
$$\overline{BC} \cong \overline{QR},$$
$$\overline{AC} \cong \overline{PR},$$

prove that $\triangle ABC \cong \triangle PQR$.



Proof:

Statement	Reason
1. $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, $\overline{AC} \cong \overline{PR}$	1. Given
2. There is a point F on the opposite side of $\overleftrightarrow{AB}$ from C such that $\angle BAF \cong \angle P$.	2. Angle Construction Theorem
3. There is a point D on $\overrightarrow{AF}$ such that $\overline{AD} \cong \overline{PR}$.	3. Segment Construction Theorem
4. $\triangle ABD \cong \triangle PQR$	4. S.A.S. Postulate (1, 2, 3)

Since C and D are on opposite sides of $\overleftrightarrow{AB}$, $\overline{CD}$ intersects $\overleftrightarrow{AB}$ in a point H. Our figure shows H to be between A and B. We could have, however, $A = H$, or $B = H$, or A-B-H, or H-A-B. Complete the proof for the case where A-H-B by using the Isosceles Triangle Theorem and the S.A.S. Postulate to prove that $\triangle ABC \cong \triangle ABD$. It will then follow from step 4 and the transitive property of congruence for triangles that $\triangle ABC \cong \triangle PQR$. Draw a figure and complete the proof for the other four cases, that is, for $A = H$, $B = H$, A-B-H, and H-A-B.

5.9 MEDIANS AND PERPENDICULAR BISECTORS

In Figure 5-18, M is the midpoint of side $\overline{BC}$ of $\triangle ABC$. Segment $\overline{AM}$ is called a *median* of $\triangle ABC$. Since each side of a triangle has exactly one midpoint, every triangle has exactly three medians. Draw a triangle and its medians. What property do the medians appear to have?

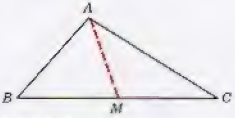


Figure 5-18

> ***Definition 5.5*** A median of a triangle is a segment whose endpoints are a vertex of the triangle and the midpoint of the side opposite that vertex.

THEOREM 5.3 The median to the base of an isosceles triangle bisects the vertex angle and is perpendicular to the base.

RESTATEMENT: In $\triangle ABC$ (Figure 5-19), $\overline{AB} \cong \overline{AC}$ and $\overline{AM}$ is the median to $\overline{BC}$.

Prove: 1. $\overrightarrow{AM}$ is the midray of $\angle BAC$.
2. $\overline{AM} \perp \overline{BC}$.

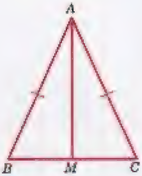


Figure 5-19

Proof: Copy and complete the following proof.

Statement	Reason
1. $\overline{AB} \cong \overline{AC}$	1. Hypothesis
2. $\angle B \cong \angle C$	2. ? (?)
3. $\overline{AM}$ is the median to $\overline{BC}$.	3. Hypothesis
4. M is the midpoint of $\overline{BC}$.	4. ? (?)
5. $\overline{BM} \cong \overline{CM}$	5. ? (?)
6. $\triangle ABM \cong \triangle ACM$	6. ? (?)
7. $\angle BAM \cong \angle CAM$	7. ? (?)
8. M is in the interior of $\angle BAC$.	8. Theorem 4.11 (4)
9. $\overrightarrow{AM}$ is the midray of $\angle BAC$.	9. Definition of midray (7, 8)
10. $\angle BMA \cong \angle CMA$	10. ? (?)
11. $\angle BMA$ and $\angle CMA$ are a linear pair.	11. Definition of linear pair (4)
12. $\angle BMA$ and $\angle CMA$ are right angles.	12. Theorem 4.12 (10, 11)
13. $\overline{AM} \perp \overline{BC}$	13. ? (?)

In Figure 5-20, $\triangle ABC$ is isosceles with vertex angle at A. Ray $\overrightarrow{AG}$ is the midray of the vertex angle. It appears from the figure that $\overrightarrow{AG}$ bisects $\overline{BC}$ and is perpendicular to $\overline{BC}$ at M. This brings us to our next theorem.

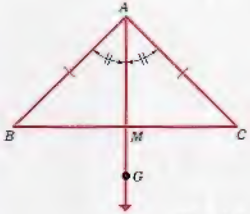


Figure 5-20

THEOREM 5.4 The midray of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

Proof: In proving this theorem it is necessary to show that ray $\overrightarrow{AG}$ intersects $\overline{BC}$ in a point M that is between B and C (as Figure 5-20 suggests). By the definition of midray, $\overrightarrow{AG}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. By the definition of betweenness for rays, B and C are in opposite half-planes with edge $\overleftrightarrow{AG}$. By the definition of opposite sides of a line, there is a point M of $\overleftrightarrow{AG}$ between B and C. Since M is between B and C, it follows from Theorem 4.11 that M is in the interior of $\angle BAC$. Since M is in the interior of $\angle BAC$ and since it is a point of $\overrightarrow{AG}$ or of *opp* $\overrightarrow{AG}$, but not both, it follows from Theorem 4.10 that M is a point of $\overrightarrow{AG}$. The rest of the proof of Theorem 5.4 is straightforward and is left as an exercise.

In Theorem 5.4, it was required to prove that a certain ray was perpendicular to a certain segment and that the ray bisected the segment. The line that contains the ray and that is in the same plane as the segment is called the *perpendicular bisector of the segment* in that plane. We state this formally in the following definition.

Definition 5.6 The perpendicular bisector of a segment in a given plane is the line in that plane which is perpendicular to the segment at its midpoint.

In Figure 5-21, line l is perpendicular to $\overline{AB}$ at M, the midpoint of $\overline{AB}$. Since l and $\overleftrightarrow{AB}$ determine exactly one plane (Why?), we say that l is the perpendicular bisector of $\overline{AB}$ in that plane. We sometimes write "$l \perp$ *bis* $\overline{AB}$" for "l is the perpendicular bisector of $\overline{AB}$."

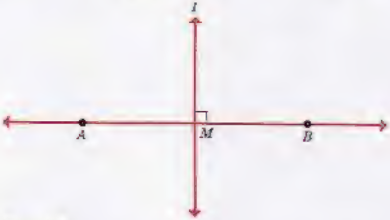


Figure 5-21

Note that, in space, there is more than one line (How many?) that is a perpendicular bisector of a given segment. However, in a given plane the perpendicular bisector of a segment is unique, since a segment has exactly one midpoint and, by Theorem 4.14, in a given plane there is exactly one line that is perpendicular to a given line at a given point on the line. The next two theorems serve to characterize the set of points in the perpendicular bisector of a segment.

THEOREM 5.5 (***The Perpendicular Bisector Theorem***) If, in a given plane α, P is a point on the perpendicular bisector of $\overline{AB}$, then P is equidistant from the endpoints of $\overline{AB}$.

RESTATEMENT: In Figure 5-22, P is a point on line l in a plane α. $l \perp$ *bis* $\overline{AB}$ at M in α.

Prove: $PA = PB$

Proof: If $P = M$, then $PA = PB$ by the definition of midpoint. If P is any point in l different from M, then $\triangle APM \cong \triangle BPM$ by S.A.S. (show this); hence $\overline{PA} \cong \overline{PB}$ by the definition of congruent triangles and $PA = PB$ by the definition of congruent segments.

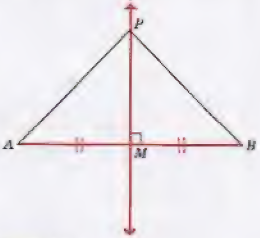
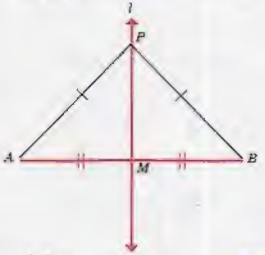


Figure 5-22

THEOREM 5.6 ***(Converse of the Perpendicular Bisector Theorem)*** If, in a given plane α, P is equidistant from the endpoints of $\overline{AB}$, then P lies on the perpendicular bisector of $\overline{AB}$.

RESTATEMENT: In plane α, $PA = PB$, M is the midpoint of $\overline{AB}$.

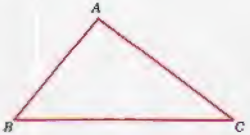
Prove: P is on l, the perpendicular bisector of $\overline{AB}$.



Proof: If P is on line $\overleftrightarrow{AB}$, then $P = M$ because $\overline{AB}$ has only one midpoint. In this case, P is on line l by the definition of the perpendicular bisector of a segment. If P is not on line $\overleftrightarrow{AB}$, then $\triangle APM \cong \triangle BPM$ by S.S.S. (show this). Therefore $\angle AMP$ is a right angle (Why?) and $\overleftrightarrow{PM}$ $\perp$ *bis* $\overline{AB}$. Since, in a given plane, a segment has only one perpendicular bisector, P is on l.

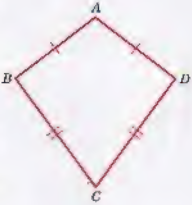
EXERCISES 5.9

1. Copy and complete the proof of Theorem 5.3.
2. Complete the proof of Theorem 5.4 by writing it in two-column form. You may assume that it has been proved that $\overrightarrow{AG}$ intersects $\overline{BC}$ in a point M which is between B and C. (See Figure 5-20.)
3. Copy $\triangle ABC$ below.
   (a) Construct the median from A to $\overline{BC}$.
   (b) On the same figure construct the midray of $\angle BAC$.
   (c) Does the midray contain the median?
   (d) What must be true about $\triangle ABC$ if the midray from A is to coincide with the median from A?

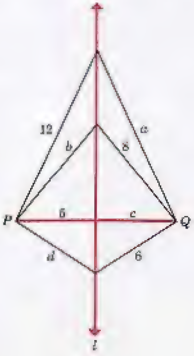


4. Copy $\triangle DEF$ in the figure.
   (a) Construct the perpendicular bisector of $\overline{EF}$ in plane DEF.
   (b) Does the perpendicular bisector of $\overline{EF}$ contain point D?
   (c) What must be true about $\triangle DEF$ if the perpendicular bisector of $\overline{EF}$ is to contain D?

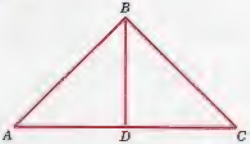
5. From which theorem may we deduce that the vertex of the angle opposite the base of an isosceles triangle lies on the perpendicular bisector of the base?
6. In the figure, A, B, C, D are distinct coplanar points, $\overline{AB} \cong \overline{AD}$, and $\overline{BC} \cong \overline{CD}$.
   (a) Is $\overleftrightarrow{BD}$ the perpendicular bisector of $\overline{AC}$?
   (b) Is $\overleftrightarrow{AC}$ the perpendicular bisector of $\overline{BD}$?
   (c) Which auxiliary segment is needed to prove $\triangle ABC \cong \triangle ADC$?
   (d) How do we know this segment exists?
   (e) Why is $\angle B \cong \angle D$?
   (f) If your answer to (a) is Yes, prove it is correct with a paragraph style proof. Do not use congruent triangles.
   (g) If your answer to (b) is Yes, prove it is correct with a paragraph style proof. Do not use congruent triangles.



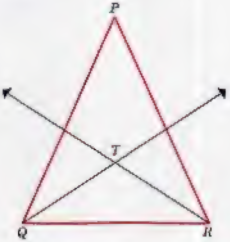
7. In the figure, l is the perpendicular bisector of $\overline{PQ}$. If the lengths of segments are as marked, find a, b, c, and d.



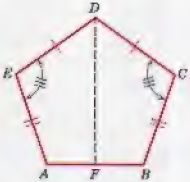
8. In the figure, $\triangle ABC$ is isosceles with $AB = BC$. If $\overline{BD}$ is a median, prove that $\triangle ABD \cong \triangle CBD$.



9. In the figure, $\triangle PQR$ is isosceles with $PQ = PR$. The midrays of $\angle QRP$ and $\angle PQR$ intersect at T. Prove that $\overleftrightarrow{PT} \perp \overline{QR}$. Do not use congruent triangles in your proof.



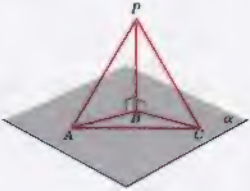
10. If $l \perp bis\ \overline{AB}$ at M, and if P is a point on l different from M, prove that $\overline{PM}$ is a median of $\triangle PAB$.
11. If S is the midpoint of $\overline{QR}$ and $\overleftrightarrow{PS} \perp \overline{QR}$, prove that $\triangle PQR$ is isosceles. (It is not necessary to use congruent triangles in your proof.)
12. Given the figure as marked with F the midpoint of $\overline{AB}$, prove that $\overline{DF} \perp \overline{AB}$.



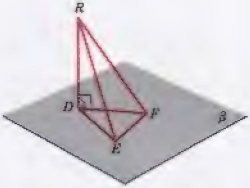
13. CHALLENGE PROBLEM. If $\overline{AB}$ and $\overline{QR}$ are coplanar and congruent segments, $n \perp bis\ \overline{AQ}$, $m \perp bis\ \overline{BR}$, and $m \cap n = P$, prove that $\triangle ABP \cong \triangle QRP$.

In Exercises 14–22, you will need to prove certain triangles congruent that are not necessarily coplanar. You should look at the figures in perspective and be aware that angles or sides that *are* congruent may not *look* congruent in the figures. In every case, carefully draw a figure on your own paper and mark on it the congruent parts before writing a proof.

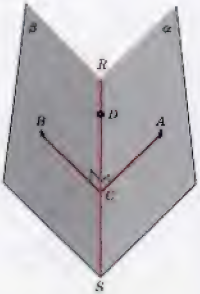
14. In the figure, points A, B, and C are in plane α. Point P is not in plane α. $\overline{PB} \perp \overline{AB}$ and $\overline{PB} \perp \overline{BC}$. If $\angle BAC \cong \angle BCA$, prove $\overline{AP} \cong \overline{CP}$.



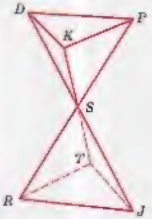
15. In the figure, points D, E, and F are in plane β. Point R is not in plane β. If $\overline{RD} \perp \overline{DE}$, $\overline{RD} \perp \overline{DF}$, and $\overline{DE} \cong \overline{DF}$, prove $\triangle REF$ is isosceles and that $\angle REF \cong \angle RFE$.

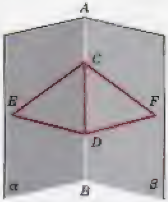


16. In dihedral angle A-RS-B, $\overline{AC}$ is in plane α, $\overline{BC}$ is in plane β, $\overline{AC} \perp \overline{RS}$, $\overline{BC} \perp \overline{RS}$, D is in $\overline{RS}$, and $\triangle ACB$ is isosceles with vertex at C. Prove that $\triangle BDA$ is isosceles.



17. Given $\overline{PR}$, $\overline{DJ}$, and $\overline{KT}$ bisect each other at S. Prove $\triangle KDP \cong \triangle TJR$.

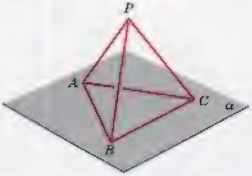




18. In dihedral angle E–AB–F, E is in plane α, F is in plane β, and $\overleftrightarrow{AB}$ contains C and D. If triangles DEC and DFC are isosceles with vertex angles at E and F, respectively, prove that $\angle EDF \cong \angle ECF$. (*Plan:* Prove $\triangle EDF \cong \triangle ECF$ by S.S.S.) (See figure above at right.)

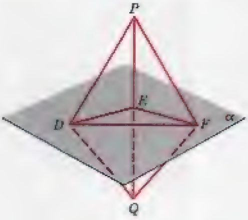
19. In Exercise 18, is $\overrightarrow{DC}$ between rays $\overrightarrow{DF}$ and $\overrightarrow{DE}$? Can we prove $\angle EDF \cong \angle ECF$ by the Angle Measure Addition Postulate?

20. In the figure, $PA = PB = PC$ and $\angle APB \cong \angle BPC \cong \angle APC$. Prove that $\triangle ABC$ is equilateral.



21. In Exercise 20, if we accept the hypothesis (but disregard the figure), is it possible that P, A, B, C are coplanar? Try to draw a figure for the "plane case."

22. CHALLENGE PROBLEM. In the figure, P and Q are on opposite sides of plane α which contains points D, E, and F. If $\overline{PQ} \perp \overline{DE}$ at E, $\overline{PQ} \perp \overline{FE}$ at E, and $\overline{PE} \cong \overline{QE}$, prove $\triangle PDF \cong \triangle QDF$.



CHAPTER SUMMARY

The Definitions

If, for $\triangle ABC$ and $\triangle DEF$,

$$\begin{array}{ll} \angle A \longleftrightarrow \angle D, & \overline{AB} \longleftrightarrow \overline{DE}, \\ \angle B \longleftrightarrow \angle E, & \overline{BC} \longleftrightarrow \overline{EF}, \\ \angle C \longleftrightarrow \angle F, & \overline{AC} \longleftrightarrow \overline{DF}, \end{array}$$

we indicate this correspondence by writing $ABC \longleftrightarrow DEF$ where the order in which the vertices are named preserves the six correspondences. We speak of the pairs in the six correspondences named as CORRESPONDING PARTS of the two triangles. Two triangles (not necessarily distinct) are CONGRUENT if and only if there exists a one-to-one correspondence between their vertices in which the corresponding parts are congruent. Such a one-to-one correspondence between the vertices of two congruent triangles is called a CONGRUENCE.

An ANGLE of a triangle is said to be INCLUDED by two sides of that triangle if the angle contains those sides. A SIDE of a triangle is said to be INCLUDED by two angles of that triangle if the endpoints of the side are the vertices of those angles.

The combination TWO SIDES AND THE INCLUDED ANGLE is abbreviated by the symbol S.A.S. If there exists a correspondence between two triangles such that S.A.S. of one triangle are congruent to S.A.S. of the second triangle, then we call this an S.A.S. CONGRUENCE.

The combination TWO ANGLES AND THE INCLUDED SIDE is abbreviated by the symbol A.S.A. If there exists a correspondence between two triangles such that A.S.A. of one triangle are congruent to A.S.A. of the second triangle, then we call this an A.S.A. CONGRUENCE.

The combination THREE SIDES is abbreviated by the symbol S.S.S. If there exists a correspondence between two triangles such that S.S.S. of one triangle are congruent to S.S.S. of the second triangle, then we call this an S.S.S. CONGRUENCE.

If a triangle has (at least) two congruent sides, it is called an ISOSCELES triangle. The VERTEX ANGLE of an isosceles triangle is the angle included by the congruent sides. The BASE of an isosceles triangle is the side that is opposite the vertex angle. The BASE ANGLES of an isosceles triangle are the angles whose vertices are the endpoints of the base. An EQUILATERAL triangle is a triangle that has three congruent sides. An EQUIANGULAR triangle is a triangle that has three congruent angles. A MEDIAN of a triangle is a segment that has for its endpoints a vertex of the triangle and the midpoint of the side opposite that vertex.

In a given plane, the PERPENDICULAR BISECTOR OF A SEGMENT is the line that is perpendicular to the segment at its midpoint.

A statement of the form "If p, then q" is called a CONDITIONAL. The statement p is called the HYPOTHESIS of the conditional and the

statement q is called the CONCLUSION. If a conditional and its hypothesis are known to be true, then it follows logically that its conclusion is also true.

The statement "If q, then p" is called the CONVERSE of the statement "If p, then q" and the statement "If p, then q" is called the converse of the statement "If q, then p." A definition in "if-then" form is to be understood as a conjunction of two statements "If p, then q" and "If q, then p." Sometimes this is abbreviated to "p if and only if q."

The Postulates

There were three postulates in this chapter having to do with congruent triangles. We list them in their abbreviated form only. Be sure that you know the complete statement of each postulate.

POSTULATE 23. The S.A.S. Postulate.
POSTULATE 24. The A.S.A. Postulate.
POSTULATE 25. The S.S.S. Postulate.

The Theorems

There were six theorems and two corollaries in this chapter. We list them in the order in which they appeared. Be sure that you know what each theorem says and that you understand its proof.

> ***THEOREM 5.1*** ***(The Isosceles Triangle Theorem)*** The base angles of an isosceles triangle are congruent.

> ***COROLLARY 5.1.1*** If a triangle is equilateral, then it is equiangular.

> ***THEOREM 5.2*** ***(Converse of the Isosceles Triangle Theorem).*** If a triangle has two congruent angles, then the sides opposite these angles are congruent and the triangle is isosceles.

> ***COROLLARY 5.2.1*** If a triangle is equiangular, then it is equilateral.

> ***THEOREM 5.3*** The median to the base of an isosceles triangle bisects the vertex angle and is perpendicular to the base.

> ***THEOREM 5.4*** The midray of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

> ***THEOREM 5.5*** ***(The Perpendicular Bisector Theorem)*** If, in a given plane α, P is a point on the perpendicular bisector of $\overline{AB}$, then P is equidistant from the endpoints of $\overline{AB}$.

> ***THEOREM 5.6*** ***(Converse of the Perpendicular Bisector Theorem)*** If, in a given plane α, P is equidistant from the endpoints of $\overline{AB}$, then P lies on the perpendicular bisector of $\overline{AB}$.

REVIEW EXERCISES

1. Write the following statement as a conjunction of two statements in the "if-then" form: A triangle is equilateral if and only if it is equiangular.
2. In your answer to Exercise 1, what is the second statement called with respect to the first? Are both statements true?
3. Write the converse of the statement: If two triangles are congruent, then their corresponding parts are congruent.
4. In Exercise 3, are both the statement and its converse true?
5. Write the converse of the statement: Vertical angles are congruent.
6. In Exercise 5, is the given statement true? Is its converse true?

In Exercises 7–18, decide if the sentence is true or false.

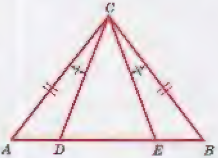
7. In the statement, "If p, then q," p is the conclusion and q is the hypothesis.
8. In the statement, "p only if q," p is the conclusion and q is the hypothesis.
9. In the statement, "p if q," p is the conclusion and q is the hypothesis.
10. If we know the statement "If p, then q" is true and if we know that p is true, then we can conclude that q is true.
11. The statement "If p, then q" may be true even though the statements p and q are both false.
12. If A is on the perpendicular bisector of $\overline{CD}$, then $AC = AD$.
13. If, in a given plane, $PE = PF$, then P is on the perpendicular bisector of $\overline{EF}$.
14. In a given plane, more than one line can be drawn which is both perpendicular to a segment and bisects the segment.
15. In a given plane, more than one line can be drawn which bisects a segment.
16. If R, S, and T are not collinear and R is on the perpendicular bisector of $\overline{ST}$, then $\triangle RST$ is isosceles.
17. Congruence of triangles has the transitive property.
18. In our formal development of geometry, we postulated that the sides opposite a pair of congruent angles in a triangle are congruent.

19. Copy and complete: If, in a given plane, R is $\boxed{?}$ from the endpoints of $\overline{ST}$, then R lies on the $\boxed{?}$ $\boxed{?}$ of $\overline{ST}$.
20. Copy and complete: If $\overline{AB}$ is a segment and if R and S are distinct points such that R, S, A, B are coplanar, $RA = RB$, and $SA = SB$, then $\boxed{?}$ is the $\boxed{?}$ $\boxed{?}$ of $\overline{AB}$.
21. If, in $\triangle ABC$, $\overrightarrow{BD}$ bisects $\angle ABC$ and intersects $\overline{AC}$ at D, $\overline{AB} \cong \overline{BC}$, prove that D is the midpoint of $\overline{AC}$.

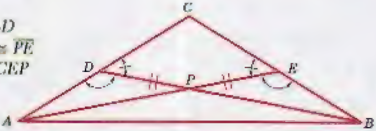
22. If, in the figure, $\overline{AB} \cong \overline{BC}$, does it follow that $\angle BAD \cong \angle BCD$ because base angles of an isosceles triangle are congruent? Why?

23. In the figure for Exercise 22, if $\overline{AB} \cong \overline{BC}$, $\angle BAD \cong \angle BCD$, and D is in the interior of $\angle ABC$, write a plan for proving $\triangle ABD \cong \triangle CBD$.

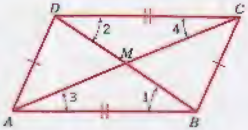
24. *Given:* $\overline{AC} \cong \overline{BC}$
$\angle ACD \cong \angle BCE$
A–D–E, D–E–B
Prove: $\overline{CD} \cong \overline{CE}$



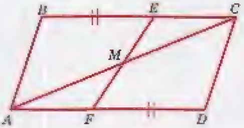
25. *Given:* A–D–C,
B–E–C, B–P–D
A–P–E, $\overline{DP} \cong \overline{PE}$
$\angle CDP \cong \angle CEP$
Prove: $\triangle APB$ is isosceles.



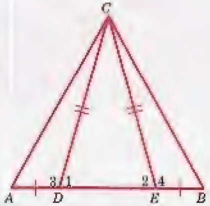
26. *Given:* $\overline{AB} \cong \overline{CD}$, A–M–C, B–M–D, and $\overline{AD} \cong \overline{BC}$
Prove: (a) $\triangle ABD \cong \triangle CDB$
(b) $\angle 1 \cong \angle 2$
(c) $\triangle ACD \cong \triangle CAB$
(d) $\angle 3 \cong \angle 4$
(e) $\triangle CMD \cong \triangle AMB$
(f) M is the midpoint of $\overline{BD}$ and $\overline{AC}$.



27. In the plane figure, M is the midpoint of segments $\overline{AC}$ and $\overline{EF}$. $\overline{BE} \cong \overline{FD}$. Prove
(a) $\triangle AMF \cong \triangle CME$.
(b) $\triangle ABC \cong \triangle CDA$.



28. *Given:* A–D–E, D–E–B
$\overline{AD} \cong \overline{BE}$
$\overline{CD} \cong \overline{CE}$
Prove: $\triangle ABC$ is isosceles.
Plan: Prove, in order,
$\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
$\triangle ACD \cong \triangle BCE$
$\overline{AC} \cong \overline{BC}$
$\triangle ABC$ is isosceles.



Chapter 6

Carl Struwe/Monkmeyer

Inequalities in Triangles

6.1 INTRODUCTION

In Chapters 3, 4, and 5 we developed the concept of congruence for segments, angles, and triangles. When we say that two segments are congruent, we mean that two numbers associated with these segments, their lengths, are the same. The concept of congruence for angles and for triangles is based also on the concept of equal measures, and hence on the concept of equality for numbers.

In this chapter we are concerned with segments and angles, in particular with segments that are not congruent to each other and with angles that are not congruent to each other. In other words, we are concerned with segments of unequal measures and with angles of unequal measures. In comparing two segments (or angles) that are not congruent to each other, it is useful to know which one has the larger measure. To express such a comparison it is convenient to use some familiar words and symbols defined formally as follows.

Definition 6.1 $\overline{AB} > \overline{CD}$ if and only if $AB > CD$; $\overline{AB} < \overline{CD}$ if and only if $AB < CD$.

Definition 6.2 $\angle ABC > \angle DEF$ if and only if $m\angle ABC > m\angle DEF$; $\angle ABC < \angle DEF$ if and only if $m\angle ABC < m\angle DEF$.

When used to compare segments, the symbol "$>$" may be read as "greater than," or "longer than," or "larger than." Similarly, "$<$" may be read as "less than," or "shorter than," or "smaller than."

When used to compare angles, the symbol "$>$" may be read as "greater than" or "larger than." Similarly, "$<$" may be read as "less than" or "smaller than."

Note that Definitions 6.1 and 6.2 express a comparison of segments, or of angles, in terms of an inequality involving numbers. It should be clear that an inequality involving segments or angles is really a statement of "not-congruence" together with a statement of which segment, or angle, has the greater, or lesser, measure. Because the properties of inequalities for numbers are essential for the development of inequalities for segments and angles, we review them in Section 6.2.

This chapter includes several important theorems, some of which involve comparisons of parts of one triangle. Others involve comparisons of parts of one triangle with parts of another triangle. To save time we shall, in some cases, give "abbreviated proofs" of these theorems, that is, just the key steps in the proofs. You should be able to supply a complete proof if asked to do so.

6.2 INEQUALITIES FOR NUMBERS

We begin by defining, formally, less than and greater than for numbers.

> ***Definition 6.3*** If a and b are numbers, then $a < b$ if and only if there is a *positive* number p such that $b = a + p$. Also $a > b$ if and only if there is a *positive* number p such that $a = b + p$.

> ***Definition 6.4*** If a and b are numbers, then $a \leq b$ if and only if $a < b$ or $a = b$.

Example 1 The statement $4 \leq 5$ is read "4 is less than or equal to 5" and, by Definition 6.4, it means $4 < 5$ or $4 = 5$. We know that a disjunction of two statements is true if either of the two statements is true. Therefore $4 \leq 5$ is a true statement because $4 < 5$ is true. Similarly, $5 \leq 5$ (which means $5 < 5$ or $5 = 5$) is a true statement because $5 = 5$ is true.

Most of the numbers in our geometry are *positive* numbers and represent lengths of segments, measures of angles, measures of plane

regions (areas), or measures of regions in space (volumes). From Definition 6.3 we can conclude that if x, y, and z are positive numbers and if $x = y + z$, then both y and z are less than x. We get this by first considering z as the positive number p in the definition and then considering y as the positive number p. This gives us the two statements

$$x = y + p \qquad \text{and} \qquad x = z + p.$$

Thus, by Definition 6.3, $y < x$ and $z < x$. That is, both numbers in a sum of two positive numbers are less than the sum. For example,

$$15 = 9 + 6.$$

Hence $9 < 15$ and $6 < 15$. From the second part of Definition 6.3, we can conclude that the sum of two positive numbers is greater than either of them. For example,

$$32 = 21 + 11.$$

Hence $32 > 21$ and $32 > 11$.

The following theorem is easy to prove using Definition 6.3.

THEOREM 6.1 If x and y are numbers, then $x < y$ if and only if $y > x$.

There are two parts to Theorem 6.1.

1. If x and y are numbers, and if $x < y$, then $y > x$.
2. If x and y are numbers, and if $y > x$, then $x < y$.

Proof of part 1:

Statement	Reason
1. x and y are numbers and $x < y$.	1. Hypothesis
2. $y = x + p$, where p is a positive number.	2. Definition 6.3
3. $y > x$	3. Statement 2 and Definition 6.3

Proof of part 2: Assigned as an exercise.

We now state five properties of order (inequalities) that are helpful in proving theorems about geometric inequalities. You may consider these properties as postulates for the real number system, although we could prove Properties O–3, O–4, and O–5 by using Definition 6.3 and Properties O–1 and O–2.

O–1 **The Positive Closure Property.** If x and y are numbers, and $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$.

O–2 **Trichotomy Property.** If x and y are numbers, then exactly one of the following is true: $x < y$, $x = y$, $x > y$.

O–3 **Transitive Property.** If x, y, and z are numbers, and if $x < y$ and $y \leq z$, then $x < z$. Also, if $x > y$ and $y \geq z$, then $x > z$.

O–4 **Addition Property.** If a, b, x, and y are numbers, and if $x < y$ and $a \leq b$, then $x + a < y + b$. Also, if $x > y$ and $a \geq b$, then $x + a > y + b$.

O–5 **Multiplication Property.** If x, y, and z are numbers and if $x < y$ and $z > 0$, then $xz < yz$. Also, if $x > y$ and $z > 0$, then $xz > yz$.

We now prove several theorems which are useful in the sections that follow.

THEOREM 6.2 $\overline{AB} > \overline{CD}$ if and only if $\overline{CD} < \overline{AB}$.

There are two parts to Theorem 6.2.

1. If $\overline{AB} > \overline{CD}$, then $\overline{CD} < \overline{AB}$.
2. If $\overline{CD} < \overline{AB}$, then $\overline{AB} > \overline{CD}$.

Proof of part 1:

Statement	Reason
1. $\overline{AB} > \overline{CD}$	1. Hypothesis
2. $AB > CD$	2. Definition 6.1
3. $CD < AB$	3. Theorem 6.1
4. $\overline{CD} < \overline{AB}$	4. Definition 6.1

Proof of part 2: Assigned as an exercise.

THEOREM 6.3 $\angle ABC > \angle DEF$ if and only if $\angle DEF < \angle ABC$.

Proof: Assigned as an exercise.

THEOREM 6.4 Let three distinct collinear points A, B, C be given. Then A–C–B if and only if $AB > AC$ and $AB > BC$.

There are two parts to Theorem 6.4. (Draw a figure showing the relationship between points A, B, and C.)

1. Given three distinct collinear points A, B, C, if point C is between points A and B, then $AB > AC$ and $AB > BC$.
2. Given three distinct collinear points A, B, C, if $AB > AC$ and $AB > BC$, then C is between A and B.

Proof of part 1: Since C is between A and B, $AB = AC + BC$. Why? But AC and BC are positive. Hence by Definition 6.3 $AB > AC$ and $AB > BC$.

Proof of part 2: Points A, B, C are collinear (given), so exactly one of them is between the other two. Why? That is, we must have exactly one of the following three betweenness relations: B–A–C, A–B–C, or A–C–B. We shall show that the first two of these are impossible; hence the only remaining possibility is A–C–B. Suppose that A is between B and C. Then, by the first part of Theorem 6.4, we must have $BC > AB$. But this contradicts the hypothesis that $AB > BC$; hence A is not between B and C. Now suppose B is between A and C. Again, by part 1 of Theorem 6.4, we must have $AC > AB$ and this contradicts the hypothesis that $AB > AC$; hence B is not between A and C. The only remaining possibility is that C is between A and B. This completes the proof.

THEOREM 6.5 If point D is in the interior of $\angle ABC$, then

$$m\angle ABC > m\angle ABD \qquad \text{and} \qquad m\angle ABC > m\angle DBC.$$

Proof: Since D is in the interior of $\angle ABC$, ray $\overrightarrow{BD}$ is between rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. By the Angle Measure Addition Postulate,

$$m\angle ABC = m\angle ABD + m\angle DBC.$$

Since $m\angle ABD$ and $m\angle DBC$ are positive, it follows from Definition 6.3 that

$$m\angle ABC > m\angle ABD \qquad \text{and} \qquad m\angle ABC > m\angle DBC.$$

EXERCISES 6.2

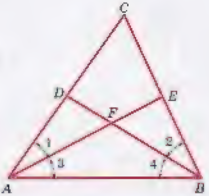
In Exercises 1–10, identify the order property that is illustrated.

1. If $AB < 6$, then $AB \neq 6$.
2. If $a - b < 15$ and $b < 3$, then $a < 18$.
3. If $x < 7$ and $7 < y$, then $x < y$.

4. If $a < 5$, then $4a < 20$.
5. If $AB < RS$ and $BC < ST$, then $AB + BC < RS + ST$.
6. If $\frac{1}{2}m\angle ABC > \frac{1}{2}m\angle RST$, then $m\angle ABC > m\angle RST$.
7. If $AB > CD$ and $CD = EF$, then $AB > EF$.
8. If $x + 3 < 8$, then $x < 5$.
9. If $x - y > 12$ and $y = 7$, then $x > 19$.
10. If $3 > 0$ and $2 > 0$, then $2 + 3 > 0$.

11. In the figure, $AD > BE$ and $DC > EC$. Prove $AC > BC$.

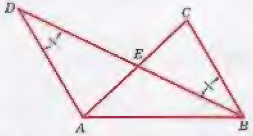
12. Given the figure for Exercise 11, if $m\angle 1 < m\angle 2$ and $m\angle 3 < m\angle 4$, prove that $m\angle ABC > m\angle BAC$.

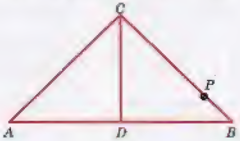


13. If $m\angle A = 90 + k$, where $k > 0$, and $\angle B$ is a supplement of $\angle A$, prove that $\angle B$ is an acute angle.

14. In the figure, $\angle D \cong \angle DBC$. Prove that $m\angle ABC > m\angle D$.

15. In the figure, $\overline{CD} \perp bis\ \overline{AB}$ and C–P–B. Prove that $\overline{AC} > \overline{CP}$.



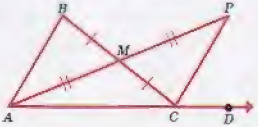


16. Prove part 2 of Theorem 6.1.
17. Prove part 2 of Theorem 6.2.
18. Prove Theorem 6.3.
19. Explain why Theorem 6.4 has the following consequence. If A, B, C are three distinct collinear points, then C is between A and B if and only if $AC < AB$ and $BC < AB$.
20. Explain why Theorem 6.5 has the following consequence. If D is a point in the interior of $\angle ABC$, then

$$m\angle ABD < m\angle ABC \qquad \text{and} \qquad m\angle DBC < m\angle ABC.$$

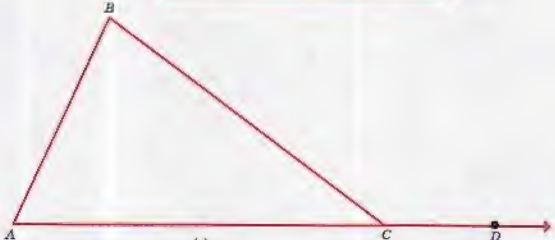
21. In the figure, M is the midpoint of $\overline{AP}$ and $\overline{BC}$. Prove that

$$m\angle BCD > m\angle B.$$



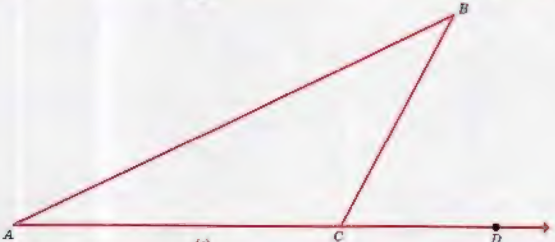
22. For each figure use your protractor to measure $\angle BCD$, $\angle A$, and $\angle B$. Record your results in a table. How does $m\angle BCD$ compare with $m\angle A$ in every case? $m\angle BCD$ with $m\angle B$ in every case?

	$m\angle BCD$	$m\angle A$	$m\angle B$
(a)	?	?	?
(b)	?	?	?
(c)	?	?	?



(a)

(b)



(c)

23. CHALLENGE PROBLEM. Given the coplanar points A, B, C, D, M, P such that A, B, C are noncollinear, $A\text{–}C\text{–}D$, $B\text{–}M\text{–}C$, and $A\text{–}M\text{–}P$, prove that P is in the interior of $\angle BCD$. (*Hint:* You must show that P is on the B-side of $\overleftrightarrow{CD}$ and that P is on the D-side of $\overleftrightarrow{BC}$.)

24. CHALLENGE PROBLEM. Prove the Transitive Property of Order (Property 6–3). (*Hint:* You will need to use Definitions 6.3 and 6.4 and the Addition Property of Equality in your proof.)

6.3 THE EXTERIOR ANGLE THEOREM

In both figures of Figure 6-1, $\angle ABC$, $\angle BCA$, and $\angle CAB$ are called *interior* angles of $\triangle ABC$. We call $\angle BCD$, which is adjacent to $\angle ACB$ and forms a linear pair with it, an *exterior* angle of $\triangle ABC$.

Figure 6-1

Both $\angle A$ and $\angle B$ are called *nonadjacent interior angles* of the exterior angle $\angle BCD$. The term "interior angle" is convenient when you want to emphasize the distinction between an exterior angle of a triangle and an angle of a triangle. Note that the adjectives "adjacent" and "nonadjacent" apply to an interior angle and describe its relation to a particular exterior angle. We formalize these ideas in the following definition.

Definition 6.5 Each angle of a triangle is called an **interior** angle of the triangle. An angle which forms a linear pair with an interior angle of a triangle is called an **exterior** angle of the triangle. Each exterior angle is said to be **adjacent** to the interior angle with which it forms a linear pair and **nonadjacent** to the other two interior angles of the triangle.

Every triangle has six exterior angles, two at each vertex, as shown in Figure 6-2. The two exterior angles at each vertex are vertical angles and hence are congruent.

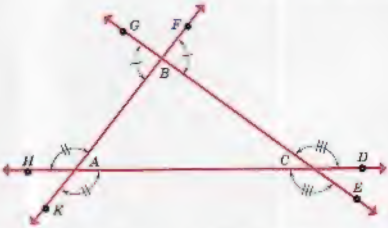


Figure 6-2

Note that $\angle DCE$ in Figure 6-2 is not an exterior angle. Why? In Figure 6-2, $\angle KAC$ and $\angle HAB$ are the two exterior angles at vertex A of $\triangle ABC$, and $\angle ABC$ and $\angle BCA$ are the two nonadjacent interior angles of each of them. Name the two exterior angles at vertex B of $\triangle ABC$ in Figure 6-2 and their nonadjacent interior angles.

If you worked Exercise 22 of Exercises 6.2, your results should suggest the following theorem.

> *THEOREM 6.6* ***(The Exterior Angle Theorem)*** Each exterior angle of a triangle is greater than either of its nonadjacent interior angles.

Proof: Let the vertices of the triangle be A, B, and C. Let D be a point on $\overrightarrow{AC}$ such that C is between A and D. (See Figure 6-3.) We must prove that

$$\angle BCD > \angle BAC$$

and that

$$\angle BCD > \angle B.$$

We first prove that

$$\angle BCD > \angle B.$$

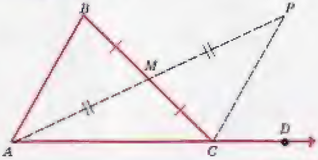


Figure 6-3

Let M be the midpoint of $\overline{BC}$ and let P be the point on $\overrightarrow{AM}$ such that A–M–P and $AM = MP$. Then $\triangle AMB \cong \triangle PMC$ by S.A.S. (show this) and $m\angle BCP = m\angle B$. Why? Since P is in the interior of $\angle BCD$, $m\angle BCD > m\angle BCP$ by Theorem 6.5. We have shown that

$$m\angle BCD > m\angle BCP$$

and that

$$m\angle BCP = m\angle B.$$

It follows from the Transitive Property (0–3) that $m\angle BCD > m\angle B$, and from Definition 6.2 we have $\angle BCD > \angle B$.

To prove that $\angle BCD > \angle BAC$, we use the midpoint of $\overline{AC}$ and show that the other exterior angle at C ($\angle ACE$ in Figure 6-4) is greater than $\angle BAC$ in the same way as in the part above. Since the two exterior angles at C are congruent (Why?), it follows that

$$\angle BCD > \angle BAC.$$

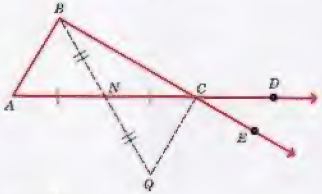


Figure 6-4

The proof that we have given for one exterior angle of the triangle can be easily modified to show that the theorem holds for any of the six exterior angles of the triangle. However, it is not necessary to go to all this trouble since our choice of the exterior angle at C was strictly an arbitrary choice. The fact that we were able to prove the theorem by choosing arbitrarily any one of the six exterior angles of $\triangle ABC$ insures us that there is no need to prove the theorem for every exterior angle.

In the proof of Theorem 6.6, we stated that point P (in Figure 6-3) is in the interior of $\angle BCD$. You are asked to prove this in the Exercises at the end of this section.

COROLLARY 6.6.1 If one of the angles of a triangle is a right angle, then the other two angles of the triangle are acute angles.

Proof: Assigned as an exercise.

COROLLARY 6.6.2 If one of the angles of a triangle is an obtuse angle, then the other two angles of the triangle are acute angles.

Proof: Assigned as an exercise.

It follows from the Exterior Angle Theorem (more directly from its corollaries, Corollary 6.6.1 and Corollary 6.6.2) that no triangle has more than one right angle or more than one obtuse angle. An important kind of a triangle is one that has one right angle. There are special names for triangles with a *right angle* and for triangles with an *obtuse angle*.

Definition 6.6 A **right triangle** is a triangle with one right angle. The **hypotenuse** of a right triangle is the side opposite the right angle. The other two sides of a right triangle are called **legs.**

Definition 6.7 An **obtuse triangle** is a triangle with one obtuse angle.

Definition 6.8 An **acute triangle** is a triangle with three acute angles.

Theorem 4.14 asserts that for each point on a line in a plane, there is one and only one line which (1) lies in the given plane, (2) contains the given point, and (3) is perpendicular to the given line. We can now use the Exterior Angle Theorem to prove a companion theorem.

THEOREM 6.7 Given a line and a point not on the line, there is one and only one line which contains the given point and which is perpendicular to the given line.

Proof: Let l be the given line and P the given point not on l. In part 1 of the proof we show that there is a line containing P and perpendicular to l. In part 2 we show there cannot be two such lines.

1. **Existence.** (See Figure 6-5.) Let A and B be any two points of line l. Then $\overleftrightarrow{PA}$ is either perpendicular to l or not perpendicular to l. If $\overleftrightarrow{PA} \perp l$, then the existence part of our proof is complete. If $\overleftrightarrow{PA}$ is not perpendicular to l, then there is a ray $\overrightarrow{AC}$, with C on the opposite side of l from P such that $\angle PAB \cong \angle BAC$. Why? There is a point D on $\overrightarrow{AC}$ such that $\overline{AD} \cong \overline{AP}$. Why? P and C are on opposite sides of l, and C and D are on the same side of l; hence P and D are on opposite sides of l. Therefore $\overline{PD}$ intersects l at some point F. $\triangle PAF \cong \triangle DAF$ by S.A.S. (show this) and so $\angle PFA \cong \angle DFA$. Therefore, by Theorem 4.12, $\angle PFA$ is a right angle and $\overleftrightarrow{PF} \perp l$.

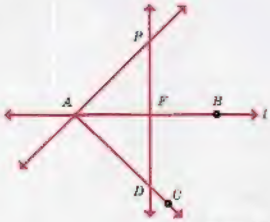


Figure 6-5

2. **Uniqueness.** (See Figure 6-6.) Let G be any point of l other than F and let H be any point of l such that G–F–H. Then $\angle PGF$ is an angle of $\triangle PGF$ and is a nonadjacent interior angle of the exterior angle $\angle PFH$. Since $m\angle PFH = 90$, it follows from the Exterior Angle Theorem that $m\angle PGF < 90$ and therefore $\overleftrightarrow{PG}$ is not perpendicular to l. It follows that $\overleftrightarrow{PF}$ is the only line through P and perpendicular to l.

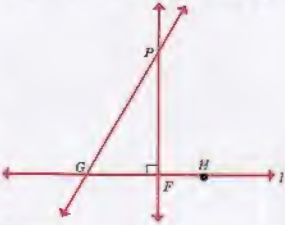
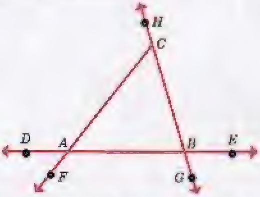


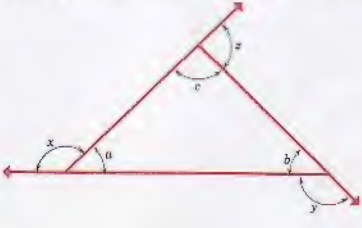
Figure 6-6

EXERCISES 6.3

1. Refer to Figure 6-4 and prove that $\angle BCD > \angle BAC$, thus completing the proof of Theorem 6.6.
2. Prove Corollary 6.6.1.
3. Prove Corollary 6.6.2.
4. Given $\triangle ABC$ with B-C-D and $m\angle ACD = 70$, what must be true about $m\angle ABC$? About $m\angle BAC$? About $m\angle ACB$?
5. In the figure, name the two nonadjacent interior angles of $\angle BAF$. Which exterior angle has $\angle CAB$ and $\angle ABC$ as its nonadjacent interior angles? Name all the exterior angles shown in the figure and their corresponding nonadjacent interior angles.

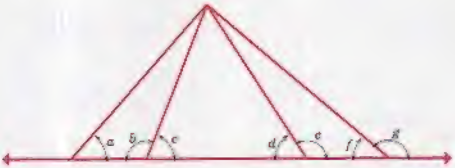


Using the figure, copy Exercises 6–11 and replace the question marks by $<$, $=$, or $>$ to make a true statement. a, b, c, x, y, z denote angle measures.



6. If $b = 40$, then x $\boxed{?}$ 40 and y $\boxed{?}$ 140.
7. If $c = 90$, then z $\boxed{?}$ 90, a $\boxed{?}$ 90, b $\boxed{?}$ 90, x $\boxed{?}$ 90, and y $\boxed{?}$ 90.
8. If $a = 40$ and $b = 60$, then z $\boxed{?}$ 60.
9. If $y = 140$, then a $\boxed{?}$ 140 and c $\boxed{?}$ 140.
10. If $x = 130$, then b $\boxed{?}$ 130.
11. If $a = 55$ and $c = 80$, then y $\boxed{?}$ 80.

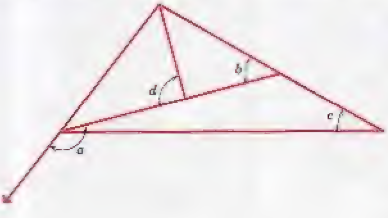
Refer to the figure for Exercises 12–18. In each exercise, arrange the numbers in order, starting with the smallest. In these exercises, a, b, c, d, e, f, g denote angles.



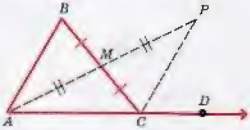
12. $m\angle a$, $m\angle c$
13. $m\angle b$, $m\angle d$
14. $m\angle e$, $m\angle c$
15. $m\angle b$, $m\angle f$
16. $m\angle c$, $m\angle a$, $m\angle e$
17. $m\angle c$, $m\angle e$, $m\angle g$
18. $m\angle c$, $m\angle a$, $m\angle g$, $m\angle e$

19. In the figure, prove that $m\angle ACB > m\angle B$.

20. List the angles marked in the figure in increasing order of size.



21. In the proof of Theorem 6.6 it was stated that the point P (Figure 6-3) was in the interior of $\angle BCD$. To prove this, we must show that P is on the D-side of $\overleftrightarrow{BC}$ and on the B-side of $\overleftrightarrow{CD}$.



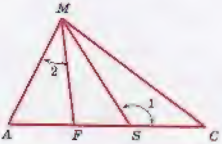
Copy and supply the missing reasons in the following proof.

Statement	Reason
1. A–C–D, A–M–P, B–M–C	1. Given
2. M and P are on the same side of $\overleftrightarrow{CD}$.	2. Theorem 2.2
3. M and B are on the same side of $\overleftrightarrow{CD}$.	3. $\boxed{?}$
4. P is on the B-side of $\overleftrightarrow{CD}$.	4. Statements 2 and 3
5. A and D are on opposite sides of $\overleftrightarrow{BC}$.	5. $\boxed{?}$
6. A and P are on opposite sides of $\overleftrightarrow{BC}$.	6. $\boxed{?}$
7. P is on the D-side of $\overleftrightarrow{BC}$.	7. Statements 5 and 6
8. P is in the interior of $\angle BCD$.	8. $\boxed{?}$ ($\boxed{?}$, $\boxed{?}$)

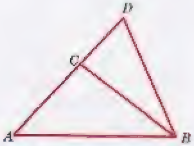
22. Is it possible for a triangle to have two right angles? Justify your answer.

23. Suppose that l is a line in plane α and that P is a point not in α. Does Theorem 6.7 still apply? Draw a figure and explain your answer.

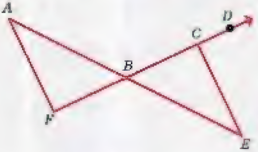
24. *Given:* $\triangle AMC$, A–F–S, F–S–C
Prove: $\angle 1 > \angle 2$



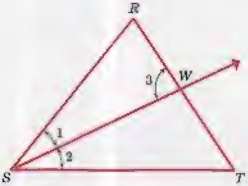
25. *Given:* $\triangle ADB$, A–C–D, $\overline{AD} \cong \overline{AB}$
Prove: $\angle ACB > \angle DBA$



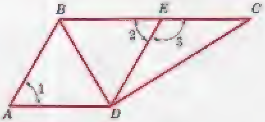
26. *Given:* B is the midpoint of $\overline{AE}$, B is the midpoint of $\overline{FC}$, B–C–D
Prove: $\angle DCE > \angle A$



27. *Given:* $\triangle RTS$, R–W–T, $\overrightarrow{SW}$ is the midray of $\angle RST$
Prove: $\angle 3 > \angle 1$



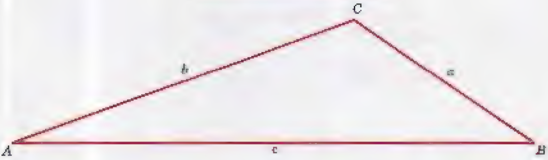
28. *Given:* Quadrilateral $ABCD$, B–E–C, $\overline{BD} \cong \overline{DE}$, $\angle 1 \cong \angle 2$
Prove: $\angle 3 > \angle 1$



29. *Given:* F is the midpoint of $\overline{AD}$ and $\overline{BE}$
Prove: $\angle 2 < \angle 1$

30. Measure the sides in centimeters and the angles in degrees of the scalene triangle in the figure. Record the measurements to the nearest tenth of a centimeter and the nearest degree in a table as shown. You will need to refer to the results of this exercise in Section 6.4.

Sides	*Angles*
$a = \boxed{?}$	$m\angle A = \boxed{?}$
$b = \boxed{?}$	$m\angle B = \boxed{?}$
$c = \boxed{?}$	$m\angle C = \boxed{?}$



31. Draw three scalene triangles of different sizes and shapes and label the vertices and sides as in Exercise 30. Also make and record measurements for each triangle. You will need to refer to the results of this exercise in Section 6.4.

32. CHALLENGE PROBLEM. If P is any point in the interior of $\triangle ABC$, prove that $m\angle APB > m\angle C$.

33. CHALLENGE PROBLEM. Prove that the sum of the measures of any two angles of a triangle is less than 180.

6.4 INEQUALITIES INVOLVING TRIANGLES

If all three sides of a triangle are congruent, then the three angles of the triangle are congruent, and if two sides of a triangle are congruent, then the angles opposite these sides are congruent. In this section we investigate how the angles of a triangle are related to each other when they are opposite sides of unequal length.

Refer to the table you prepared in Exercise 30 of Exercises 6.3 and answer the following questions.

Which side is the longest?
Which angle is the largest?
How are the longest side and the largest angle of $\triangle ABC$ situated with respect to each other?
Which side is the shortest?
Which angle is the smallest?
How are the shortest side and the smallest angle of $\triangle ABC$ situated with respect to each other?

Observe the order relation among the measures of the angles of $\triangle ABC$ and complete the following statement:

$$m\angle \boxed{?} > m\angle \boxed{?} > m\angle \boxed{?}.$$

Observe the order relation among the measures of the sides of $\triangle ABC$ and complete the following statement:

$$\boxed{?} > \boxed{?} > \boxed{?}.$$

How do the order relations among the measures of the angles of the triangle compare with the order relations among the measures of the corresponding opposite sides of the triangle?

Refer to the tables you prepared for the three triangles in Exercise 31 of Exercises 6.3 and answer the same questions as the preceding ones for each of these triangles. Make a general statement about the relative position of the longest side and the largest angle of a triangle; the shortest side and the smallest angle.

The comparisons suggested by the preceding experiment are formulated in the following two theorems.

THEOREM 6.8 (***Angle-Comparison Theorem***) If two sides of a triangle are not congruent, then the angles opposite them are not congruent and the greater angle lies opposite the greater side.

Proof:

Given: $\triangle ABC$,
with $\overline{AB} > \overline{AC}$

To Prove: $\angle ACB > \angle ABC$ (See Figure 6-7.)

Figure 6-7

Statement	Reason
1. $\overline{AB} > \overline{AC}$	1. Given
2. $AB > AC$	2. Definition 6.1
3. There is a point D on $\overrightarrow{AB}$ such that $\overline{AD} \cong \overline{AC}$.	3. Segment Construction Theorem
4. $AD = AC$	4. Definition of congruence for segments (3)
5. $AB > AD$	5. Substitution Property of Equality (2, 4)
6. D is between A and B.	6. Theorem 6.4
7. D is in the interior of $\angle ACB$.	7. Theorem 4.11
8. $m\angle ACB > m\angle ACD$	8. Theorem 6.5
9. $\angle ACD \cong \angle ADC$	9. Isosceles Triangle Theorem (3)
10. $m\angle ACD = m\angle ADC$	10. Definition of congruence for angles (9)
11. $m\angle ACB > m\angle ADC$	11. Transitive Property (8, 10)
12. $\angle ADC > \angle ABC$	12. Exterior Angle Theorem
13. $m\angle ADC > m\angle ABC$	13. Definition 6.2 (12)
14. $m\angle ACB > m\angle ABC$	14. Transitive Property (11, 13)
15. $\angle ACB > \angle ABC$	15. Definition 6.2 (14)

We now state and prove the converse of Theorem 6.8.

THEOREM 6.9 (***Side-Comparison Theorem***) If two angles of a triangle are not congruent, then the sides opposite them are not congruent and the greater side lies opposite the greater angle.

Proof: Let $\triangle ABC$ be any triangle with two angles that are not congruent. Suppose it has been named so that these noncongruent angles are $\angle B$ and $\angle C$ and such that $\angle C > \angle B$. In terms of this situation the hypothesis and the conclusion to be proved are as follows.

Hypothesis: $\angle C > \angle B$
Conclusion: $\overline{AB} > \overline{AC}$
(See Figure 6-8.)

Figure 6-8

Since AB and AC are numbers, one of the following must hold:

$$\begin{array}{ll} (1) & AB < AC \\ (2) & AB = AC \\ (3) & AB > AC \end{array}$$

Which property of numbers are we using here?

The method of proof is to show that (1) and (2) are impossible, so (3) must hold, thus proving the theorem.

(1) If $AB < AC$, then, by Theorem 6.8, $\angle C < \angle B$. This contradicts the hypothesis; hence $AB < AC$ is impossible.
(2) If $AB = AC$, then $\triangle ABC$ is isosceles and $\angle C \cong \angle B$. Again this contradicts the hypothesis; thus we see that $AB = AC$ is impossible.

It follows then that $AB > AC$ so that $\overline{AB} > \overline{AC}$ and the desired conclusion has been proved.

The following two corollaries follow immediately from Theorem 6.9 and Corollary 6.6.1.

COROLLARY 6.9.1 The hypotenuse of a right triangle is the longest side of the triangle.

Proof: Assigned as an exercise.

COROLLARY 6.9.2 The shortest segment from a point to a line not containing the point is the segment perpendicular to the line.

RESTATEMENT: Given a line l and a point P that is not on l, if $\overline{PA} \perp l$ at A and B is any other point of l, then $\overline{PA} < \overline{PB}$. (See Figure 6-9.)

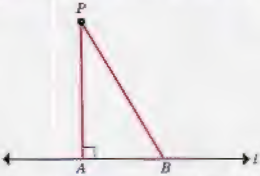


Figure 6-9

Proof: Assigned as an exercise.

When we speak of the distance between a point P and a line l, we naturally mean the *shortest* distance from P to l. It follows from Corollary 6.9.2 that there is such a shortest distance, and so we make the following definition.

Definition 6.9 The **distance** between a point and a line not containing the point is the length of the perpendicular segment joining the point to the line. The distance between a line and a point on the line is defined to be zero.

It is customary to associate three "distances between a point and a line" with every triangle. With $\triangle ABC$ there is associated the distance between A and $\overleftrightarrow{BC}$, the distance between B and $\overleftrightarrow{CA}$, and the distance between C and $\overleftrightarrow{AB}$. Any side (or its length) of a triangle may be thought of as the base. Associated with each base is the segment (or its length) joining the opposite vertex to a point of the line containing the base.

The following definition has two parts. In (1), we define base and altitude, thought of as segments. In (2), we define base and altitude, thought of as numbers (lengths of segments or distances).

Definition 6.10
1. Any side of a triangle is a **base** of that triangle. Given a base of a triangle, the segment joining the opposite vertex to a point of the line containing its base, and perpendicular to the line containing the base, is the **altitude** corresponding to that base.
2. The length of any side of a triangle is a **base** of that triangle. The distance between the opposite vertex and the line containing that side is the corresponding **altitude**.

Figure 6-10 shows two triangles, $\triangle ABC$ and $\triangle A'B'C'$. In each triangle the segments from the vertices to the lines containing the opposite sides have been drawn. These segments are the altitudes of the triangles. Thus $\overline{AD}$ is an altitude of $\triangle ABC$. It is the altitude from A to side $\overline{BC}$. The point D is the foot of the altitude from A to $\overline{BC}$. Note for an acute triangle, such as $\triangle ABC$ in the figure, that the foot of each altitude is an interior point of a side. Note for an obtuse triangle, such as $\triangle A'B'C'$ in the figure, that the feet of two altitudes are not points of the triangle. Even though point E', for example, is not a point of side $\overline{A'C'}$, it is frequently called the foot of the altitude from B' to side $\overline{A'C'}$.

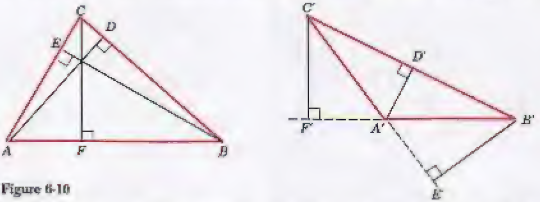


Figure 6-10

Draw a right triangle, $\triangle ABC$, with the right angle at C. Draw the altitude from C to the hypotenuse. Is the foot of this altitude a point of the triangle? Name the other two altitudes of $\triangle ABC$. Are the feet of these altitudes points of the triangle?

In Chapter 3 we postulated that if A, B, C are noncollinear points, then for distances in any system, $AB + BC > AC$. The postulate was called the Triangle Inequality Postulate. We now have the necessary geometric properties to prove this statement as a theorem.

THEOREM 6.10 (***Triangle Inequality Theorem***) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Proof: Given any triangle, it follows from the Trichotomy Property for numbers that there is one side of the triangle which is at least as long as each of the other two sides. Suppose, in $\triangle ABC$, that $BC \geq AB$ and that $BC \geq AC$. (See Figure 6-11.)

Figure 6-11

We must prove the following three statements:

$$\begin{aligned} &(1) \quad AB + BC > AC \\ &(2) \quad AC + BC > AB \\ &(3) \quad AB + AC > BC \end{aligned}$$

(1) By hypothesis, $BC \geq AC$. Since $AB > 0$, (that is, AB is a positive number), we have

$$AB + BC > AC$$

by the Addition Property of Order (O-4).

(2) By hypothesis, $BC \geq AB$. Since $AC > 0$, we have

$$AC + BC > AB. \qquad \text{Why?}$$

(3) On *opp* $\overrightarrow{AC}$ choose point D so that $AD = AB$. (See Figure 6-12.)

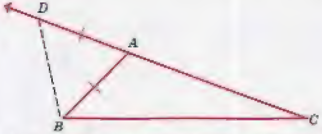


Figure 6-12

Since A is between C and D, A is in the interior of $\angle DBC$. Also $m\angle DBC > m\angle ABD$ by Theorem 6.5. But $m\angle ABD = m\angle ADB$ by the Isosceles Triangle Theorem, and so

$$m\angle DBC > m\angle ADB.$$

Therefore, by the Side-Comparison Theorem (applied to $\triangle DBC$),

$$DC > BC.$$

Since $DC = DA + AC$ (Why?) and $DA = AB$, we have

$$DC = AB + AC.$$

We have shown that $DC > BC$ and that $DC = AB + AC$. Therefore, by the Substitution Property,

$$AB + AC > BC.$$

It follows from Theorem 6.10 that, in informal language, the shortest distance between two points is the length of the segment joining them. In formal geometry, of course, once a unit of distance is given, there is only one distance between two points.

It follows from Theorems 6.8 and 6.9 that, in any one triangle, the greater angle lies opposite the greater side and conversely, the greater side lies opposite the greater angle. Let us now consider two companion theorems involving two triangles.

In Figure 6-13, $AB = A'B'$ and $BC = B'C'$. What can you say about these triangles if $m\angle B = m\angle B'$? How do AC and $A'C'$ compare?

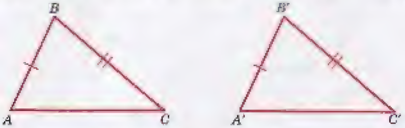


Figure 6-13

In Figure 6-14, we again have $AB = A'B'$ and $BC = B'C'$, but this time $m\angle B > m\angle B'$. Is the correspondence $ABC \longleftrightarrow A'B'C'$ a congruence? How do AC and $A'C'$ compare in this case?

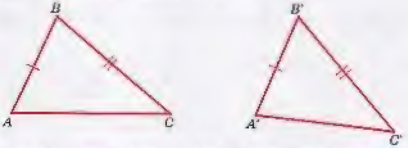


Figure 6-14

In Figure 6-15, we have $AB = A'B'$, $BC = B'C'$, and $m\angle B < m\angle B'$. Is the correspondence $ABC \longleftrightarrow A'B'C'$ a congruence? How do AC and $A'C'$ compare in this case?

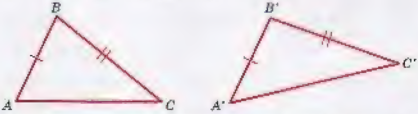


Figure 6-15

On the basis of the preceding discussion, it would appear that if two sides of one triangle are congruent, respectively, to two sides of a second triangle and if the corresponding included angles are not congruent, then the sides opposite these included angles are not congruent and the side opposite the larger angle is the larger side.

It may help you to understand this last statement by examining a pair of compasses. Observe that, as the angle formed by the pointers of the compasses gets larger, so does the distance between the ends of the pointers. We make this idea formal with our next theorem.

THEOREM 6.11 (***Side-Comparison Theorem for Two Triangles***) If two sides of one triangle are congruent, respectively, to two sides of a second triangle, and if the angle included by the sides of the first triangle is greater than the angle included by the sides of the second triangle, then the third side of the first triangle is greater than the third side of the second triangle.

RESTATEMENT: Given $\triangle ABC$ and $\triangle A'B'C'$ with $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$. If $\angle B > \angle B'$, then $\overline{AC} > \overline{A'C'}$. (See Figure 6-16.)

Figure 6-16

Proof: By the Angle Construction Theorem, there is a point Q on the C-side of $\overleftrightarrow{AB}$ such that $\angle ABQ \cong \angle A'B'C'$. (See Figure 6-17.) Since $\angle ABC > \angle A'B'C'$, Q is in the interior of $\angle ABC$. Choose point P on $\overrightarrow{BQ}$ such that $\overline{BP} \cong \overline{B'C'}$. There are now three cases to consider.

Case 1. P is in the exterior of $\triangle ABC$.
Case 2. P is on $\overline{AC}$.
Case 3. P is in the interior of $\triangle ABC$.

We consider only Case 1 here. The proofs of Cases 2 and 3 are left as exercises.

Proof of Case 1: In Figure 6-17, $\triangle ABP \cong \triangle A'B'C'$ by S.A.S. Let $\overrightarrow{BR}$ be the midray of $\angle CBP$ intersecting $\overline{AC}$ at E. Then $\triangle PBE \cong \triangle CBE$ by S.A.S. Applying the Triangle Inequality Theorem to $\triangle AEP$, we have $AE + EP > AP$.

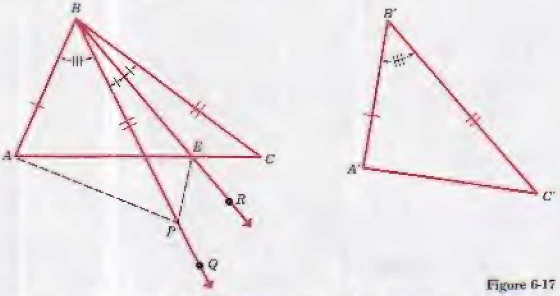


Figure 6-17

But $AP = A'C'$ (Why?) and $EP = EC$. Why? By the Substitution Property, we get $AE + EC > A'C'$, and by the Distance Betweenness Postulate, $AC > A'C'$. Therefore $\overline{AC} > \overline{A'C'}$ and this completes the proof for Case 1.

The converse of Theorem 6.11 is also true and we state it as our last theorem of this section.

THEOREM 6.12 **(*Angle-Comparison Theorem for Two Triangles*)** If two sides of one triangle are congruent, respectively, to two sides of a second triangle, and if the third side of the first triangle is greater than the third side of the second triangle, then the angle included by the two sides of the first triangle is greater than the angle included by the two sides of the second triangle.

RESTATEMENT: Given $\triangle ABC$ and $\triangle A'B'C'$ with $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, if $\overline{AC} > \overline{A'C'}$, then $\angle B > \angle B'$. (See Figure 6-18.)

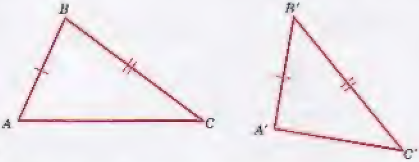


Figure 6-18

Proof: Since $m\angle B$ and $m\angle B'$ are numbers, one of the following must hold:

$$
\begin{array}{ll}
(1) & m\angle B < m\angle B' \\
(2) & m\angle B = m\angle B' \\
(3) & m\angle B > m\angle B'
\end{array}
$$

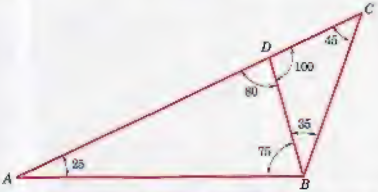
Which property of numbers are we using here?

The method of proof is to show that (1) and (2) are impossible, so (3) must hold, and the theorem will be proved. The problem of showing that (1) and (2) are impossible is left as an exercise.

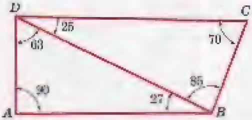
EXERCISES 6.4

1. Given $\triangle ABC$ with $AB = 12$, $BC = 15$, and $AC = 10$, name the angles in order of size beginning with the angle of least measure.
2. In $\triangle SKM$, $m\angle S = 47$, $m\angle K = 85$, and $m\angle M = 48$. Name the shortest side; the longest side.
3. Name the longest side of $\triangle ABC$ if (a) $m\angle A = 44$, $m\angle B = 90$; (b) $m\angle A = 120$, $m\angle B = 40$.
4. Does a triangle exist with the following numbers as side lengths? Why? (a) 5, 3, 10 (b) 5, 3, 8 (c) 5, 3, 4
5. Given $\triangle ABC$ with A-B-D, if $m\angle ABC > m\angle CBD$, prove $AC > BC$.

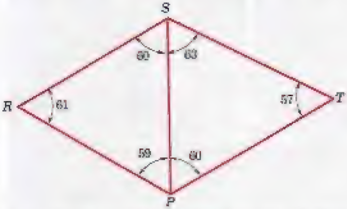
6. Given the following figure with angle measures as marked, for each of the three triangles name the sides of the triangle in order of increasing length.



7. In the figure for Exercise 6, which segment is the shortest?
8. Given the following figure with angle measures as marked, prove that $\overline{CD}$ is the longest segment.



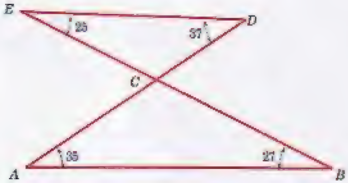
9. In the figure, if the angles have the indicated measures, which segment is shortest?



In Exercises 10–12, get the "best" answer you can in the sense that the smaller number is as large as possible and the larger number is as small as possible.

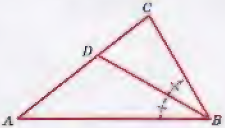
10. Copy and complete: If the lengths of two sides of a triangle are 7 and 12, then the length of the third side must be greater than $\boxed{?}$ and less than $\boxed{?}$.
11. Copy and complete: If the lengths of two sides of a triangle are 6 and 9, then the third side must have a length less than $\boxed{?}$ and greater than $\boxed{?}$.
12. Between what two numbers must the length of the third side of a triangle lie if two of its sides have lengths of 17 and 28?

13. Given A–C–D, E–C–B, and with angle measures as marked, prove that $BE > AD$.

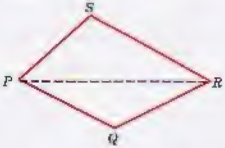


14. Prove that the sum of the lengths of the diagonals of a convex quadrilateral is less than the perimeter of the quadrilateral.

15. Given $\triangle ABC$ with D a point on $\overline{AC}$ such that $\overrightarrow{BD}$ bisects $\angle ABC$, prove that $AB > AD$.



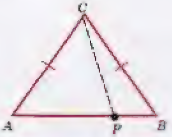
16. In the figure, $PS < SR$ and $PQ < QR$. Prove that $m\angle SPQ > m\angle SRQ$.



17. Prove Corollary 6.9.1. (*Hint:* You will need to use Corollary 6.6.1 and Theorem 6.9 in your proof.)

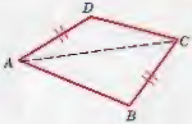
18. Prove Corollary 6.9.2. (See Figure 6.9.)

19. Given $AP > PB$, prove that $m\angle ACP > m\angle BCP$. (*Hint:* Use Theorem 6.12.)



20. Given $\triangle PQR$ with M the midpoint of $\overline{PQ}$, if $m\angle RMQ > m\angle PMR$, prove that $QR > PR$.

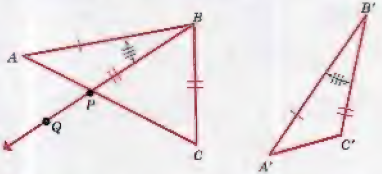
21. Given a convex quadrilateral $ABCD$ with $AD = BC$, if $AB > DC$, compare $m\angle CAD$ with $m\angle BCA$.



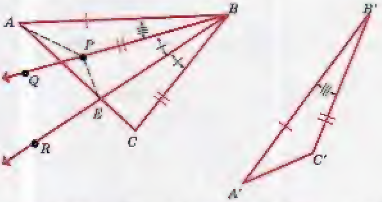
22. Prove the following theorem.

THEOREM If a, b, c are the side lengths of a triangle and if h_a is the altitude corresponding to base a, then $h_a \leq b$ and $h_a \leq c$.

23. CHALLENGE PROBLEM. Prove Case 2 of Theorem 6.11. This is the Case where point P is on $\overline{AC}$. (*Hint:* In Figure 6-17 for Theorem 6.11, recall that $\triangle ABP \cong \triangle A'B'C'$. You must show that $AC > A'C'$.)

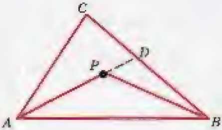


24. CHALLENGE PROBLEM. Prove Case 3 of Theorem 6.11. This is the Case where point P is in the interior of $\triangle ABC$. (*Hint:* Let $\overrightarrow{BR}$ be the bisector ray of $\angle CBP$, intersecting $\overline{AC}$ at E. Why is $\triangle PBE \cong \triangle CBE$? Why is $AE + EC > AP$? Prove $\overline{AC} > \overline{A'C'}$.)



25. CHALLENGE PROBLEM. Complete the proof of Theorem 6.12 by showing that Cases 1 and 2 are impossible. (*Hint:* Use Theorem 6.11 for Case 1 and the S.A.S. Postulate for Case 2.)

26. CHALLENGE PROBLEM. If P is any point in the interior of $\triangle ABC$, prove that $AP + PB < AC + CB$. (*Hint:* Let $\overrightarrow{AP}$ intersect $\overline{BC}$ at D. Apply the Triangle Inequality Theorem to $\triangle ACD$ and to $\triangle BDP$.)



CHAPTER SUMMARY

In this chapter we dealt with geometric inequalities involving angles and sides for any one triangle and also for two triangles. We defined GREATER THAN and LESS THAN for angles in terms of their measures and for segments in terms of their lengths.

We stated some order properties for numbers which are listed below by name only. You should know the complete statement of each of these properties.

- O–1 POSITIVE CLOSURE PROPERTY
- O–2 TRICHOTOMY PROPERTY
- O–3 TRANSITIVE PROPERTY
- O–4 ADDITION PROPERTY
- O–5 MULTIPLICATION PROPERTY

The key theorem concerning geometric inequalities is the EXTERIOR ANGLE THEOREM which states that an exterior angle of a triangle is greater than either of its two nonadjacent interior angles.

Other important theorems involving inequalities in any one triangle are listed below by name only. It is important that you be able to state these theorems in your own words and that you understand their proofs.

THEOREM 6.8 (*The Angle Comparison Theorem*)

THEOREM 6.9 (*The Side-Comparison Theorem*)

THEOREM 6.10 (*The Triangle Inequality Theorem*)

The following two theorems give inequalities concerning two triangles. Be sure that you know the complete statement of these theorems.

THEOREM 6.11 (*The Side-Comparison Theorem for Two Triangles*)

THEOREM 6.12 (*The Angle-Comparison Theorem for Two Triangles*)

We defined the DISTANCE between a point and a line not containing the point to be the length of the perpendicular segment joining the point to the line. The distance between a line and a point on the line is defined to be zero.

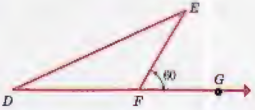
We proved that the hypotenuse of a right triangle is the longest side of the triangle and that the shortest segment from a point not on a line to the line is the segment perpendicular to the line.

REVIEW EXERCISES

In Exercises 1–7, identify the order property that is illustrated.

1. If $m\angle A > m\angle B$ and $m\angle B = m\angle C$, then $m\angle A > m\angle C$.
2. If $x - y > 5$ and $y = 4$, then $x > 9$.
3. If $m\angle A > m\angle B$, then $\frac{1}{2}m\angle A > \frac{1}{2}m\angle B$.
4. If $x \neq y$, then $x > y$ or $x < y$.
5. If $AB < RS$ and $BC < ST$, then $AB + BC < RS + ST$.
6. If $x + y > z$, then $x > z - y$.
7. If $a + b > c$ and $a + b < d$, then $d > c$.

8. In the figure, what must be true about $m\angle D$? About $m\angle DFE$? About $m\angle E$?



Refer to Figure 6-19 for Exercises 9–14. Copy each exercise and replace the question marks by the symbol $<$, $=$, or $>$ to make a true statement.

9. b [?] d
10. a [?] c [?] e
11. c [?] d
12. e [?] f
13. a [?] f
14. b [?] d [?] f [?] c [?] a

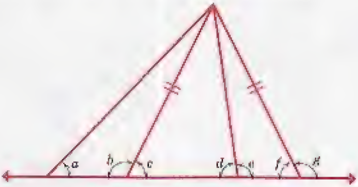
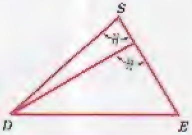
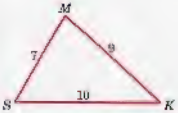


Figure 6-19

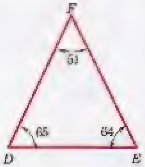
15. Which theorem do the markings on the figure contradict?



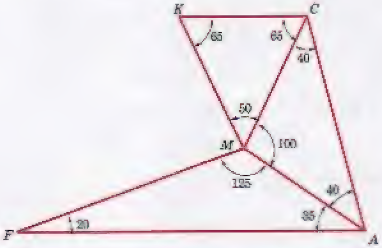
16. In the figure, which angle is the largest? The smallest?



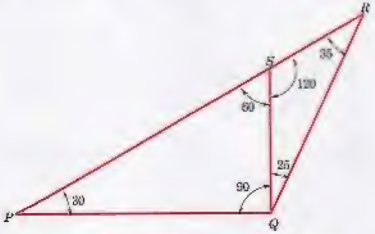
17. In the figure, which side is the longest? The shortest?



18. In the figure, if the angles have the indicated measures, which segment is shortest? What theorem did you use to decide?



19. In the figure, if the angles have the indicated measures, arrange the segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{QS}$, $\overline{PS}$, and $\overline{PR}$ in order of increasing length.



20. Copy and complete the following statement, putting the largest number in (a) and the smallest number in (b) which will make a true statement.

If the lengths of two sides of a triangle are 8 and 15, then the length of the third side must be greater than (a) [?] and less than (b) [?].

Exercises 21–26 refer to Figure 6-20 showing $\triangle ABC$ with A–D–C, B–E–C, B–F–D, and A–F–E. You should be able to defend your answers using theorems that you know.

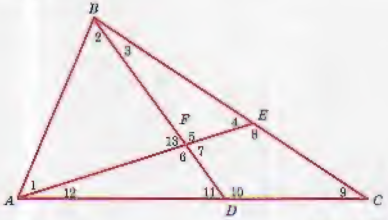
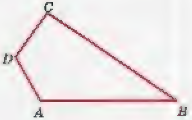


Figure 6-20

21. Name five angles whose measures are less than $m\angle 10$.
22. Name two angles whose measures are less than $m\angle 5$.
23. Name four angles whose measures are greater than $m\angle 1$.
24. $\angle 7$ is an exterior angle of which two triangles?
25. Is $\angle 6$ an exterior angle of $\triangle BFE$?
26. Is $m\angle 8 > m\angle 2$? Explain why.

27. In the figure, if $\overline{BC}$ is the longest side and $\overline{DA}$ is the shortest side, prove that $m\angle A > m\angle C$. (*Hint:* Draw $\overline{CA}$ and use the Angle-Comparison Theorem.)

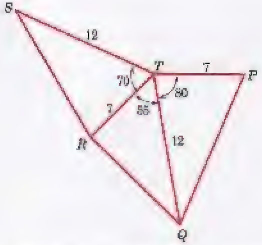


28. *Given:* $SK = KM$
    Prove: $KM > KJ$
    (*Hint:* Use the Exterior Angle Theorem, the Isosceles Triangle Theorem, and the Side-Comparison Theorem.)

29. For $\triangle ABC$ prove that $AB - BC < AC$.

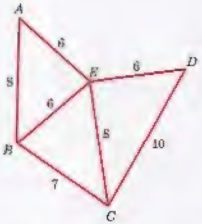
30. If, in $\triangle ABC$, $BC = AC$, prove that $AC > \frac{1}{2}AB$.

31. Given the figure with angle and side measures as marked, name the sides $\overline{SR}$, $\overline{RQ}$, and $\overline{QP}$ in order of increasing length. State a theorem that justifies your conclusions.



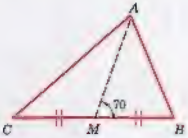
32. Given the figure with sides as marked, copy and insert $<$ or $>$ in (a), (b), and (c) so that each is a true statement.

(a) $m\angle A$ $\boxed{?}$ $m\angle BEC$
(b) $m\angle DEC$ $\boxed{?}$ $m\angle BEC$
(c) $m\angle A$ $\boxed{?}$ $m\angle DEC$

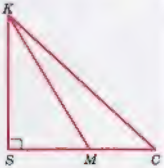


33. State theorems that justify your conclusions in Exercise 32.

34. *Given:* $\triangle ABC$ with median $\overline{AM}$
$m\angle AMB = 70$
Prove: $m\angle B > m\angle C$



35. If $\triangle SKM$ is a right triangle with S–M–C, prove that $KC > KM$.



36. CHALLENGE PROBLEM. If A, B, C, D are four distinct points in space and if no three of these points lie on a line, prove that

$$AB + BC + CD > DA.$$

37. CHALLENGE PROBLEM. Prove that in any triangle there are two sides of lengths r and s such that

$$1 \leq \frac{r}{s} < 2.$$

Chapter 7

David Plowden/Photo Researchers

Parallelism

7.1 INTRODUCTION

We are now ready to consider one of the most basic ideas in geometry, the idea of parallel lines. What do you think of when someone says "parallel lines?" Perhaps the lines which separate lanes on a straight running track, or the cracks between the boards in a floor, or the strings on a violin? What are some of the properties of parallel lines?

If two ships are sailing parallel courses close to one another as suggested in Figure 7-1, what is the relationship of the angles marked in the figure?

Figure 7-1

The congruence of these angles is an example of an idea from informal geometry that suggests a property of parallel lines in formal geometry.

The set of rails on a railroad track must fit wheels which are fixed so that the distance between them cannot change as they roll down the track. This suggests another property of parallel lines in formal geometry, the property of being the same distance apart everywhere.

What are the basic properties of parallel lines? Can you think of one or more properties such that if lines have these properties they should then be considered to be parallel lines? It would be natural to use such basic properties in deciding on a definition for parallel lines.

The most basic properties of points, lines, and planes are the Incidence Properties which were discussed in Chapter 1. Two different lines either intersect or they do not. If they do not intersect and are coplanar, we call them *parallel lines*. If they do not intersect and are not coplanar, we call them *skew lines*. We know that two distinct intersecting lines determine a plane. It follows that, if two lines are not coplanar, then they must be nonintersecting lines and hence are skew lines. We shall find it convenient to consider a line as parallel to itself. We begin our formal treatment with several definitions based on the foregoing ideas.

7.2 DEFINITIONS

Definition 7.1 Two distinct lines which are coplanar and nonintersecting are parallel lines, and each is said to be parallel to the other. Also, a line is parallel to itself. The lines in a set of lines are said to be parallel lines if each two in the set are parallel.

Definition 7.2 Two lines which do not lie in the same plane are called skew lines.

Notation. If p and q are lines, then $p \parallel q$ means that p is parallel to q, and $p \nparallel q$ means that p is not parallel to q.

Note that every pair of distinct parallel lines are coplanar, that every pair of distinct intersecting lines are coplanar, and that every pair of skew lines are noncoplanar. If p and q are lines, there are four distinct possibilities:

1. p and q are noncoplanar, in which case they are skew.
2. p and q are coplanar but not parallel, in which case they intersect in exactly one point and there is exactly one plane which contains them.
3. p and q are distinct parallel lines, in which case they do not intersect and there is exactly one plane which contains them.
4. p and q are nondistinct parallel lines, in which case p and q are the same line and there are infinitely many different planes containing p and q.

EXERCISES 7.2

Exercises 1–5 pertain to the rectangular box suggested in Figure 7-2.

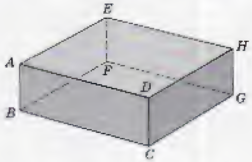


Figure 7-2

1. Using $A, B, C, \ldots$, designate two distinct edges which lie on parallel lines.
2. Using $A, B, C, \ldots$, designate two distinct edges which lie on intersecting lines.
3. Using $A, B, C, \ldots$, designate two distinct edges which lie on skew lines.
4. How many edges of the box lie on lines which are parallel to $\overleftrightarrow{AB}$?
5. How many edges of the box lie on lines which are skew to $\overleftrightarrow{AB}$?

6. If p and q are skew lines, why is there no plane which contains both?
7. If p is a line in a plane α, how many different lines in α are parallel to p?
8. If p is a line in a plane α, how many different lines in α are perpendicular to p?
9. If p is a line in a plane α, how many different lines in α are skew to p?
10. Prove that two distinct parallel lines determine exactly one plane.
11. Given that m is a line and P is a point on m, prove that there is one and only one line through P and parallel to m.
12. Let m be a line in a plane α and P a point in α but not on m. Use your knowledge regarding the existence of perpendicular lines and the Exterior Angle Theorem for triangles to prove that there is at least one line through P and parallel to m.

7.3 EXISTENCE OF PARALLEL LINES

Answering questions regarding existence is an important part of building a formal geometry. How do we know that there are such things as points, lines, and planes? We know that they exist because of the postulates and the theorems of Chapter 1 on incidence relations. Given two distinct points A and B on a line l, how do we know that there is a point C on l such that A–C–B and $AC = CB$? We know it because we can prove it! Our proof depends in a very essential way on the Ruler Postulate. Given a line $\overleftrightarrow{AB}$ in a plane α, how do we know that there exists a ray $\overrightarrow{AC}$ in α such that $m\angle BAC = 30$? We know it because we can prove it! Of course, the proof is easy once we adopt the Protractor Postulate.

Given a line l and a point P not on l, how do we know that there exists at least one line through P and parallel to l? It is true in some geometries that there are no parallel lines. But in our geometry, the geometry that we inherited from Euclid, there are parallel lines, and furthermore, we can prove it. In fact, you were asked to prove it in Exercises 11 and 12 of Exercises 7.2. These exercises are combined in the next theorem.

> *THEOREM 7.1* (***Existence of Parallel Lines Theorem***) If l is a line and P is a point, then there is at least one line through P and parallel to l. If P is on l, there is exactly one line through P and parallel to l.

Proof: There are two cases to consider: (1) P is on l and (2) P is not on l.

Case 1. We suppose first that P is on l. Since l is parallel to l, it follows that there is a line through P and parallel to l. If m is any line different from l and through P, then it intersects l in exactly one point and hence is not parallel to l. Therefore there is one and only one line through P and parallel to l.

Case 2. Suppose next that P is not on l. (See Figure 7-3.) Then P and l determine a plane. Why? Call it α.

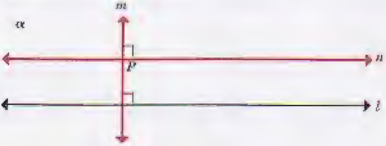


Figure 7-3

In α there is a unique line m through P and perpendicular to l (see Theorem 6.7) and a unique line n through P and perpendicular to m. We shall prove that n and l are parallel.

Suppose, contrary to what we want to prove, that n and l are not parallel. Then m, n, and l are distinct intersecting lines forming a triangle with an exterior angle and a nonadjacent interior angle both of which are right angles. (See Figure 7-4.) But this is impossible in view of the Exterior Angle Theorem. It follows that n and l are parallel. Therefore there is at least one line through P and parallel to l. This completes the proof.

Figure 7-4

Along with questions regarding existence in mathematics there are sometimes questions regarding uniqueness. Theorem 7.1 settles the matter of existence of parallel lines, but it does not settle the matter of uniqueness. If l is a line and P is a point *on* l, we know that there is *one and only one* line through P and parallel to l. If l is a line and P is a point *not on* l, we do not know yet that there is one and only one line through P and parallel to l. For about 2000 years following the time of Euclid, mathematicians tried to prove that, given a line and a point not on the line, there is a unique line through the given point and parallel to the given line. Finally two mathematicians, a Russian named Nikolai Ivanovitch Lobachevsky (1793–1856) and a Hungarian named Janos Bolyai (1802–1860), proved independently that it is impossible using only the postulates of Euclid (other than his parallel postulate) to prove the uniqueness of parallels. Since we want uniqueness of parallels, we follow in the footsteps of Euclid and adopt a "parallel postulate." We defer the statement of this postulate, however, to Section 7.6.

In Sections 7.4 and 7.5 we introduce the concept of a transversal and develop some theorems whose proofs do not depend on the Parallel Postulate. These are theorems that belong both to the ordinary geometry of Euclid and to the non-Euclidean geometry of Lobachevsky and Bolyai in which the Euclidean Parallel Postulate is replaced by a postulate which says that parallels are not unique. More specifically, the Bolyai-Lobachevsky Postulate states that if l is any line and P is any point not on l, then there are at least two distinct lines through P and parallel to l.

7.4 TRANSVERSALS AND ASSOCIATED ANGLES

Let p, q, and t be three distinct lines in a plane α. The line t may intersect both p and q or it may not. If t intersects both p and q, then it may intersect them in different points or it may intersect them in the same point. In Figure 7-5, p and q are intersecting lines and t intersects p and q in different points. In Figure 7-6, p and q are parallel lines and t intersects p and q in different points. In Figure 7-7, t intersects both p and q in the same point. In Figure 7-8, t intersects q but does not intersect p.

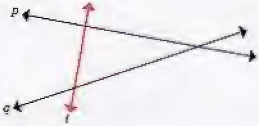


Figure 7-5

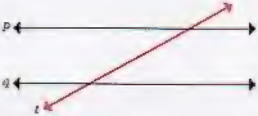


Figure 7-6

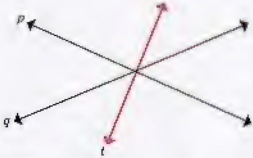


Figure 7-7

Figure 7-8

In situations like those in Figures 7-5 and 7-6 we say that t is a *transversal* of p and q. In each of these figures t intersects the union of p and q in a set consisting of two distinct points. On the other hand, in Figures 7-7 and 7-8, t does not intersect the union of p and q in a set consisting of two distinct points. As we said, in Figure 7-5, t is a transversal of p and q. In Figure 7-5 q is a transversal of p and t, and p is a transversal of q and t. We are now ready for the following formal definition.

Definition 7.3 A transversal of two distinct coplanar lines is a line which intersects their union in exactly two distinct points.

In Figure 7-9, A, B, C, D are four of the vertices of a rectangular box.

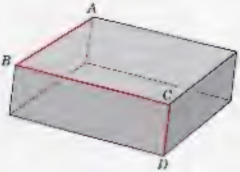


Figure 7-9

Note that

$\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$ are three distinct lines.
$\overleftrightarrow{BC}$ intersects the union of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ in two distinct points. Name them.
$\overleftrightarrow{BC}$ is *not* a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Why not?

Is it true that if a line is a transversal of two other lines, then the three lines are distinct coplanar lines? Give a reason for your answer.

A transversal of two lines forms with these lines eight distinct angles. For convenience we give names to certain pairs of these angles.

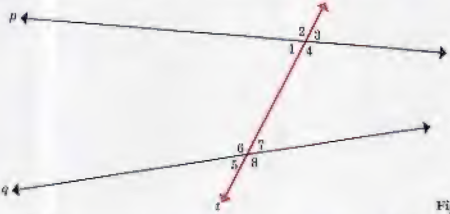


Figure 7-10

In Figure 7-10, t is a transversal of p and q; angles 1, 2, 3, 4 are the angles formed by p and t; angles 5, 6, 7, 8 are the angles formed by q and t. Angles 1, 4, 6, 7 are called interior angles and angles 2, 3, 5, 8 are called exterior angles.

Angles 1 and 6 are one pair of *consecutive interior angles*. Figure 7-11 shows these angles with several points labeled. Notice that $\angle 1$ is the union of $\overrightarrow{BA}$ and $\overrightarrow{BD}$ and that $\angle 1$ is $\angle ABD$. Express $\angle 6$ in terms of rays and write a name for it involving names of points.

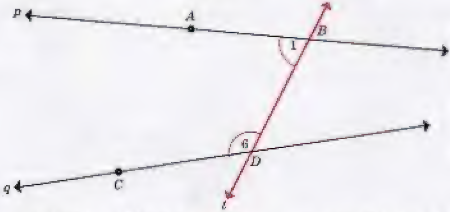


Figure 7-11

Notice that $\angle 1$ and $\angle 6$ have a segment in common and that their interiors are on the same side of the transversal. Name the segment that is the intersection of these two angles.

Lines p and q, which we are considering in connection with a transversal t, might be parallel or they might not be. If p and q are not parallel, then they intersect in a point. This point might be on the opposite side of the transversal from the interiors of a pair of consecutive interior angles, as it would be for $\angle 1$ and $\angle 6$ in Figure 7-11, or it might be on the same side, as it is for the pair of consecutive interior angles 4 and 7 in Figure 7-12. Note that although the intersection of $\angle 1$ and $\angle 6$ is a segment, the intersection of $\angle 4$ and $\angle 7$ is the union of a segment and a set consisting of a single point. Name that segment and that point. Thus the intersection of two consecutive interior angles may be a segment or it may be the union of a segment and a set consisting of a single point.

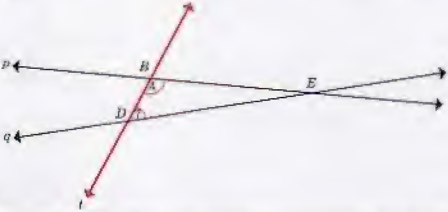


Figure 7-12

We shall give a formal definition of consecutive interior angles after we have introduced two other phrases for angles associated with a pair of lines and a transversal.

In Figure 7-13, $\angle 1$ and $\angle 7$ are interior angles but not consecutive interior angles and not adjacent angles. We call them *alternate interior angles*. The intersection of $\angle 1$ and $\angle 7$ is a segment and their interiors are on opposite sides of the transversal.

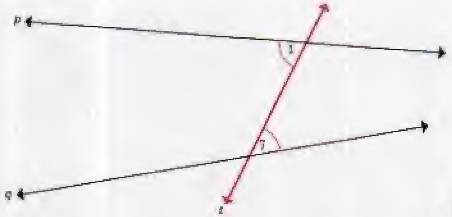


Figure 7-13

Another pair of alternate interior angles are $\angle 4$ and $\angle 6$. (See Figure 7-14.) Their intersection is a segment and their interiors lie on opposite sides of the transversal.

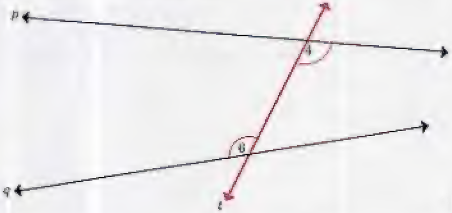


Figure 7-14

In Figure 7-15, $\angle 1$ and $\angle 6$ are consecutive interior angles. Angles 1 and 2 are adjacent angles. Angles 1 and 7 are alternate interior angles. This brings us to $\angle 1$ and $\angle 5$.

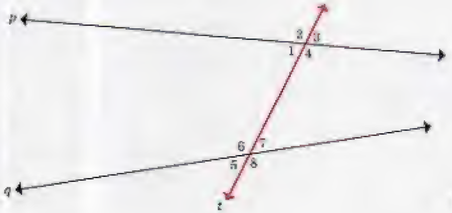


Figure 7-15

Angle 1 is an interior angle and $\angle 5$ is not. Their interiors lie on the same side of the transversal. In Figure 7-16, these angles are shown with several points labeled. Examine the intersection of $\angle 1$ and $\angle 5$.

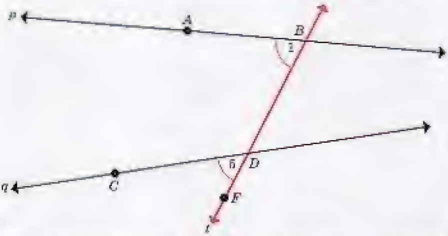


Figure 7-16

Is the intersection of these angles a ray, the ray $\overrightarrow{BF}$? Observe that the intersection of the interiors of $\angle 1$ and $\angle 5$ is the interior of $\angle 5$. Angles 1 and 5 are called *corresponding angles.*

Another pair of corresponding angles associated with the lines p and q and their transversal t are $\angle 4$ and $\angle 8$ as shown in Figure 7-17. The intersection of these angles is the union of a ray and a set consisting of a single point. The interiors of these angles lie on the same side of the transversal. Although neither interior contains the other, they do intersect.

Figure 7-17

Thus far in this section we have introduced alternate interior angles, consecutive interior angles, and corresponding angles in connection with two coplanar lines and a transversal. Actually, it is not necessary to refer to these lines and the transversal in describing these angles. Indeed, no such reference is made in the following definitions.

The phrase "a segment and a point" appears in the following definitions. We accept these phrases as a short way of saying "the union of two sets of points, one of them a segment and the other a set which consists of a single point." We accept "a ray and a point" to mean "the union of two sets, one of them a ray and the other a set consisting of a single point."

Definition 7.4 Two coplanar angles are alternate interior angles if their intersection is a segment and if their interiors do not intersect.

Definition 7.5 Two coplanar angles are consecutive interior angles if their intersection is a segment, or a segment and a point, and if their interiors intersect.

Definition 7.6 Two coplanar angles are corresponding angles if their intersection is a ray, or a ray and a point, and if their interiors intersect.

EXERCISES 7.4

1. How do you know that parallel lines exist in Euclidean geometry? Is your reason a postulate or a theorem or a definition?
2. If p is a line, if q is a line, and if p is parallel to q, is it possible that the intersection of p and q is a set which contains more than one point? Explain.
3. If p is a line, if q is a line, and if the intersection of p and q is the null set, is it possible that p and q are not parallel? Explain.
4. If p is a line, if q is a line, and if p is parallel to q, is it possible that there is no plane containing the union of p and q? Explain.

Figure 7-18 shows two coplanar lines p and q and a transversal t with several points labeled. Copy Exercises 5–10 and replace the question marks with one word, two words, or three capital letters so that the resulting statement is a true sentence concerning Figure 7-18.

Figure 7-18

5. $\angle BDC$ and $\angle$ [?] are corresponding angles.
6. $\angle BDC$ and $\angle$ [?] are alternate interior angles.
7. $\angle BDC$ and $\angle$ [?] are consecutive interior angles.
8. $\angle FDC$ and $\angle ACD$ are [?] angles.
9. $\angle HDF$ and $\angle HCE$ are [?] angles.
10. $\angle FDC$ and $\angle ECD$ are [?] angles.

■ Figure 7-19 shows two coplanar lines p and q and a transversal t. Eight angles are labeled. Copy Exercises 11–20 and replace the question marks with one word or two words or a letter or a number so as to make a true statement about Figure 7-19.

Figure 7-19

11. a and c are vertical [?].
12. a and d are [?] angles.
13. a and u are [?] angles.
14. v and c are [?] angles.
15. v and [?] are corresponding angles.
16. x and [?] are corresponding angles.
17. d and [?] are alternate interior angles.
18. There are [?] pairs of alternate interior angles among the eight angles.
19. There are [?] pairs of consecutive interior angles among the eight angles.
20. There are [?] pairs of corresponding angles among the eight angles.

■ Figure 7-20 shows three noncollinear points A, B, C, the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and 12 angles formed by them. Copy Exercises 21–23 and replace the question marks so as to make a true statement about the figure.

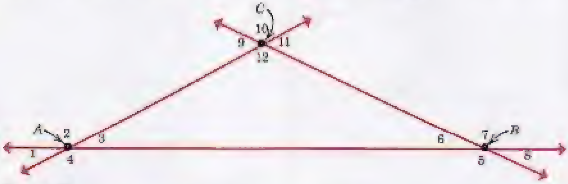


Figure 7-20

21. $\angle 1$ and [?] are corresponding angles; also $\angle 1$ and [?] are corresponding angles.
22. $\angle 12$ and [?] are alternate interior angles; also $\angle 12$ and [?] are alternate interior angles.

23. $\angle 12$ and $\boxed{?}$ are consecutive interior angles; also $\angle 12$ and $\boxed{?}$ are consecutive interior angles.

■ Figure 7-21 shows two distinct parallel lines, a transversal, and eight associated angles. Copy Exercises 24–28 and replace the question marks so as to make a true statement about the figure.

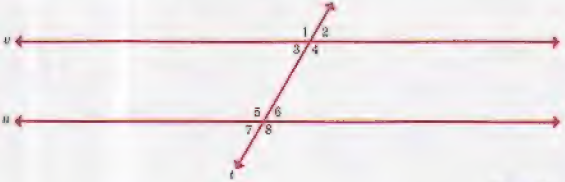
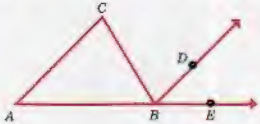


Figure 7-21

24. Angles 1 and 5 are corresponding angles; their intersection is $\boxed{?}$; their interiors lie on $\boxed{?}$ of t.
25. Angles 3 and 5 are consecutive interior angles; their intersection is $\boxed{?}$; their interiors lie on $\boxed{?}$ of t.
26. Angles 3 and 6 are alternate interior angles; their intersection is $\boxed{?}$; their interiors lie on $\boxed{?}$ of t.
27. Angles 6 and 7 are vertical angles; their union is the $\boxed{?}$ of a pair of lines; their intersection is $\boxed{?}$; the intersection of their interiors is $\boxed{?}$.
28. Angles 5 and 6 are a linear pair of angles; their union is the union of a line and a $\boxed{?}$; their intersection is $\boxed{?}$; the union of these angles and their interiors is the union of a halfplane and $\boxed{?}$.

29. The figure shows $\triangle ABC$ and two rays in the same plane such that B is between A and E, and $\overrightarrow{BD}$ is between $\overrightarrow{BE}$ and $\overrightarrow{BC}$. Considering only angles that can be named in terms of three points labeled in the figure, identify all pairs of alternate interior angles.



30. Under the same conditions as those in Exercise 29, identify all pairs of consecutive interior angles.
31. Under the same conditions as those in Exercise 29, identify all pairs of corresponding angles.

■ Figure 7-22 shows a quadrilateral and one of its diagonals. In Exercises 32–35, copy and complete each statement so that it will be true.

Figure 7-22

32. $\angle ACB$ and $\angle CAD$ are alternate interior angles determined by transversal $\overleftrightarrow{AC}$ and lines $\boxed{?}$ and $\boxed{?}$.
33. $\angle BAC$ and $\angle DCA$ are alternate interior angles determined by transversal $\boxed{?}$ and lines $\boxed{?}$ and $\boxed{?}$.
34. $\angle ABC$ and $\angle BCD$ are $\boxed{?}$ angles determined by transversal $\boxed{?}$ and lines $\boxed{?}$ and $\boxed{?}$.
35. $\angle BAD$ and $\angle \boxed{?}$ are consecutive interior angles determined by transversal $\boxed{?}$ and lines $\boxed{?}$ and $\boxed{?}$.

■ Figure 7-23 is a plane figure with five angles labeled, including two pairs of vertical angles. Let $m\angle a = 65$ and $m\angle a = m\angle e$. In Exercises 36–38, find the measure of the given angle.

36. $\angle i$
37. $\angle u$
38. $\angle v$

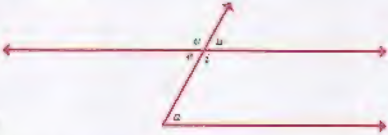


Figure 7-23

■ Figure 7-24 is a plane figure showing two lines and a transversal. Refer to this figure for Exercises 39–42.

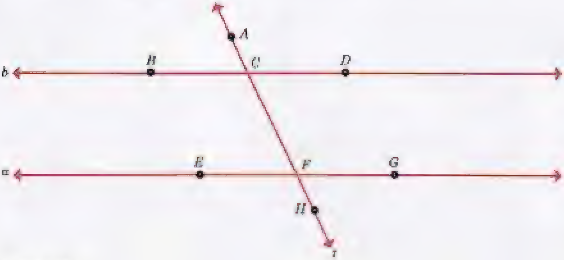


Figure 7-24

39. Name
    (a) the pairs of alternate interior angles;
    (b) the pairs of consecutive interior angles;
    (c) the pairs of corresponding angles.
40. *Prove:* If one pair of alternate interior angles are congruent, then
    (a) the other pair of alternate interior angles are congruent;
    (b) each pair of consecutive interior angles are supplementary;
    (c) each pair of corresponding angles are congruent.
41. *Prove:* If one pair of consecutive interior angles are supplementary, then
    (a) the other pair of consecutive interior angles are supplementary;
    (b) each pair of alternate interior angles are congruent;
    (c) each pair of corresponding angles are congruent.
42. *Prove:* If one pair of corresponding angles are congruent, then
    (a) each pair of corresponding angles are congruent;
    (b) each pair of alternate interior angles are congruent;
    (c) each pair of consecutive interior angles are supplementary.

7.5 SOME PARALLEL LINE THEOREMS

This section contains several theorems that are sometimes helpful in proving that two lines are parallel.

THEOREM 7.2 Let two distinct coplanar lines and a transversal be given. If the transversal is perpendicular to both lines, then the lines are parallel.

Proof: Let a and b be distinct coplanar lines and let t be a transversal of them. (See Figure 7-25.) We wish to prove that if a and b are perpendicular to t, then a and b are parallel. Suppose, contrary to this assertion and as suggested by the figure, that $a \perp t$, that $b \perp t$, and that a is not parallel to b. Then a and b intersect in some point P forming $\triangle PVQ$. This triangle has $\angle 1$ as an exterior angle and $\angle 2$ as a nonadjacent interior angle. By hypothesis, both of these angles are right angles, so $\triangle PVQ$ is a triangle with one exterior angle congruent to a nonadjacent interior angle.

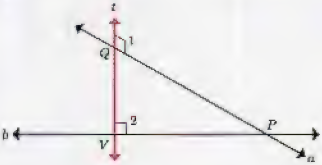


Figure 7-25

According to the Exterior Angle Theorem, however, the measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle. Hence we have two angles which are congruent by one line of reasoning and which are not congruent by another line of reasoning. We arrived at this contradiction after we had supposed that the lines a and b are not parallel. Since we cannot have a contradiction in our system, we have proved that lines a and b cannot be not parallel. Hence they are parallel. This completes the proof of the theorem.

We pause briefly to comment on this proof. It is an example of an *indirect* proof. The theorems in our formal geometry are true statements. Their truth rests on a foundation of postulates and definitions. In proving theorems, that is, in establishing their truth, we may use definitions, postulates, and theorems proved previously.

How did we prove Theorem 7.2? Consider the given situation involving lines a, b, and t. Lines a and b are distinct and coplanar, and t is perpendicular to both of them. We do not know that a and b are parallel, nor do we know that a and b are not parallel. But we do know from our definitions that a and b are either parallel or they are not parallel. Since one of these possibilities leads to a contradiction, we conclude that the other possibility is the valid conclusion.

It may be helpful to see this reasoning in skeleton form using symbols. We start with a situation in which a and b are coplanar lines and t is a transversal of them. Let P and Q stand for statements as follows:

P: $t \perp a$ and $t \perp b$.
Q: $a \parallel b$.

Then our theorem and proof are essentially as follows.

THEOREM If P, then Q.

Proof: It is given that P is true. Either Q is true or Q is false. If Q is false, then it follows that P is false. This contradicts the hypothesis that P is true. Therefore Q is true, and the proof is complete.

If P denotes a statement, it is convenient to denote the opposite statement by not-P. If P is a true statement, then not-P is a false statement. If P is a false statement, then not-P is a true statement. If a statement has the form "If P, then Q," then there is always a related statement of the form "If not-Q, then not-P." This related statement is called the contrapositive of the given statement. If the contrapositive of a statement is true, then the statement is true. For, if not-Q implies not-P, then the only way it is possible for P to be true is for Q also to be

true. In other words, if not-Q implies not-P, then P implies Q. This means that we can prove a theorem by proving its contrapositive. If it is easier to prove the contrapositive, then this is what we should do. A proof using the contrapositive is one form of *indirect proof*.

For example, suppose we wish to prove

$$\text{if} \quad a^2 \neq b^2, \qquad \text{then} \quad a \neq b.$$

An easy way to prove this is to prove its contrapositive

$$\text{if} \quad a = b, \qquad \text{then} \quad a^2 = b^2.$$

The proofs of the following three theorems are assigned as exercises.

THEOREM 7.3 (***Alternate Interior Angle Theorem***) If two alternate interior angles determined by two distinct coplanar lines and a transversal are congruent, then the lines are parallel.

THEOREM 7.4 (***Corresponding Angle Theorem***) If two corresponding angles determined by two distinct coplanar lines and a transversal are congruent, then the lines are parallel.

THEOREM 7.5 (***Consecutive Interior Angle Theorem***) If two consecutive interior angles determined by two distinct coplanar lines and a transversal are supplementary, then the lines are parallel.

EXERCISES 7.5

Figure 7-26 shows a transversal of two distinct coplanar lines and eight associated angles. In Exercises 1–4, measures of two of the angles are given. Explain why m must be parallel to n.

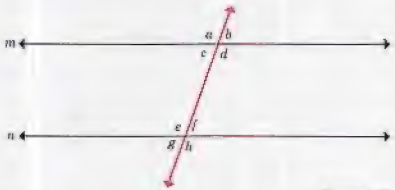


Figure 7-26

1. $m\angle a = 105$, $m\angle e = 105$
2. $m\angle d = 105$, $m\angle e = 105$
3. $m\angle c = 75$, $m\angle e = 105$
4. $m\angle c = 75$, $m\angle f = 75$

Figure 7-27 shows five coplanar segments and several associated angles. In Exercises 5–9, state which lines must be parallel on the basis of the given information.

5. $\angle u \cong \angle v$
6. $\angle p \cong \angle q$
7. $m\angle ADC + m\angle DCB = 180$
8. $m\angle ADC + m\angle BAD = 180$
9. $m\angle DAB + m\angle ABC = 180$

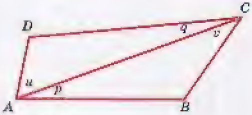


Figure 7-27

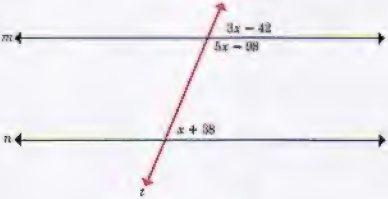
Figure 7-28 shows five distinct coplanar points A, B, C, D, E. Points A, B, C are collinear, $\overline{AD} \perp \overline{AC}$, and $\overline{CE} \perp \overline{AC}$. In Exercises 10–14, state whether this given information implies the stated conclusion.

10. $\overleftrightarrow{AB} \parallel \overleftrightarrow{AC}$
11. $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$
12. $\overleftrightarrow{AD} \parallel \overleftrightarrow{CE}$
13. $\overleftrightarrow{AD} \parallel \overleftrightarrow{EC}$
14. $\overleftrightarrow{AD} \parallel \overleftrightarrow{BE}$

Figure 7-28

In Exercises 15–20, the figures show three coplanar lines. The measures of three angles, expressed in terms of a number x, are given in the figures. Find x and determine whether m is parallel to n. Give a reason for your answer.

15.



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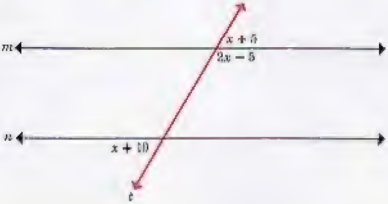

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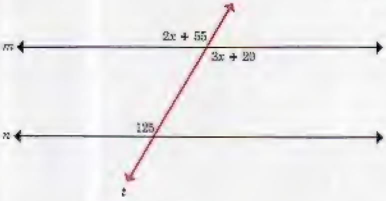

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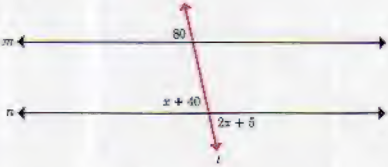

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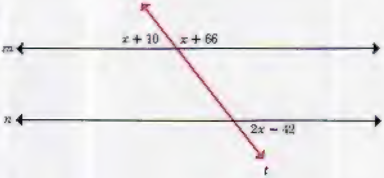
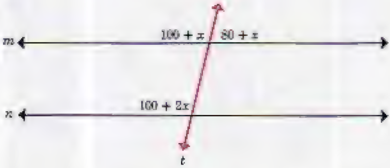


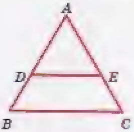
20.



21. *Given:* A plane figure with $AB = CD$, $AD = BC$.
Prove: $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$

22. *Given:* A figure with $A\text{-}D\text{-}B$, $A\text{-}E\text{-}C$,
$AD = AE$, $AB = AC$,
$\angle ACB \cong \angle ADE$

Prove: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$



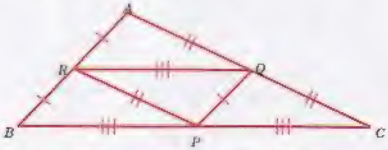
23. *Given:* A figure in which $\overline{AC}$ and $\overline{BD}$ bisect each other at M.

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$

24. *Given:* A figure with $A\text{-}R\text{-}B$, $B\text{-}P\text{-}C$, $C\text{-}Q\text{-}A$.

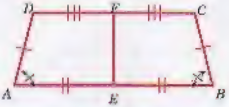
$AQ = QC = RP$
$CP = PB = QR$
$BR = RA = PQ$

Prove: $m\angle A + m\angle B + m\angle C = 180$



25. *Given:* A plane figure, with $\angle A \cong \angle B$, $AD = BC$, $AE = EB$, $DF = FC$.

Prove: $\overline{EF} \perp \overline{DC}$, $\overline{AB} \perp \overline{EF}$, $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$



26. Prove Theorem 7.3.

27. Prove Theorem 7.4.

28. Prove Theorem 7.5.

29. Write the contrapositive of Theorem 7.3. Is this contrapositive a true statement?

30. Write the contrapositive of Theorem 7.4. Is this contrapositive a true statement?

31. Write the contrapositive of Theorem 7.5. Is this contrapositive a true statement?

7.6 THE PARALLEL POSTULATE AND SOME THEOREMS

Theorem 7.1 states that through a given point not on a given line there is at least one line parallel to the given line. As we pointed out following the proof of that theorem, it is impossible, using only Postulates 1–25, to prove that this parallel line is unique. Since we want it to be unique, we adopt the following Parallel Postulate. As we stated in Section 7.3 it is this postulate which makes our geometry, Euclidean geometry, different from that of Lobachevsky and Bolyai.

POSTULATE 26 (*Parallel Postulate*) There is at most one line parallel to a given line and containing a given point not on the given line.

This postulate and Theorem 7.1 tell us that if a line and a point not on it are given, then there is exactly one line through the given point and parallel to the given line. Furthermore, as we said in Section 7.3, we know that if a line and a point on it are given, then there is exactly one line through the given point and parallel to the given line, namely the given line itself. Hence, in every case, given a line and a point, there is exactly one line through the given point and parallel to the given line.

In Section 7.5 there are some parallel line theorems, actually theorems stating conditions that imply that lines are parallel. These theorems are useful in proving lines parallel. In this section we have the converses of several of these theorems. These converses were deferred until now because the Parallel Postulate is essential for their proof.

If a theorem is of the form "If P, then Q," then its converse is the associated statement "If Q, then P." It should be clear that the converses of some true statements are not true statements. For example, "If a number is greater than 100, then it is greater than 6" is a true statement, whereas its converse, "If a number is greater than 6, then it is greater than 100," is certainly a false statement. The converses of some theorems are theorems; the converses of other theorems are not theorems. We shall prove that the converses of Theorems 7.3, 7.4, and 7.5 are theorems.

Note first, however, that the converse of Theorem 7.2 is not true. To see this, suppose m and n are distinct parallel lines and that t is a transversal of them which makes an angle of measure 30 with m. Then, according to Theorem 7.7 given later in this section, it also makes an angle of measure 30 with line n. Thus we see that lines m and n may be parallel even though a transversal is not perpendicular to both of them.

We proceed now to the converses of the other theorems in Section 7.5.

THEOREM 7.6 (*Converse of Alternate Interior Angle Theorem*) If two distinct lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent.

Proof: (See Figure 7-29.) Let m and n be distinct parallel lines. Let t be a transversal that intersects them in P and Q, respectively. Let R and S be points on m and n, respectively, such that R and S are on opposite sides of t. We shall prove that $\angle RPQ \cong \angle SQP$.

Figure 7-29

It follows from the Angle Construction Theorem that there is a ray $\overrightarrow{PR'}$, with R' on the R-side of t, such that $\angle R'PQ \cong \angle SQP$. Since $\angle R'PQ$ and $\angle SQP$ are congruent alternate interior angles, it follows from Theorem 7.3 that lines $\overleftrightarrow{R'P}$ and n are parallel. Therefore $\overleftrightarrow{R'P}$ and m are lines through P and parallel to n. From the Parallel Postulate, however, it follows that there is only one line through P and parallel to n. Therefore $\overleftrightarrow{R'P}$ and m are the same line. Then $\angle R'PQ$ and $\angle RPQ$ are the same angle and it follows that $\angle RPQ \cong \angle SQP$.

THEOREM 7.7 (*Converse of Corresponding Angle Theorem*) If two distinct lines are parallel, then any two corresponding angles determined by a transversal of the lines are congruent.

Proof: Assigned as an exercise.

THEOREM 7.8 (*Converse of Consecutive Interior Angle Theorem*) If two distinct lines are parallel, then any two consecutive interior angles determined by a transversal of the lines are supplementary.

Proof: Assigned as an exercise.

As we noted above, the converse of Theorem 7.2 is not true. It should be noted, however, that a theorem similar to Theorem 7.2 does have a true converse. This is Theorem 7.9. Its converse is Theorem 7.10. Theorem 7.9 follows immediately from Theorem 7.2. Theorem

7.2 begins with a "situation statement" (Let two distinct coplanar lines and a transversal be given) followed by an "If P, then Q" type of statement. In Theorem 7.9 some of the P has been put into the "situation," but the meaning of the two sentences taken together is the same as in Theorem 7.2. Theorem 7.10 follows immediately from Theorem 7.7.

THEOREM 7.9 Let a and b be two distinct coplanar lines, and let t be a transversal of them that is perpendicular to a. If t is perpendicular to b, the lines a and b are parallel.

THEOREM 7.10 Let a and b be two distinct coplanar lines, and let t be a transversal of them that is perpendicular to a. If a and b are parallel, then t is perpendicular to b.

The next theorem provides another useful method for proving lines parallel.

THEOREM 7.11 Two coplanar lines parallel to the same line are parallel to each other.

Proof: Let p and q be coplanar lines each parallel to a line r. If $p = q$, then p is certainly parallel to q. Suppose, then, that p and q are distinct lines. There are two possibilities as indicated in Figure 7-30. Either p and q are parallel lines or else they have exactly one point, say P, in

Figure 7-30

common. If they have exactly one point P in common, then there are two distinct lines through P and parallel to r. Since this contradicts the Parallel Postulate, it follows that possibility (b) in Figure 7-30 is impossible, and therefore p is parallel to q.

If we consider only lines lying in a given plane, we see that the relationship of parallelism for lines is an equivalence relation. That is, it is reflexive, symmetric, and transitive. (1) It is reflexive since every line is parallel to itself. (2) It is symmetric since if line p is parallel to line q, then line q is parallel to line p. (3) To show that it is transitive, suppose that line p is parallel to line r and that line r is parallel to line q. Then, since parallelism in a plane is symmetric, it follows that line q is parallel to line r. Then p and q are both parallel to line r and it follows from

Theorem 7.11 that p is parallel to q. Later we shall see that parallelism for lines is an equivalence relation even without the restriction that the lines all lie in one plane.

> *THEOREM 7.12* Let three distinct coplanar lines with two of them parallel be given. If the third line intersects one of the two parallel lines, then it intersects the other also.

Proof: Assigned as an exercise.

> *THEOREM 7.13* Let two sets $\mathcal{S}$ and $\mathcal{T}$ of parallel lines in a plane α be given. (This means that every two lines in $\mathcal{S}$ are parallel and that every two lines in $\mathcal{T}$ are parallel.) If one line in $\mathcal{S}$ is perpendicular to one line in $\mathcal{T}$, then every line in $\mathcal{S}$ is perpendicular to every line in $\mathcal{T}$.

Proof: Let s be a line in $\mathcal{S}$ and let t be a line in $\mathcal{T}$ such that s is perpendicular to t. Let u be any line in $\mathcal{S}$ and let v be any line in $\mathcal{T}$. We want to prove that u is perpendicular to v. Suppose, first, that s and u are distinct lines and that t and v are distinct lines as indicated in Figure 7-31. Since s intersects t and t is parallel to v, it follows from Theorem 7-12 that s intersects v and therefore s is a transversal of the parallel lines t and v. Then it follows from Theorem 7.10 that s is perpendicular to v. Similarly, it follows from Theorems 7.12 and 7.10 that v is a transversal of the parallel lines s and u and that v is perpendicular to u. This completes the proof for the case in which s and u are distinct lines and t and v are distinct lines.

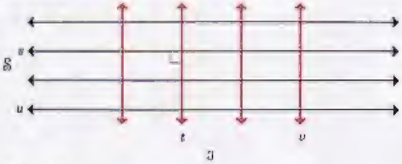


Figure 7-31

Suppose next that $s = u$ and that s is perpendicular to t. Then it follows as above (really the halfway point in the reasoning of the preceding paragraph) that s is perpendicular to v.

The case in which s and u are distinct and $t = v$ is assigned as an exercise.

The case in which $s = u$ and $t = v$ requires no proof since the hypothesis and the conclusion are the same in this instance.

EXERCISES 7.6

1. Use Theorem 7.6 to prove Theorem 7.7.
2. Use Theorem 7.6 to prove Theorem 7.8.
3. Prove Theorem 7.7 without using Theorem 7.6. (This proof will be similar to the one for Theorem 7.6 that uses the Parallel Postulate.)
4. Prove Theorem 7.8. (This proof will be similar to the one for Theorem 7.6 that uses the Parallel Postulate.)
5. Explain the following statement: Of the three theorems, 7.6, 7.7, 7.8, any one of them may be considered as the basic theorem and the other two as corollaries of it.
6. Prove Theorem 7.13 for the case in which $s \neq u$ and $t = v$.

In Figure 7-32, lines a, b, c, and d are coplanar and the measures of several angles are marked. In Exercises 7–9, justify the given assertion regarding the lines in this figure.

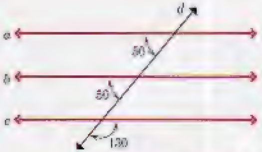


Figure 7-32

7. a is parallel to b.
8. a is parallel to c.
9. b is parallel to c.

10. If a, b, c are coplanar lines such that a is parallel to b and b is parallel to c, does it follow that a is parallel to c? Is this an instance of the reflexive property of parallelism for lines? Of the symmetric property? Of the transitive property?
11. If a, b, c are coplanar lines such that a is parallel to b and b is parallel to c, does it follow that b is parallel to a? Is this an instance of the reflexive property of parallelism for lines? Of the symmetric property? Of the transitive property?
12. If a, b, c are coplanar lines such that a is parallel to b and b is parallel to c, does it follow that c is parallel to c? Is this an instance of the reflexive property of parallelism for lines? Of the symmetric property? Of the transitive property?

In Figure 7-33, a, b, c, d are coplanar parallel lines, t is a transversal of them, and the measures of several angles are marked. In Exercises 13–18, justify the given assertion regarding angle measure.

13. $x = 117$
14. $y = 63$
15. $z = 117$
16. $u = 63$
17. $v = 63$
18. $w = 63$

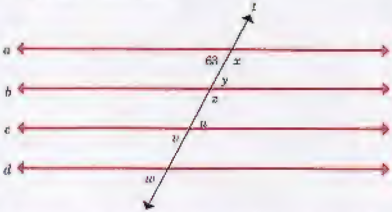


Figure 7-33

In Exercises 19–21, use Figure 7-34 with E-A-D, $m\angle EAB = 118$, $\overleftrightarrow{ED} \parallel \overleftrightarrow{BC}$.

19. Find $m\angle ABD$ if $2m\angle ABD = 3m\angle DBC$.
20. Find $m\angle DBC$ and $m\angle BDA$ if $2m\angle ABD = 3m\angle DBC$.

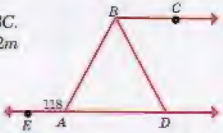
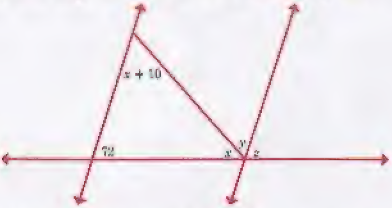
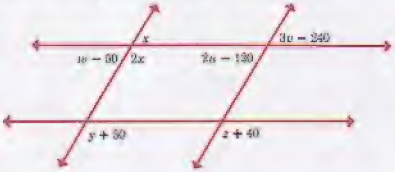


Figure 7-34

22. Given a plane figure made up of two parallel lines, a transversal, and a segment, with angle measures as marked, find x, y, and z.



23. Given a plane figure made up of two pairs of parallel lines, with angle measures as marked, find x, y, z, u, v, and w.

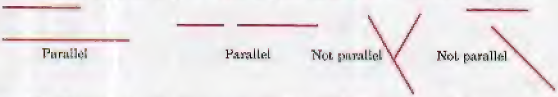


24. Given a plane figure with $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, with A, C, E all on the same side of $\overleftrightarrow{BD}$, with C on the E-side of $\overleftrightarrow{AB}$, and with C on the A-side of $\overleftrightarrow{ED}$, prove that

$$m\angle BCD = m\angle ABC + m\angle CDE.$$

(*Hint:* Copy the figure and draw the line through C parallel to $\overleftrightarrow{AB}$.)

25. In the figure for Exercise 24, what is the sum of the measures of $\angle ABC$, $\angle CBD$, $\angle BDC$, and $\angle CDE$?
26. In the figure for Exercise 24, what is the sum of the measures of $\angle CBD$, $\angle BDC$, and $\angle DCB$?
27. The figure below shows some segments that are parallel and some that are not parallel. Try to write a good definition of parallel segments.



28. The figure below shows some quadrilaterals that are parallelograms and some that are not. Try to write a good definition of parallelogram.

7.7 PARALLELISM FOR SEGMENTS; PARALLELOGRAMS

In this section we introduce some definitions and theorems concerning an important type of quadrilateral called a *parallelogram*. Since the sides of a quadrilateral and, hence, of a parallelogram are segments, we need a formal definition that extends the concept of parallelism to segments. In Section 7.8 we extend the concept also to rays.

> ***Definition 7.7*** If the lines which contain two segments are parallel, then the segments are said to be parallel segments, and each is said to be parallel to the other. The segments in a set of segments are parallel if every two of them are parallel.

The lines which contain parallel segments need not be distinct. In other words, a segment of a line is parallel to every segment of that line. As a special case of a special case, we note that every segment is parallel to itself.

Let A, B, C, D be four points with $A \neq B$ and $C \neq D$. (See Figure 7-35.) Then $\overline{AB}$ is parallel to $\overline{CD}$ if and only if $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. We use the same symbol to denote parallelism for segments that we use for lines. Thus $\overline{AB} \parallel \overline{CD}$ means that $\overline{AB}$ is parallel to $\overline{CD}$.

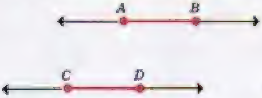


Figure 7-35

> ***Definition 7.8*** A parallelogram is a quadrilateral each of whose sides is parallel to the side opposite it.

Consider a parallelogram $ABCD$ as shown in Figure 7-36. Since $ABCD$ is a quadrilateral, it follows that $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{CD}, \overleftrightarrow{DA}$ are four distinct lines. Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are distinct parallel lines, it follows that C and D lie on the same side of $\overleftrightarrow{AB}$. Similarly, D and A lie on the same side of $\overleftrightarrow{BC}$, A and B lie on the same side of $\overleftrightarrow{CD}$, and B and C lie on the same side of $\overleftrightarrow{DA}$. Therefore each side of a parallelogram lies on a line which is the edge of a halfplane that contains all of the parallelogram

Figure 7-36

except that side. Therefore every parallelogram is a convex quadrilateral. (The word *convex* is used here in the sense of a convex polygon. See Definition 4.16.)

The next three theorems state some important properties of parallelograms.

THEOREM 7.14 If a convex quadrilateral is a parallelogram, then its opposite sides are congruent.

Proof: Let parallelogram $ABCD$ be given. (See Figure 7-37.) Then $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$. We shall prove that $AB = CD$ and that $BC = DA$. First draw the segment $\overline{AC}$. Our plan is to use congruent triangles.

Figure 7-37

Statement	Reason
1. $\angle BAC \cong \angle DCA$	1. Alternate interior angles determined by a transversal of two parallel lines are congruent.
2. $\overline{AC} \cong \overline{CA}$	2. Why?
3. $\angle ACB \cong \angle CAD$	3. Why?
4. $\triangle ABC \cong \triangle CDA$	4. Why?
5. $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$	5. Why?

THEOREM 7.15 If a convex quadrilateral is a parallelogram, then its opposite angles are congruent.

Proof: Assigned as an exercise.

THEOREM 7.16 If a convex quadrilateral is a parallelogram, then its diagonals bisect each other.

Proof: Assigned as an exercise.

The next three theorems are useful in proving that certain quadrilaterals are parallelograms.

THEOREM 7.17 If two sides of a convex quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

Proof: Assigned as an exercise.

THEOREM 7.18 If the diagonals of a convex quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Proof: Assigned as an exercise.

THEOREM 7.19 If each two opposite sides of a convex quadrilateral are congruent, then the quadrilateral is a parallelogram.

Proof: Assigned as an exercise.

Just as parallelograms are convex quadrilaterals with a special property, so are trapezoids convex quadrilaterals with a special property (only not quite as special as that for parallelograms).

Definition 7.9 A trapezoid is a convex quadrilateral with at least two parallel sides.

Note that we do not say two and only two sides parallel. A trapezoid may have only one pair of parallel sides or it may have two pairs of parallel sides. In other words, every parallelogram is a trapezoid, but not every trapezoid is a parallelogram. In some books trapezoids are restricted to have only two parallel sides. In these instances the set of all trapezoids and the set of all parallelograms do not intersect. In this book the set of all parallelograms is a subset of the set of all trapezoids.

Rhombuses, rectangles, and squares are all parallelograms with special properties. Their formal definitions come next.

Definition 7.10 A rhombus is a parallelogram with two adjacent sides congruent.

Definition 7.11 A rectangle is a parallelogram with a right angle.

Definition 7.12 A square is a rectangle with two adjacent sides congruent.

THEOREM 7.20 A rhombus is an equilateral parallelogram.

Proof: Assigned as an exercise.

THEOREM 7.21 A rectangle is a parallelogram with four congruent angles.

Proof: Assigned as an exercise.

THEOREM 7.22 A square is an equilateral rectangle.

Proof: Assigned as an exercise.

THEOREM 7.23 A square is an equiangular rhombus.

Proof: Assigned as an exercise.

THEOREM 7.24 The diagonals of a rhombus are perpendicular.

Proof: Assigned as an exercise.

In order to prove that a figure is a rhombus it is sufficient to prove that it is a parallelogram with two adjacent sides that are congruent (Definition 7.10). If we know that a figure is a rhombus, then we may conclude that all four of its sides are congruent (Theorem 7.20). If we want to show that a parallelogram is a rectangle, it is sufficient to show that it has one right angle, since it then necessarily has four right angles (Definition 7.11 and Theorem 7.21). To show that a rectangle is a square, it is sufficient to show that two of its adjacent sides are congruent, since then all four of its sides are congruent (Definition 7.12 and Theorem 7.22). To show that a rhombus is a square, it is sufficient to show that it has a right angle since it is easy to show that an equilateral parallelogram with a right angle is an equilateral rectangle, that is, a square.

In Chapter 3 we introduced the concept of distance. We restricted ourselves there to the idea of the distance between two points. We are ready now to extend the idea of distance to the distance between two parallel lines. Of course, we agree that the distance between a line and itself is zero. So let us consider the idea of the distance between two distinct parallel lines m and n as suggested in Figure 7-38. The length of a segment (see the "dashed" segment in the figure) joining a point of one line to a point of the other line might be very long depending on how the endpoints are picked.

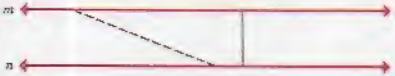


Figure 7-38

It seems natural to think of the distance between two parallel lines as the length of the shortest segment joining the two lines, and it seems that this segment should be perpendicular to both lines. If we pick any point on m, say M, then there is a line t in the plane of m and n that is perpendicular to m at M. (See Figure 7-39.) Also t intersects n in some point, say N, and $\overline{MN}$ is perpendicular to both m and n. If t' is another transversal of m and n perpendicular to both of them and intersecting them in M' and N', respectively, how do we know that $\overline{MN}$ and $\overline{M'N'}$ are congruent? Our idea for the distance between two parallel lines will not be any good unless we can show that $\overline{MN}$ and $\overline{M'N'}$ are congruent. The next theorem shows that this is possible.

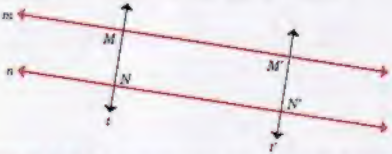


Figure 7-39

THEOREM 7.25 For every two distinct parallel lines there is a number that is the common length of all segments perpendicular to both of the given lines and with one endpoint on one of the given lines and one endpoint on the other one.

Proof: (See Figure 7-40.) Let m and n be two distinct parallel lines.

Figure 7-40

Let A and B be two distinct points of m, and C and D two distinct points of n such that the segments $\overline{AC}$ and $\overline{BD}$ are perpendicular to both m and n. Then m is a transversal of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ and is perpendicular to both. It follows that $\overline{AC}$ and $\overline{BD}$ are parallel (Theorem 7.2); hence $ABCD$ is a parallelogram (by definition) and $AC = BD$ (Theorem 7.14).

If we think of segment $\overline{AC}$ in Figure 7-40 as a fixed segment and $\overline{BD}$ as a variable segment (one that we can pick anywhere as long as B is on m, D is on n, and $\overline{BD}$ is perpendicular to m and to n), then the number we are looking for, the one whose existence we wanted to prove, is the number AC. The idea of Theorem 7.25 is sometimes expressed by saying that two parallel lines are everywhere equidistant.

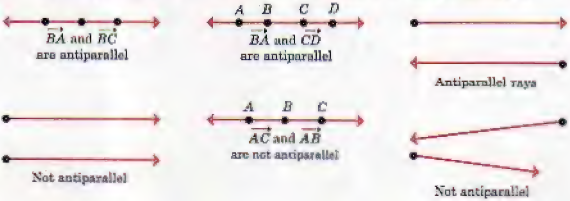
We are now ready for the following definition.

Definition 7.13 The distance between two distinct parallel lines is the length of a segment which is perpendicular to both lines and whose endpoints lie on these lines, one endpoint on one line and the other endpoint on the other line. The distance between a line and itself is zero.

EXERCISES 7.7

1. *Prove:* If the convex quadrilateral $ABCD$ is a parallelogram, then $\angle A \cong \angle C$ and $\angle B \cong \angle D$. (This is Theorem 7.15.)
2. *Prove:* If the convex quadrilateral $ABCD$ is a parallelogram, then $\overline{AC}$ and $\overline{BD}$ bisect each other. (This is Theorem 7.16.)
3. *Prove:* If $ABCD$ is a convex quadrilateral, if $\overline{AB} \cong \overline{CD}$, and if $\overline{AB} \parallel \overline{CD}$, then $ABCD$ is a parallelogram. (This is Theorem 7.17.)
4. *Prove:* If $ABCD$ is a convex quadrilateral, and if $\overline{AC}$ and $\overline{BD}$ bisect each other, then $ABCD$ is a parallelogram. (This is Theorem 7.18.)
5. *Prove:* If $ABCD$ is a convex quadrilateral, if $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, then $ABCD$ is a parallelogram. (This is Theorem 7.19.)
6. If $ABCD$ is a rhombus and if $\overline{AB} \cong \overline{AD}$, then $AB = BC = CD = DA$ and $ABCD$ is an equilateral parallelogram. (This is Theorem 7.20.)
7. Prove Theorem 7.21.
8. Prove Theorem 7.22.
9. Prove Theorem 7.23.
10. Prove Theorem 7.24.
11. Why are opposite sides of a parallelogram parallel?
12. Why are opposite sides of a parallelogram congruent?
13. The figure below shows some rays that are parallel and some that are not parallel. Try to write a good definition of parallel rays.

14. The figure below shows some antiparallel rays and some rays that are not antiparallel. Try to write a good definition of antiparallel rays.



7.8 PARALLELISM FOR RAYS

In this section we extend the concept of parallelism to rays. Since segments are parallel if the lines containing them are parallel, we might consider rays parallel if the lines containing them are parallel. But it is useful to distinguish between the case suggested by rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and the case suggested by rays $\overrightarrow{EF}$ and $\overrightarrow{GH}$ in Figure 7-41. In both cases the rays lie on parallel lines. In one case, however, they point in the same direction, whereas in the other case they point in opposite directions. (We use the word "direction" here in an intuitive sense; it is not a part of our formal development.)

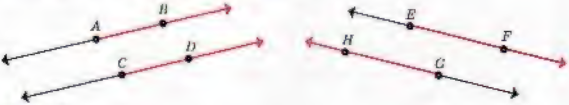


Figure 7-41

Definition 7.14 (See Figure 7-42.) Two noncollinear rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines and if B and D lie on the same side of $\overleftrightarrow{AC}$. Two collinear rays are parallel if one of them is a subset of the other.

Definition 7.15 (See Figure 7-42.) Two noncollinear rays $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are antiparallel if $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are parallel lines and if F and H lie on opposite sides of $\overleftrightarrow{EG}$. Two collinear rays are antiparallel if neither is a subset of the other.

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

$\overrightarrow{IJ}$ and $\overrightarrow{JK}$ are parallel.
$\overrightarrow{JI}$ and $\overrightarrow{IK}$ are antiparallel.

$\overrightarrow{EF}$ and $\overrightarrow{GH}$ are antiparallel.

Figure 7-42

The proofs of the following three theorems are assigned as exercises.

THEOREM 7.26 If $\angle ABC$ and $\angle DEF$ are coplanar angles with $\overrightarrow{BA}$ and $\overrightarrow{ED}$ parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ parallel, then $\angle ABC \cong \angle DEF$.

THEOREM 7.27 If $\angle ABC$ and $\angle DEF$ are coplanar angles with $\overrightarrow{BA}$ and $\overrightarrow{ED}$ parallel and with $\overrightarrow{BC}$ and $\overrightarrow{EF}$ antiparallel, then $\angle ABC$ and $\angle DEF$ are supplementary.

THEOREM 7.28 If $\angle ABC$ and $\angle DEF$ are coplanar angles with $\overrightarrow{BA}$ and $\overrightarrow{ED}$ antiparallel and with $\overrightarrow{BC}$ and $\overrightarrow{EF}$ antiparallel then $\angle ABC \cong \angle DEF$.

EXERCISES 7.8

Draw a line and on it mark five points A, B, C, D, E in the order named. In Exercises 1–20, state whether the given statement is true or false.

1. $\overleftrightarrow{AB} = \overleftrightarrow{AE}$
2. $\overrightarrow{AB} = \overrightarrow{AE}$
3. $\overline{AB} \cong \overline{BA}$
4. $\overline{AB} = \overline{BA}$
5. $\overline{AB} \parallel \overline{BA}$
6. $\overline{AB} \parallel \overline{AC}$
7. $\overline{AB} \parallel \overline{CD}$
8. $\overline{AB} \subset \overline{AC}$
9. $\overline{AB} \subset \overline{BC}$
10. $\overrightarrow{AB} \subset \overrightarrow{AC}$
11. $\overrightarrow{AC} \subset \overrightarrow{BC}$
12. $\overrightarrow{BC} \subset \overrightarrow{AB}$
13. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are parallel.
14. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are antiparallel.
15. $\overrightarrow{AB}$ and $\overrightarrow{ED}$ are parallel.
16. $\overrightarrow{BC}$ and $\overrightarrow{BA}$ are antiparallel.
17. $\overrightarrow{BC} \cap \overrightarrow{BA} = \{B\}$
18. $\overrightarrow{BC} \cap \overrightarrow{DA} = \overrightarrow{BD}$
19. $\overrightarrow{BC} \cap \overrightarrow{DA} = \overline{DB}$
20. $\overrightarrow{DE} \cap \overrightarrow{CD} = \varnothing$

$A, B, C, \ldots, K, L$ are distinct coplanar points, as in Figure 7-43, so that $\overleftrightarrow{DE} \parallel \overleftrightarrow{HI}$ and $\overleftrightarrow{DH} \parallel \overleftrightarrow{EI}$. In Exercises 21–30, state whether the given rays are parallel or antiparallel.

21. $\overrightarrow{DE}$ and $\overrightarrow{HI}$
22. $\overrightarrow{DE}$ and $\overrightarrow{IH}$
23. $\overrightarrow{KH}$ and $\overrightarrow{BE}$
24. $\overrightarrow{DA}$ and $\overrightarrow{LI}$
25. $\overrightarrow{CD}$ and $\overrightarrow{EF}$
26. $\overrightarrow{CF}$ and $\overrightarrow{FC}$
27. $\overrightarrow{CF}$ and $\overrightarrow{IJ}$
28. $\overrightarrow{CF}$ and $\overrightarrow{JI}$
29. $\overrightarrow{GI}$ and $\overrightarrow{HJ}$
30. $\overrightarrow{GI}$ and $\overrightarrow{JH}$

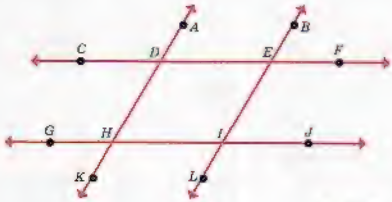


Figure 7-43

In Exercises 31–36, copy and complete the statements.

31. The intersection of two parallel collinear rays is $\boxed{?}$.
32. The intersection of two parallel noncollinear rays is $\boxed{?}$.
33. The intersection of two antiparallel collinear rays is $\boxed{?}$ or $\boxed{?}$ or $\boxed{?}$.
34. The union of two parallel collinear rays is $\boxed{?}$.
35. The union of two antiparallel collinear rays is a line with the interior points of a segment deleted or it is an entire $\boxed{?}$.
36. If $ABCD$ is a parallelogram, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are $\boxed{?}$ rays.

37. Prove Theorem 7.26 for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are collinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are noncollinear and parallel. (See the figure below.) The case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are noncollinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are collinear and parallel can be proved in a similar way.

38. Prove Theorem 7.26 for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are noncollinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are noncollinear and parallel as suggested in the figure. (*Hint:* Think of the figure formed by the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{EF}$. Do you see that this figure contains a parallelogram? Then use Theorem 7.15.)

39. In Exercises 37 and 38, you were asked to prove Theorem 7.26 except for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are collinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are collinear and parallel. Draw a figure for this special case. Note that in this special case the theorem amounts to saying that equal angles are congruent angles.
40. Draw an appropriate figure and prove Theorem 7.27 for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are collinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are collinear and antiparallel.
41. Draw an appropriate figure and prove Theorem 7.27 for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are noncollinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are collinear and antiparallel.

42. Draw an appropriate figure and prove Theorem 7.27 for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are collinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are noncollinear and antiparallel.
43. Draw an appropriate figure and prove Theorem 7.27 for the case in which $\overrightarrow{BA}$ and $\overrightarrow{ED}$ are noncollinear and parallel and $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are noncollinear and antiparallel.
44. Prove Theorem 7.28. Do it by cases. In which special case does the assertion of the theorem amount to an assertion that vertical angles are congruent?
45. $ABCD$ is a convex quadrilateral. *Prove:* If

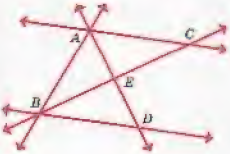
$$m\angle A + m\angle B + m\angle C + m\angle D = 360,$$

if $m\angle A = m\angle C$, and if $m\angle B = m\angle D$, then $ABCD$ is a parallelogram.
46. *Prove:* If $\angle A$ and $\angle B$ are consecutive angles of a parallelogram, then they are supplementary. (How many pairs of consecutive angles does a parallelogram have?)

In Exercises 47–52, write a sentence justifying the truth of the given statement.

47. If two alternate interior angles determined by a transversal of two given coplanar lines are not congruent, then the given lines are not parallel.
48. If two corresponding angles determined by a transversal of two given coplanar lines are not congruent, then the given lines are not parallel.
49. If two consecutive interior angles determined by a transversal of two given coplanar lines are not supplementary, then the given lines are not parallel.
50. If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.
51. If two opposite sides of a quadrilateral are not parallel, then the quadrilateral is not a parallelogram.
52. If $ABCD$ is a parallelogram, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are everywhere equidistant.

53. Given two alternate interior angles determined by a transversal of two parallel lines, prove that the angle bisectors of these angles are antiparallel.
54. The figure shows five coplanar lines and some (perhaps all) of their points of intersection. If $m\angle ABE + m\angle BEA + m\angle EAB = 180$, $\angle BAE \cong \angle EAC$, $\angle ABE \cong \angle DBE$, and $m\angle BEA = 90$, prove that $\overleftrightarrow{BD}$ is parallel to $\overleftrightarrow{AC}$.



7.9 SOME THEOREMS ON TRIANGLES AND QUADRILATERALS

One of the most important consequences of the Parallel Postulate is the following theorem.

THEOREM 7.29 The sum of the measures of the angles of a triangle is 180.

Proof: (See Figure 7-44.) Let $\triangle ABC$ be given. Let m be the unique line through C and parallel to $\overleftrightarrow{AB}$. Let D and E be points on m such that D–C–E and such that B and D are on opposite sides of $\overleftrightarrow{AC}$ as in the figure. Then

$$m\angle A + m\angle B + m\angle ACB$$
$$= m\angle ACD + m\angle BCE + m\angle ACB = 180.$$

(Which theorems justify these equations?)

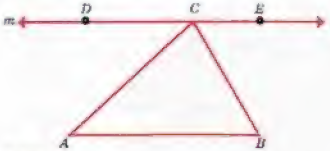


Figure 7-44

It may be of interest to mention here that in the non-Euclidean geometry of Lobachevsky and Bolyai the sum of the measures of the angles of a triangle is less than 180. The sum is not the same number for all triangles in that geometry. The smaller the triangle is the closer the sum is to 180.

THEOREM 7.30 The measure of an exterior angle of a triangle is equal to the sum of the measures of its nonadjacent interior angles.

Proof: Assigned as an exercise.

THEOREM 7.31 Let a one-to-one correspondence between the vertices of two triangles be given. If two angles of one triangle are congruent, respectively, to the corresponding angles of the other triangle, then the third angles of the two triangles are also congruent.

Proof: Assigned as an exercise. (Note Figure 7-45 which suggests that the correspondence need not be a congruence. One of the triangles may be larger than the other.)

Figure 7-45

THEOREM 7.32 (*The S.A.A. Theorem*) Let a one-to-one correspondence between the vertices of two triangles be given. If two angles and a side opposite one of them in one triangle are congruent, respectively, to the corresponding parts of the second triangle, then the correspondence is a congruence.

Proof: Assigned as an exercise.

THEOREM 7.33 The sum of the measures of the angles of a convex quadrilateral is 360.

Proof: Let $ABCD$ be a convex quadrilateral as in Figure 7-46. Draw diagonal $\overline{BD}$. Since $ABCD$ is a convex quadrilateral, D is in the interior of $\angle ABC$ and B is in the interior of $\angle CDA$. Then, using the notation of the figure, we have

$$\begin{aligned} m\angle A + m\angle ABC + m\angle C + m\angle CDA \\ &= m\angle A + (m\angle 1 + m\angle 2) + m\angle C + (m\angle 4 + m\angle 3) \\ &= (m\angle A + m\angle 1 + m\angle 3) + m\angle 2 + (m\angle C + m\angle 4) \\ &= 180 + 180 = 360. \end{aligned}$$

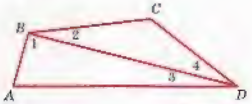


Figure 7-46

THEOREM 7.34 The acute angles of a right triangle are complementary.

Proof: Assigned as an exercise.

THEOREM 7.35 (***The Hypotenuse-Leg Theorem***) Let there be a one-to-one correspondence between the vertices of two right triangles in which the vertices of the right angles correspond. If the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

Proof: (See Figure 7-47.) Let $\triangle ABC$ with $m\angle A = 90$ and $\triangle DEF$ with $m\angle D = 90$ be given. Let it be given that $\overline{BC} \cong \overline{EF}$ and $\overline{AB} \cong \overline{DE}$. We shall prove that $ABC \longleftrightarrow DEF$ is a congruence. You will be asked in Exercise 19 below to supply the missing reasons.

Figure 7-47

Statement	Reason
1. Let G be a point on *opp* $\overrightarrow{DF}$ such that $DG = AC$.	1. The Segment Construction Theorem
2. $m\angle EDG = 90$	2. $\angle EDG$ is a supplement of a right angle
3. $ED = BA$ and $m\angle A = 90$	3. Given
4. $\triangle DEG \cong \triangle ABC$	4. S.A.S.
5. $\overline{EG} \cong \overline{BC}$	5. $\boxed{?}$
6. $\overline{BC} \cong \overline{EF}$	6. Given
7. $\overline{EG} \cong \overline{EF}$	7. $\boxed{?}$
8. $\angle EGD \cong \angle EFD$	8. $\boxed{?}$
9. $\triangle DEG \cong \triangle DEF$	9. S.A.A.
10. $\triangle ABC \cong \triangle DEF$	10. The equivalence properties of congruence for triangles

EXERCISES 7.9

In Exercises 1–6, the measures of two angles of a triangle are given. Find the measure of the third angle of that triangle.

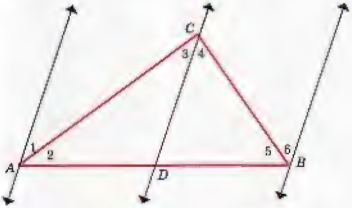
1. 93 and 80.
2. a and b. (Express the answer in terms of a and b.)
3. 25 and x. (Express the answer in terms of x.)
4. $90 + k$ and $90 - 2k$. (Express the answer in terms of k.)

5. $45 + x$ and $45 + x$. (Express the answer in terms of x.)
6. $60 + u$ and $60 + v$. (Express the answer in terms of u and v.)

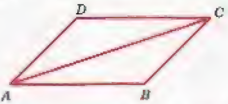
In Exercises 7–11, the measures of the three angles of a triangle are expressed in terms of a number x. In each case, find the value of x and check your answer by adding the angle measures.

7. $x, x - 5, x + 5$
8. $x, 2x, 3x$
9. $x + 1, x + 2, x + 3$
10. $x + 50, 2x + 30, 3x + 80$
11. $3x + 5, 2x - 3, 5x - 8$

12. The figure shows a triangle $\triangle ABC$, a point D between A and B, the line $\overleftrightarrow{CD}$, and the lines parallel to $\overleftrightarrow{CD}$ through A and B. What is the sum of the measures of angles 1, 2, 5, and 6? How does the sum of the measures of $\angle 1$ and $\angle 6$ compare with the sum of the measures of $\angle 3$ and $\angle 4$? How does the sum of the measures of $\angle 3$ and $\angle 4$ compare with the measure of $\angle ACB$? Use these ideas to write an alternate proof of Theorem 7.29.



13. The figure shows a parallelogram $ABCD$ and one of its diagonals $\overline{AC}$.



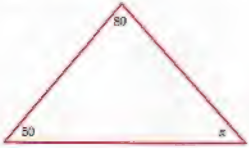
What is the sum of the measures of angles $\angle DAB$ and $\angle B$? Why?
What is the sum of the measures of angles $\angle DCB$ and $\angle D$?
What is the sum of the measures of the angles of a parallelogram?
Why is $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{DA}$, and $\overline{CA} \cong \overline{AC}$?
Why is $\triangle ABC \cong \triangle CDA$?
Use these ideas to write an alternate proof of Theorem 7.29.

14. Prove Theorem 7.30.
15. Prove Theorem 7.31.
16. Prove Theorem 7.32. (Start with two triangles, $\triangle ABC$ and $\triangle A'B'C'$, and a correspondence $ABC \longleftrightarrow A'B'C'$.)

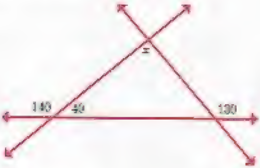
17. Justify the equations at the end of the proof of Theorem 7.33.
18. Prove Theorem 7.34.
19. Copy and supply reasons for steps 5, 7, 8 in the proof of Theorem 7.35.
20. Let $\triangle ABC$ and numbers a, b, c be given. Suppose that $m\angle A = ka$, $m\angle B = kb$, $m\angle C = kc$ for some number k, the same number k in all three cases. If $a = 1$, $b = 2$, $c = 3$, find $m\angle A$, $m\angle B$, $m\angle C$. Check by addition.
21. Same as Exercise 20, except that $a = 2$, $b = 3$, $c = 5$.
22. Same as Exercise 20, except that $a = 100$, $b = 250$, $c = 300$.
23. Same as Exercise 20, except that $a = 180$, $b = 180$, $c = 180$.
24. Same as Exercise 20, except that $a = 1$, $b = 1$, $c = 100$.
25. Same as Exercise 20, except that $a = 100$, $b = 1$, $c = 100$.

In Exercises 26–30, there is a labeled figure in which x denotes the measure of an angle and there is an assertion about x. Justify the assertion. In the figure for Exercise 30, y and z also denote angle measures.

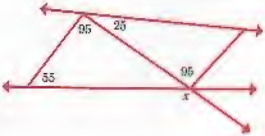
26. $x = 50$



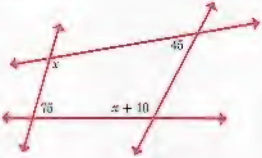
28. $x = 90$



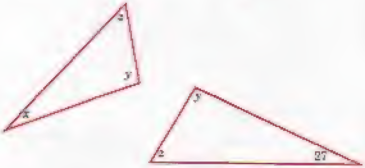
27. $x = 150$



29. $x = 115$

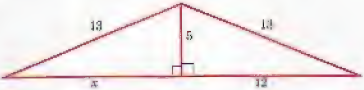


30. $x = 27$

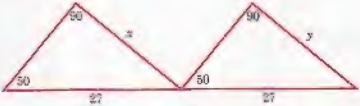


In Exercises 31–35, there is a labeled figure in which x denotes the length of a segment. At the left, there is an assertion about x. Justify the assertion. If y appears in a figure, it also denotes a length.

31. $x = 12$



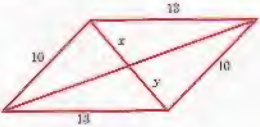
32. $x = y$



33. $x = 10$

34. $x = 11$

35. $x = y$



36. If the measure of the vertex angle of an isosceles triangle is 12.76, find the measure of each base angle.
37. If the measure of each base angle of an isosceles triangle is 83, find the measure of an exterior angle at the vertex of the triangle.
38. Find the sum of the measures of the exterior angles (one at each vertex) of a triangle whose angles have measures as follows: 27.32, 59.39, and 93.29.
39. Given a parallelogram $ABCD$ with $AC = BD$, find the measures of the four angles of the parallelogram.
40. Can a convex quadrilateral have three obtuse angles? Explain your answer.

41. Given a convex quadrilateral $ABCD$ with $m\angle A = 90$, find $m\angle B + m\angle C + m\angle D$. Can $\angle B$ and $\angle C$ both be obtuse angles?

42. Let $ABCD$ be a convex quadrilateral with $m\angle A = 90$ and $m\angle B = 90$. Can $\angle C$ and $\angle D$ both be obtuse angles?

43. Let $\overline{AB}$ and $\overline{CD}$ intersect at E, an interior point of each segment. If $AD = BC$ and $\overline{AD} \parallel \overline{BC}$, prove that $\overline{DC}$ and $\overline{AB}$ bisect each other.

44. The measures of the angles of a triangle are 36, x, and y. We conclude that x is a number less than $\boxed{?}$, that y is a number less than $\boxed{?}$, and that $x + y = \boxed{?}$. (In each case find the smallest number that will make the statement true.)

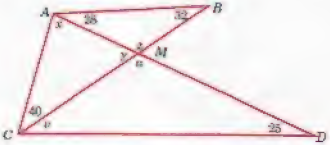
45. The measures af the angles of a convex quadrilateral are a, b, c, d. We conclude that each of these four numbers is less than $\boxed{?}$. (Find the smallest number that will make the statement true.)

46. The measures of the angles of a convex quadrilateral are 20, x, x, and y. We conclude that x is a number between $\boxed{?}$ and $\boxed{?}$, that y is a number between $\boxed{?}$ and $\boxed{?}$. (Make the smaller number as large as possible and the larger number as small as possible in each case.)

47. Given the plane figure in which $\overrightarrow{AC}$ is the bisector of $\angle BAD$, $\overline{CB} \perp \overline{AB}$, and $\overline{CD} \perp \overline{AD}$, prove that $BC = DC$.

48. *Prove:* If each angle of a convex quadrilateral is congruent to the angle opposite to it, then the quadrilateral is a parallelogram. (*Hint:* Given quadrilateral $ABCD$ with $\angle A \cong \angle C$, $\angle B \cong \angle D$, find $m\angle A + m\angle B$ and deduce that two sides are parallel. Find $m\angle A + m\angle D$ and deduce that the other two sides are parallel.)

49. Given $\triangle ABC$ and $\triangle ADC$ with $\overline{AD}$ intersecting $\overline{BC}$ at M, an interior point of $\overline{AD}$ and $\overline{BC}$ as indicated in the figure, find the angle measures x, y, z, u, v.



CHAPTER SUMMARY

The central theme of this chapter is parallel lines, and the high point of the chapter is the PARALLEL POSTULATE. Before introducing the Parallel Postulate we proved several theorems on conditions which imply that lines are parallel. These are theorems in EUCLIDEAN GEOMETRY as well as in the NON-EUCLIDEAN GEOMETRY of Lobachevsky and Bolyai. These theorems are concerned with lines and TRANSVERSALS and angles associated with them. The Parallel Postulate gives us what we need to prove the converses of these theorems.

PARALLELISM for lines in a given plane is an equivalence relation, that is, it is reflexive, symmetric, and transitive. The same is true for parallelism of segments in a given plane and for rays in a given plane.

With parallel lines it is natural to associate PARALLELOGRAMS. This chapter includes several theorems regarding properties of parallelograms and several others regarding conditions under which quadrilaterals are parallelograms.

An important consequence of the Parallel Postulate is the theorem on the sum of the measures of the angles of a triangle. An easy corollary of this theorem is the theorem on the sum of the measures of the angles of a convex quadrilateral. Another important consequence of the Parallel Postulate is the result that parallel lines are everywhere equidistant.

Along with the idea of PARALLEL RAYS we introduced the idea of ANTIPARALLEL RAYS. There were several theorems regarding the relationship of two angles with parallel or antiparallel sides.

Early in the book we introduced the S.A.S., A.S.A., and S.S.S. Postulates. In this chapter there were two results which extend these congruence ideas, namely the S.A.A. THEOREM and the HYPOTENUSE-LEG THEOREM for right triangles.

REVIEW EXERCISES

Copy and complete the definitions in Exercises 1–10.

1. Line m is parallel to line n if and only if $\boxed{?}$.
2. Two lines are skew lines if and only if $\boxed{?}$.
3. Two segments are parallel if and only if $\boxed{?}$.
4. Two collinear rays are parallel if and only if $\boxed{?}$.
5. Two noncollinear rays are parallel if and only if $\boxed{?}$.
6. Two collinear rays are antiparallel if and only if $\boxed{?}$.
7. Two noncollinear rays are antiparallel if and only if $\boxed{?}$.
8. If distinct lines m, n, t are coplanar, then t is a transversal of m and n if and only if $\boxed{?}$.

9. A parallelogram is a convex quadrilateral whose $\boxed{?}$.
10. The distance between two distinct parallel lines is $\boxed{?}$.

■ Exercises 11–20 pertain to the geometrical figure suggested by Figure 7-48. It is given that lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ are parallel and noncoplanar. Also,

$$AB = CD = EF.$$

In each exercise tell whether the given statement is true or false, and explain why it is true or why it is false.

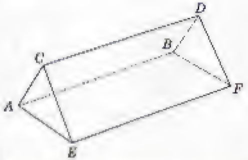


Figure 7-48

11. There is more than one plane containing A and B.
12. There is more than one plane containing A, B, and C.
13. There is exactly one plane containing A, B, E, and F.
14. There is exactly one plane containing A, E, C, and F.
15. Quadrilateral $ABDC$ is a parallelogram.
16. Quadrilateral $ABFE$ is a parallelogram.
17. Quadrilateral $CDFE$ is a parallelogram.
18. $\triangle ACE \cong \triangle BDF$
19. $\triangle ABC \cong \triangle DCB$
20. $m\angle EAC + m\angle ACE + m\angle CEA + m\angle FBD + m\angle BDF + m\angle DFB$
    $= m\angle CAB + m\angle ABD + m\angle BDC + m\angle DCA$

■ Exercises 21–30 pertain to the geometrical figure suggested by Figure 7-49. Line t is a transversal of lines m and n. The measures of eight angles formed by m, n, and t have been marked in the figure. In each exercise, tell whether the statement is true or false. If it is true, state a theorem that justifies your answer.

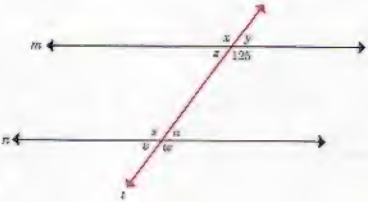


Figure 7-49

21. If $s = 125$, then $m \parallel n$.
22. If $u = 65$, then $m \parallel n$.
23. If $w = 125$, then $m \parallel n$.
24. If $m \parallel n$, then $s = 125$.
25. If $m \parallel n$, then $u = 65$.
26. If $m \parallel n$, then $w = 125$.
27. If $u = 60$, then m is not parallel to n.
28. If $s = 140$, then m is not parallel to n.
29. If $y \neq u$, then m is not parallel to n.
30. $x = 126$

31. In your own words write a sentence that tells of a significant difference between the geometry of Euclid and the geometry of Lobachevsky and Bolyai.
32. Explain what we mean when we say that the equals relation is reflexive, symmetric, and transitive.
33. Explain what we mean when we say that the relation of parallelism for lines is reflexive, symmetric, and transitive.
34. Is the relation of parallelism for rays an equivalence relation?
35. Is the relation of antiparallelism for rays an equivalence relation?

In Exercises 36–42, you are asked to prove a statement. You may use anything that we have had in our formal geometry structure up to this point (except the theorem you are asked to prove) in writing your proof.

36. Prove that opposite sides of a parallelogram are congruent.
37. Prove that if two opposite sides of a convex quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.
38. Prove that if opposite sides of a convex quadrilateral are congruent, then the quadrilateral is a parallelogram.
39. Prove that opposite angles of a parallelogram are congruent.
40. Prove that two consecutive angles of a parallelogram are supplementary.
41. Prove that the two acute angles of a right triangle are complementary angles.
42. State and prove the theorem regarding the angle measure sum of a triangle.

43. In the geometry of Lobachevsky and Bolyai is it true that two alternate interior angles formed by two parallel lines and a transversal are congruent?
44. In the geometry of Euclid is it true that two alternate interior angles formed by two parallel lines and a transversal are congruent?
45. In the geometry of Lobachevsky and Bolyai is it true that if two alternate interior angles formed by two distinct lines and a transversal of them are congruent, then the lines are parallel?
46. In the geometry of Euclid is it true that if two alternate interior angles formed by two distinct coplanar lines and a transversal of them are congruent, then the lines are parallel?

Chapter 8

Fritz Henle/Photo Researchers

Perpendicularity and Parallelism in Space

8.1 INTRODUCTION

This chapter is concerned mostly with figures that do not lie in one plane. We use the word *figure* in two ways: as a set of points (geometrical figure) and as a picture or drawing that represents a geometrical figure. The figures in this book and the figures that you draw as you study geometry are important. In communicating information regarding geometrical figures one drawing may be worth 473 words.

When drawing a figure, the perspective view is the best one from the standpoint of communicating how the figure looks, and that is the view used in the illustrations in this book. However, the oblique view is the easiest to draw; therefore instructions for drawing figures in oblique views are given on the following two pages.

A plane may be suggested by a parallelogram as in the figure which illustrates two intersecting planes. Note that the plane suggested by a parallelogram includes more than a parallelogram and its interior. It includes all of the points in the plane of the parallelogram, the points in the exterior of the parallelogram as well as those on the parallelogram and in its interior.

PROCEDURES FOR
DRAWING GEOMETRIC
FORMS IN
OBLIQUE VIEWS

Draw front plane, edge or edges of form, in normal elevation view.

Project from the corners, at a 45° angle, receding parallel lines, whose unit of measurement should be 2/3 of those in the elevation view.

The black dashes are guide lines to be erased later.

Continue projecting backwards from all corners.

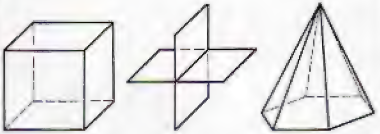
Draw vertical guide line for the center of the pyramid.

Draw horizontal lines on bottom planes.

Draw all vertical lines.

Finish drawing horizontal lines.

Lines that are inside or behind an object are by convention drawn as dashes.



A horizontal plane is oriented like the top of the cube. A vertical plane is oriented like the right face of the cube. Note that horizontal and vertical are not defined formally in our geometry. They are descriptive terms that we use to describe figures.

The main topics in this chapter are perpendicularity and parallelism for lines and planes. In the preceding chapters of this book we have given detailed proofs for most of the theorems that were stated. Proofs not included in the text were assigned as exercises. In this chapter, however, we sometimes state theorems without giving proofs and without assigning these proofs in the exercises. In such cases we may include an outline of the main steps or a short statement of the strategy or main idea of a possible proof. In all cases it is possible to write detailed proofs based on the postulates of our formal geometry. Our abbreviated presentation is appropriate for a first course in formal geometry.

8.2 A PERPENDICULARITY DEFINITION

Definition 8.1 A line and a plane are perpendicular if the line intersects the plane and is perpendicular to every line in the plane through the point of intersection.

If a line l and a plane α are perpendicular, we say that l is perpendicular to α and that α is perpendicular to l.

Notation. $l \perp \alpha$, or $\alpha \perp l$, means that α and l are perpendicular; $l \perp \alpha$ at P, or $\alpha \perp l$ at P, means that $l \perp \alpha$ and that P is their point of intersection.

Figure 8-1 represents a line l and a plane α that are perpendicular at P. The figure shows four of the lines through P and in α. Actually, there are infinitely many such lines and they are all perpendicular to l at P.

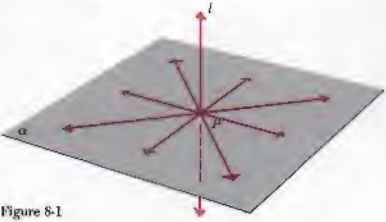


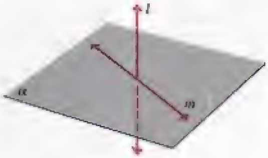
Figure 8-1

EXERCISES 8.2

In Exercises 1–9, draw the figure that is described by the following statements.

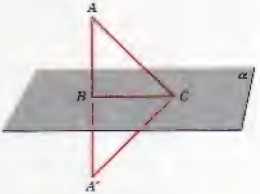
1. Two distinct horizontal planes.
2. Two distinct vertical planes.
3. A horizontal plane and a vertical line.
4. A vertical plane and a horizontal line.
5. A vertical plane and a line perpendicular to it.
6. Two distinct parallel planes and a line that intersects both of them.
7. Two distinct parallel lines and a plane that is parallel to both of the lines.
8. Two distinct intersecting planes, one vertical plane and one horizontal plane.
9. Two congruent triangles lying in distinct parallel planes and the segments connecting corresponding vertices.

10. See the figure. If $l \perp m$, and m lies in α, does it follow that $l \perp \alpha$? Explain.

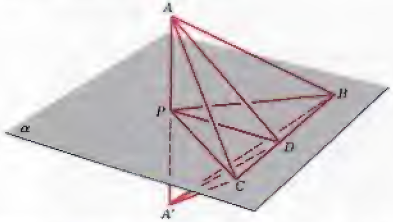


11. In the figure, A-B-A', $AB = A'B$, $\overleftrightarrow{AA'} \perp \alpha$ at B, $B \neq C$, and $C \in \alpha$. Prove that

$$AC = A'C.$$



12. In the following figure, $\overleftrightarrow{AA'} \perp \alpha$ at P, A'–P–A, $AP = A'P$, and B–D–C. On the basis of Exercise 11, what can you say about the lengths AB and $A'B$, AC and $A'C$, AD and $A'D$? Now prove that $\triangle ABD \cong \triangle A'BD$, $\triangle ACD \cong \triangle A'CD$, $\triangle ABC \cong \triangle A'BC$.



13. In the figure in Exercise 12, $\overline{AP} \perp \overline{PB}$; $\overline{AP} \perp \overline{PC}$; A'–P–A; B–D–C; α is the plane containing the noncollinear points B, C, P; $AB = A'B$; $AC = A'C$. State the postulate which implies that D is in α. Prove that $AD = A'D$.
14. *(An informal geometry exercise.)* Let l and m be two distinct lines in the plane of a table top and intersecting at a point P. Hold a yardstick so that it appears to be perpendicular to l and to m at P. Use another yardstick to represent a third line n in the plane of the table top and passing through P. Does the first yardstick appear to be perpendicular to line n? Try n in various positions and see if you can find a position such that the yardstick and line n are no longer perpendicular. Does it appear that the yardstick is perpendicular to line n in all cases? Does it appear that the yardstick is perpendicular to the table?

8.3 A BASIC PERPENDICULARITY THEOREM

Theorem 8.1 is introduced to help us prove a basic perpendicularity theorem, Theorem 8.2. Theorem 8.2 is suggested by our experiences with perpendicular lines and planes. See Exercise 14 of Exercises 8.2.

THEOREM 8.1 If A, A', B, C, D are distinct points with B and C each equidistant from A and A' and with D on $\overleftrightarrow{BC}$, then D is equidistant from A and A'.

Proof: Given distinct points A, A', B, C, D, with $AB = A'B$, $AC = A'C$, and D on $\overleftrightarrow{BC}$ as suggested in Figure 8.2, we want to prove that $AD = A'D$. Then D–B–C, or B–D–C, or B–C–D. The proof may be completed by showing that $\triangle ABC \cong \triangle A'BC$, $\triangle ABD \cong \triangle A'BD$, $\triangle ACD \cong \triangle A'CD$, and hence that $AD = A'D$.

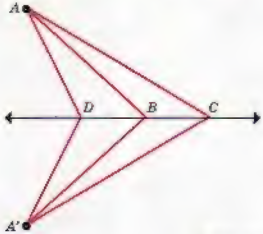


Figure 8-2

THEOREM 8.2 If a line is perpendicular to each of two distinct intersecting lines at their point of intersection, then it is perpendicular to the plane that contains them.

Proof: Let lines l, m, n be given with m, n in plane α and such that $l \perp m$ at A, $l \perp n$ at A, as suggested in Figure 8-3. Let p be any line distinct from m and n that lies in α and passes through A. We want to prove that $l \perp p$. Let q be a line in α that is not parallel to m, or to n, or to p, and that does not pass through A. Then q intersects m, p, n in three distinct points; call them M, P, N, respectively.

Let Q and Q' be two points of l such that Q–A–Q' and $QA = Q'A$. The proof may be completed by showing that M is equidistant from Q and Q', N is equidistant from Q and Q', P is equidistant from Q and Q',

$$\triangle QPA \cong \triangle Q'PA, \qquad \angle QAP \cong \angle Q'AP, \qquad \text{and} \qquad l \perp p.$$

Since p is an arbitrary line other than m and n in the plane α and passing through the point A, we have $l \perp \alpha$.

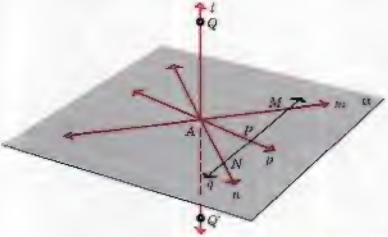
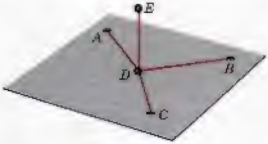


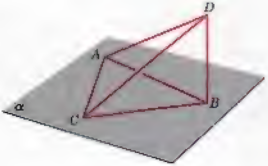
Figure 8-3

EXERCISES 8.3

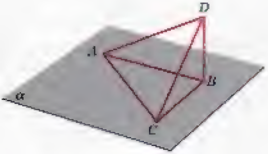
1. Let the distinct points A, B, C, D, E be given as suggested in the figure. A, B, C, D are coplanar points with no three of them collinear. Also $\overline{ED} \perp \overline{DB}$ and $\overline{ED} \perp \overline{DC}$. Prove that $\overline{ED} \perp \overline{DA}$.



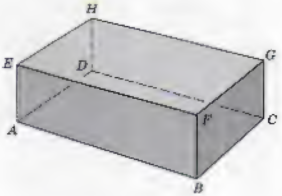
2. Let the distinct points A, B, C, D be given as suggested in the figure. A, B, C are noncollinear points in a plane α, $\overleftrightarrow{BD} \perp \alpha$, and $AB = BC$. Prove that $\angle DAC \cong \angle DCA$.



3. Let the distinct points A, B, C, D be given as suggested in the figure. $\overline{AB} \perp \overline{BD}$, and $\overline{AB} \perp \overline{BC}$. A, B, C are noncollinear points in α, and D is a point not in α. Prove that $\overleftrightarrow{AD}$ is not perpendicular to α.



4. For the situation in Exercise 3, prove that $\overleftrightarrow{AD}$ is not perpendicular to the plane determined by B, C, D. Is $\overleftrightarrow{AB}$ perpendicular to the plane BCD?
5. Let four noncoplanar points A, B, C, D be given such that
$$AB = AC = AD,$$
$\overleftrightarrow{AB} \perp$ plane ACD, $\overleftrightarrow{AC} \perp$ plane BAD, $\overleftrightarrow{AD} \perp$ plane BAC. Draw an appropriate figure and prove that $\triangle BCD$ is an equilateral triangle.
6. A rectangular solid has the property that if two of its edges intersect, then they intersect at right angles. The following figure shows a rectangular solid with the eight vertices labeled. Prove that $\overleftrightarrow{EA}$ is perpendicular to the plane that contains B, C, D.



7. Assume the same situation as in Exercise 6. Since $\overleftrightarrow{EA}$ is perpendicular to the plane that contains the quadrilateral $ABCD$, we say that edge $\overline{EA}$ is perpendicular to face $ABCD$. Name six other combinations of an edge and a face that are perpendicular. (There are 24 such combinations altogether.)
8. In the proof of Theorem 8.1, suppose that D–B–C. Which Congruence Postulate would you use to show that $\triangle ABC \cong \triangle A'BC$? That $\triangle ABD \cong \triangle A'BD$?
9. Explain how the proof of Theorem 8.1 for the case in which B–D–C differs from the proof for the case in which D–B–C. Draw an appropriate figure.
10. Draw a figure like Figure 8-3, except much larger. Draw the segments connecting Q and Q' with M, P, N. In completing the proof of Theorem 8.2, which theorem, postulate, or definition plays a key role in proving that
    (a) M is equidistant from Q and Q' and that N is equidistant from Q and Q'?
    (b) P is equidistant from Q and Q'?
    (c) $\triangle QPA \cong \triangle Q'PA$?
    (d) $\angle QAP \cong \angle Q'AP$?
    (e) $l \perp p$?

8.4 OTHER PERPENDICULARITY THEOREMS

THEOREM 8.3 If a line and a plane are perpendicular, then the plane contains every line perpendicular to the given line at the point of intersection of the given line and the given plane.

Proof: (See Figure 8-4.) Let $l \perp \alpha$ at P. Let l_1 be any line such that $l_1 \perp l$ at P.

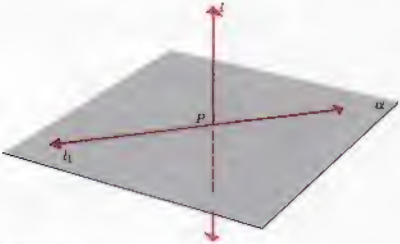


Figure 8-4

We want to prove that l_1 lies in α. Let β be the plane that contains l and l_1. (See Figure 8-5.) Then $\beta \neq \alpha$ since β contains l and α does not. The intersection of β and α is a line; call it l_2. Then $l_2 \perp l$. Why? Also $l_1 \perp l$. Why? Then $l_2 = l_1$. Why? Therefore l_1 lies in α. Since l_1 is any arbitrary line perpendicular to l at P, the proof is complete.

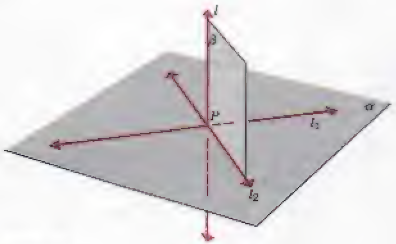


Figure 8-5

THEOREM 8.4 Given a line and a point, there is a unique plane perpendicular to the line and containing the point.

Proof: Let l be a given line and P a given point. We divide the proof into four parts.

1. If $P \in l$, we must prove that there is at least one plane perpendicular to l at P.
2. If $P \in l$, we must prove that there is at most one plane perpendicular to l at P.
3. If $P \notin l$, we must prove that there is at least one plane perpendicular to l and passing through P.
4. If $P \notin l$, we must prove that there is at most one plane perpendicular to l and passing through P.

Parts 1 and 3 are existence proofs; parts 2 and 4 are uniqueness proofs.

Proof of 1: If P is a point on a line l as in Figure 8-6, let α be a plane containing l and let m be the line in α that is perpendicular to l at P.

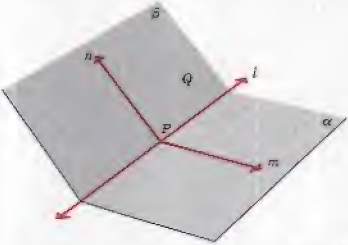


Figure 8-6

Let Q be a point not in α and let β be the plane that contains Q and l. Let n be the line in β that is perpendicular to l at P. Let γ be the plane that contains m and n. Then $l \perp \gamma$ at P. Why? This proves that there is at least one plane perpendicular to l at P.

Proof of 2: If point P lies on a line l as in Figure 8-7, we shall prove that there is at most one plane perpendicular to l at P. Suppose, con-

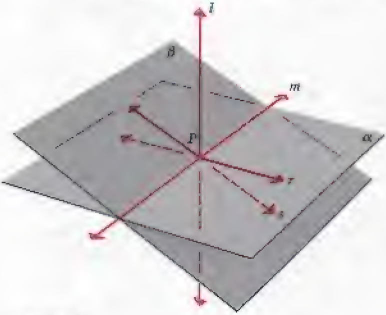


Figure 8-7

trary to what we assert, that there are two distinct planes α and β each perpendicular to l at P. Then α and β intersect. Why? Their intersection is a line. Why? Call this line m. Let r and s be the lines such that r is in α and $r \perp m$ at P, and s is in β and $s \perp m$ at P. Let γ be the plane that contains r and s. Then $m \perp \gamma$ at P (Why?), $m \perp l$ at P (Why?), and l lies in γ. Then l, r, s are three distinct coplanar lines with $r \perp l$ and $s \perp l$. But this is impossible. There cannot be two distinct planes each perpendicular to l at P. Therefore there is at most one plane perpendicular to l at P.

Proof of 3: Given a line l and a point P not on l as in Figure 8-8, let Q be the foot of the perpendicular from P to l. Let α be the unique plane such that $\alpha \perp l$ at Q. How do we know that there is one and only one such plane α? Then P lies in α. Why? Therefore there is at least one plane containing P and perpendicular to l.

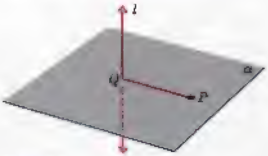


Figure 8-8

Proof of 4: Given a line l and a point P not on it, as in Figures 8-9 and 8-10, we shall prove that there is at most one plane perpendicular to l and containing P. Suppose, contrary to our assertion, that there are two distinct planes α and β each perpendicular to l and each containing P. Then the intersection of α and β is a line; call it m.

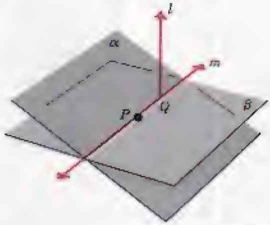


Figure 8-9

Suppose l intersects m as in Figure 8-9. Then l intersects m in a point Q different from P. Why? Then α and β are each perpendicular to l at Q. But this is impossible. Why? Therefore l does not intersect m. Therefore there are two distinct points Q and R, as in Figure 8-10, such that $l \perp \alpha$ at Q and $l \perp \beta$ at R. Then $\triangle RPQ$ has two right angles.

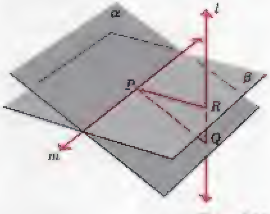


Figure 8-10

But this is impossible. Why? Therefore there cannot be two distinct planes each perpendicular to l and each containing P. Hence there is at most one plane perpendicular to l and containing P.

Definition 8.2 If A and B are distinct points, the unique plane that is perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$ is called the **perpendicular bisecting plane** of $\overline{AB}$.

THEOREM 8.5 The perpendicular bisecting plane of a segment is the set of all points equidistant from the endpoints of the segment.

Proof: Let α be the perpendicular bisecting plane of $\overline{AB}$ and let C be the midpoint of $\overline{AB}$. (See Figure 8-11.) Let P be a point. There are two things to prove.

1. If $P \in \alpha$, then $AP = PB$.
2. If $AP = PB$, then $P \in \alpha$.

You are asked to prove these two statements as exercises.

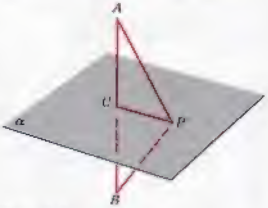


Figure 8-11

THEOREM 8.6 Given two perpendicular lines, there is a unique line that is perpendicular to each of the given lines at their point of intersection.

Proof: Let l and m be lines that are perpendicular to each other at P as in Figure 8-12. Let α be the unique plane that is perpendicular to l at P. Let n be the unique line in α that is perpendicular to m at P. Then n is also perpendicular to l. Why? Therefore there is at least one line perpendicular to both l and m at their point of intersection.

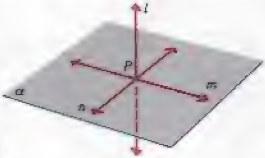


Figure 8-12

Suppose that n and n' are distinct lines each perpendicular to m and to l at P as in Figure 8-13.

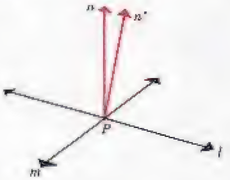


Figure 8-13

Let α be the unique plane that is perpendicular to l at P. Then n, n', m are three distinct lines in α and $n \perp m$, $n' \perp m$. Since this is impossible, there is at most one line perpendicular to both l and m at their point of intersection.

Since there is at least one line and at most one line that is perpendicular to l and m at P, it follows that there is one and only one line that is perpendicular to both l and m at their point of intersection.

THEOREM 8.7 If two lines are perpendicular to the same plane, they are parallel.

Proof: Let l and m be lines that are perpendicular to plane α, as in Figure 8-14. If $l = m$, then $l \parallel m$ and there is nothing more to prove. Suppose, then, that $l \neq m$. Then it can be shown that l and m are nonintersecting lines. (If l and m intersect at a point A not in α, then A and the two points where l and m cut α are the vertices of a triangle with two right angles. If l and m intersect at a point A in α, then the plane containing l and m intersects α in a line n such that $n \perp l$, $n \perp m$.)

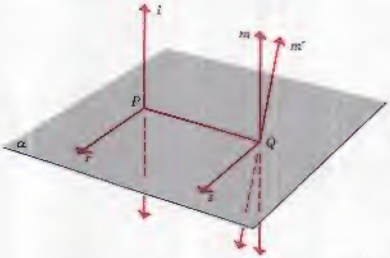


Figure 8-14

Let P and Q be the points in which l and m, respectively, intersect α. Suppose, contrary to what we shall prove, that l and m are skew lines. Let m' be the unique line through Q and parallel to l. Then $l \perp \overline{PQ}$ and $m' \perp \overline{PQ}$. Why? Let r and s be the unique lines in α that are perpendicular to $\overline{PQ}$ at P and at Q, respectively. Then $l \perp r$ and it follows from Theorem 7.26 that $m' \perp s$. Since $m' \perp \overline{PQ}$ and $m' \perp s$, it follows that $m' \perp \alpha$. Let β be the plane that contains m and m' and let t be the line in which β and α intersect. Then $m \perp t$, $m' \perp t$, and the three distinct lines m, m', t are coplanar. Since this is impossible, it follows that l and m are not skew lines. Therefore they are coplanar nonintersecting lines, that is, they are parallel.

THEOREM 8.8 If one of two distinct parallel lines is perpendicular to a plane, then the other line is also perpendicular to that plane.

Proof: Given two distinct parallel lines l and m and a plane α perpendicular to l, let P and Q be the points in which l and m intersect α, as suggested in Figure 8-15. Let $r = \overleftrightarrow{PQ}$, let s be the unique line in α perpendicular to r at Q, and let m' be the unique line perpendicular to r and s at Q (see Theorem 8.6). It follows from Theorem 8.2 that m' is perpendicular to α, from Theorem 8.7 that m' is parallel to l, and from the Parallel Postulate that $m' = m$. Since m' is perpendicular to α, it follows that m is perpendicular to α.

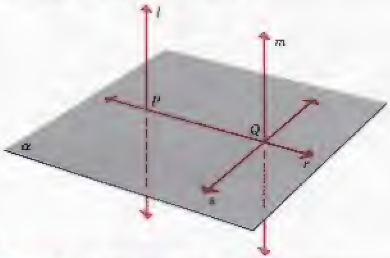


Figure 8-15

THEOREM 8.9 Given a plane and a point, there is a unique line containing the given point and perpendicular to the given plane.

Proof: Let a plane α and a point P be given. We shall prove that there is a unique line through P and perpendicular to α. We divide the proof into four parts: two existence proofs 1 and 3 and two uniqueness proofs 2 and 4.

1. If $P \in \alpha$, we must prove that there is at least one line perpendicular to α at P.
2. If $P \in \alpha$, we must prove that there is at most one line perpendicular to α at P.
3. If $P \notin \alpha$, we must prove that there is at least one line perpendicular to α and containing P.
4. If $P \notin \alpha$, we must prove that there is at most one line perpendicular to α and containing P.

Proof of 1: Given a plane α and a point P in α as suggested in Figure 8-16, let m and n be two lines in α that are perpendicular to each other at P. Then it follows from Theorem 8.6 that there is a unique line l that is perpendicular to both m and n at P. Then $l \perp \alpha$ at P. This proves that there is at least one line perpendicular to α at P.

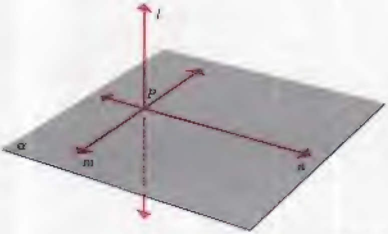


Figure 8-16

Proof of 2: Given a plane α and a point P in α, as suggested in Figure 8-17, suppose contrary to what we shall prove, that m and n are distinct lines and that each is perpendicular to α at P. Let β be the plane that contains m and n and let l be the line in which β intersects α. Then $m \perp l$, $n \perp l$, and the three lines l, m, n are coplanar. This is impossible. Therefore there cannot be two distinct lines through P each perpendicular to α.

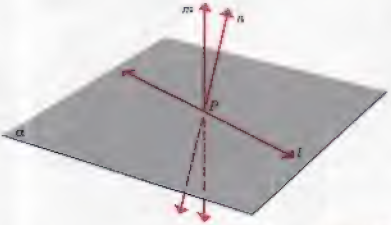


Figure 8-17

Proof of 3: Given a plane α and a point P not in α, as suggested in Figure 8-18, we shall prove that there is at least one line through P and perpendicular to α. Let Q be any point of α. If $\overleftrightarrow{PQ} \perp \alpha$, there is nothing more to prove. Suppose, then, that $\overleftrightarrow{PQ}$ is not perpendicular to α.

Let l be the unique line such that $l \perp \alpha$ at Q. How do you know there is one and only one such line l? Let m be the unique line through P and parallel to l. Then it follows from Theorem 8.8 that $m \perp \alpha$. Therefore there is at least one line through P and perpendicular to α.

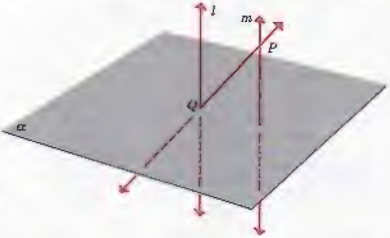


Figure 8-18

Proof of 4: Let a plane α and a point P not in α be given. If there were two distinct lines through P each perpendicular to α, then there would be a triangle with two right angles. Since this is impossible, it follows that there is at most one line through P and perpendicular to α.

EXERCISES 8.4

1. Let four noncoplanar points A, B, C, D such that $\overline{AB} \cong \overline{AC} \cong \overline{AD}$, $\overline{AB} \perp \overline{AC}$, $\overline{AB} \perp \overline{AD}$, $\overline{AC} \perp \overline{AD}$ be given. Draw an appropriate figure and prove that $\triangle BCD$ is not a right triangle.
2. Let four noncoplanar points A, B, C, D such that $\overline{AB} \perp \overline{AC}$, $\overline{AB} \perp \overline{AD}$, $\overline{AC} \perp \overline{AD}$ be given. If $\overrightarrow{AE}$ is the bisector of $\angle BAC$, prove that $\angle DAE$ is a right angle.
3. Let l, m, n be three distinct lines, not necessarily coplanar, and such that $l \parallel m$ and $m \parallel n$. Use Theorems 8.4, 8.7, and 8.8 to prove that $l \parallel n$. (This proves that parallelism for lines is a transitive relation.)
4. Let A, B, C, D, E, F be distinct points such that A-B-C, $AB = BC$, $AD = DC$, $AE = EC$, $AF = FC$. Draw an appropriate figure and explain why B, E, D, F are coplanar points.
5. Let segments $\overline{AB}$ and $\overline{CD}$ and three points E, F, G be given. If $\overline{AB}$ and $\overline{CD}$ are perpendicular and intersect at E, if F–E–G and $\overline{FG} \perp \overline{AB}$, does it follow necessarily that $\overleftrightarrow{FG}$ is perpendicular to plane ABC? Draw an appropriate figure.
6. Let segments $\overline{AB}$ and $\overline{CD}$ and three points E, F, G be given. If $\overline{AB}$ and $\overline{CD}$ are perpendicular and intersect at E, if F–E–G, $\overline{FG} \perp \overline{AB}$, and $\overline{FG} \perp \overline{CD}$, does it follow necessarily that $\overline{FG}$ is perpendicular to plane ABC? Draw an appropriate figure.

7. In part 2 of the proof of Theorem 8.4, how do we know that the intersection of α and β is a line?
8. In part 2 of the proof of Theorem 8.4, how do we know that $m \perp \gamma$ at P?
9. In part 3 of the proof of Theorem 8.4, how do we know that P lies in α?
10. Given two distinct points A and B in a plane α describe the set of all points in α that are equidistant from A and B.
11. Given two distinct points A and B describe the set of all points that are equidistant from A and B.
12. Given a line l and the set $\mathcal{S}$ of all lines parallel to l, prove that if P is any point in space, then P lies on one and only one of the lines in $\mathcal{S}$.
13. Give a plane α and the set $\mathcal{S}$ of all lines perpendicular to α, prove that if P is any point in space, then P lies on one and only one of the lines in $\mathcal{S}$.
14. See the proof of Theorem 8.5. Prove that if $P \in \alpha$, then $AP = PB$.
15. See the proof of Theorem 8.5. Prove that if $AP = PB$, then $P \in \alpha$.
16. Given $\triangle ABC$ and two points D and E such that $\overline{DA} \perp \overline{AB}$, $\overline{DA} \perp \overline{AC}$, $\overline{EB} \perp \overline{AB}$, $\overline{EB} \perp \overline{BC}$, prove that D, E, A, B are coplanar points.
17. See the proof of Theorem 8.9. In part 4 of this proof we asserted that if there were two distinct lines through P each perpendicular to α, then there would be a triangle with two right angles. Draw an appropriate figure and prove this assertion.

8.5 PARALLELISM FOR LINES AND PLANES

In this section we shall investigate the properties of planes parallel to planes and of lines parallel to planes. We begin with two definitions.

Definition 8.3 Two planes are parallel if their intersection is not a line.

Definition 8.4 A line and a plane are parallel if their intersection is not a point.

Let us consider Definition 8.3 first. If α and β are planes, then the intersection of α and β is (1) the null set, or (2) a line, or (3) a plane.

If $\alpha \cap \beta = \varnothing$, then $\alpha \neq \beta$ and $\alpha \parallel \beta$.

If $\alpha \cap \beta$ is a plane, then $\alpha \cap \beta = \alpha = \beta$ and $\alpha \parallel \beta$.

If $\alpha \cap \beta$ is a line, then α is not parallel to β.

Therefore α is parallel to β if and only if

$$\alpha = \beta \qquad \text{or} \qquad \alpha \cap \beta = \varnothing.$$

It follows immediately from Definition 8.3 that parallelism for planes is reflexive and symmetric. Later we shall see that parallelism for planes is also transitive, and therefore an equivalence relation.

The fact that parallelism for lines is an equivalence relation follows from Definition 7.1, and Theorems 8.4, 8.7, and 8.8. See Exercises 10, 11, 12 of Exercises 7.6 and Exercise 3 of Exercises 8.4.

Now let us consider Definition 8.4. If l is a line and α a plane, then the intersection of l and α is (1) the null set, or (2) a point, or (3) a line. If $l \cap \alpha = \varnothing$, then $l \parallel \alpha$. If $l \cap \alpha$ is a line, then that line is l and $l \parallel \alpha$. If $l \cap \alpha$ is a point, then l and α are not parallel. Therefore $l \parallel \alpha$ if and only if $l \subset \alpha$ or $l \cap \alpha = \varnothing$.

THEOREM 8.10 If a plane intersects one of two distinct parallel lines but does not contain it, then it intersects the other line and does not contain it.

Proof: (See Figure 8-19.) Let l and m be two distinct parallel lines. Let α be the plane that contains l and m. Let β be a plane that intersects one of the lines, say l, in a single point P. Now P lies in both β and α. Since α contains l and β does not contain l, it follows that $\alpha \neq \beta$. Therefore the intersection of β and α is a line; call it n. Since n lies in β and l does not lie in β, it follows that n is different from l. Also, m does not lie in β. (Suppose m lies in β. Then m lies in α and β, and $m = n$. Since n is not parallel to l, it follows that m is not parallel to l. Contradiction. Therefore m does not lie in β.)

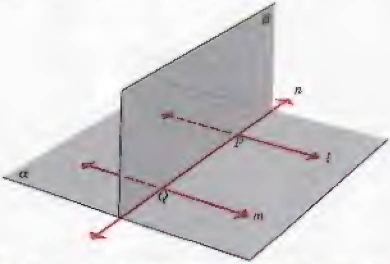


Figure 8-19

Therefore m, n, l are distinct lines in plane α with l parallel to m and with n intersecting l in a single point. It follows from the Parallel Postulate that n intersects m in a single point; call it Q. (Suppose n does not intersect m. Then n and l are two distinct lines through P and parallel to m. Contradiction.) It follows that β contains Q and hence that β intersects m in a single point.

THEOREM 8.11 If a plane is parallel to one of two parallel lines, it is parallel to the other also.

Proof: Assigned as an exercise.

THEOREM 8.12 If a plane intersects two distinct parallel planes, the intersections are two distinct parallel lines.

Proof: Let α and β be two distinct parallel planes and γ a plane that intersects both of them, as suggested in Figure 8-20. Then γ is distinct from α and from β, and it intersects each of them in a line. Let l and m be the lines in which γ intersects α and β, respectively; then l and m are distinct coplanar lines which do not intersect. Why do they not intersect? Therefore l and m are distinct parallel lines.

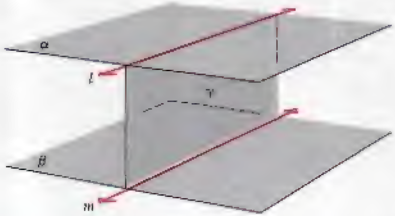


Figure 8-20

THEOREM 8.13 If α, β, γ are three distinct planes such that β is parallel to γ and such that α intersects β, then α intersects γ.

Proof: (See Figure 8-21.) Suppose that α, β, γ are distinct planes, that β and γ are parallel, and that α intersects β.

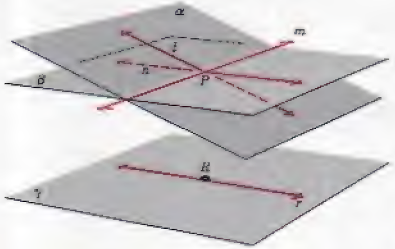


Figure 8-21

Suppose, contrary to the assertion of the theorem, that α does not intersect γ. Let m be the line in which α and β intersect. Let P be a point of m; let l be a line through P that lies in α but not in β, and R a point of γ. Let δ be the plane containing l and R. Then δ intersects β in a line n distinct from m, and δ intersects γ in a line r. Then l and n are distinct coplanar lines through P and parallel to r. Since this contradicts the Parallel Postulate, it follows that α intersects γ.

THEOREM 8.14 If a line intersects one of two distinct parallel planes in a single point, then it intersects the other plane in a single point.

Proof: Let α and β be two distinct parallel planes and let l be a line that intersects α in a single point; call it P. (See Figure 8-22.) Let γ be any plane that contains l. Then γ and α are distinct intersecting planes. Their intersection is a line; call it m. It follows from Theorem 8.13 and Theorem 8.12 that γ intersects β in a line; call it n. Then m and n are distinct parallel lines that are coplanar with l. Since l intersects m and is distinct from it, it follows from the Parallel Postulate that l intersects n and is distinct from it. Therefore the intersection of l and n is a single point; call it Q. Since Q is a point of n, it is also a point of β. Therefore Q is the unique point in which l and β intersect.

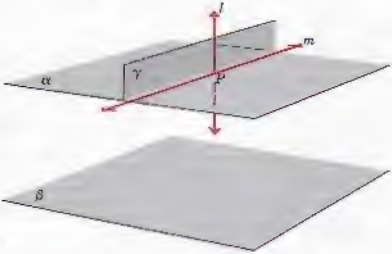


Figure 8-22

THEOREM 8.15 If a line is parallel to one of two distinct parallel planes, it is parallel to the other plane.

Proof: Assigned as an exercise.

THEOREM 8.16 There is a unique plane that contains a given point and is parallel to a given plane.

Proof: (We give only a plan for a proof.) Let a point P and a plane α be given. Divide the proof into four major parts.

1. If P is in α, explain why there is at least one plane containing P and parallel to α.
2. If P is in α, explain why there is only one plane containing P and parallel to α.
3. If P is not in α, let l be the unique line through P and perpendicular to α. Let β be the unique plane that is perpendicular to l at P. Prove that $\alpha \parallel \beta$.
4. If P is not in α, prove that there is at most one plane through P and parallel to α.

THEOREM 8.17 Given a point and a plane, then every line containing the given point and parallel to the given plane lies in the plane containing the given point and parallel to the given plane.

Proof: (Plan only.) Let a point P and a plane α be given. Suppose first that P lies in α. Then α itself is the unique plane containing P and parallel to α. Show that the assertion of the theorem is true in this case.

Suppose next that P is not in α. Let β be the unique plane containing P and parallel to α. Then $\alpha \neq \beta$. Let l be any line through P and parallel to α, as suggested in Figure 8-23. Use Theorem 8.14 to complete the proof.

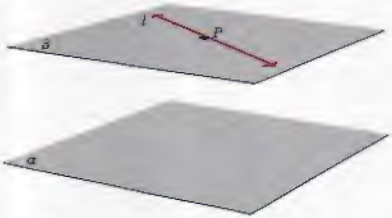


Figure 8-23

EXERCISES 8.5

1. If α and β are planes such that $\alpha \cap \beta = \varnothing$, is $\alpha \parallel \beta$? Why?
2. If l is a line and α is a plane and $l \cap \alpha = \varnothing$, is it possible that (a) l is parallel to α? (b) l is not parallel to α? Explain.
3. Let α, β, γ be three distinct planes with α parallel to β and α not parallel to γ. From which theorem of this section may we conclude that β is not parallel to γ?

4. Prove Theorem 8.11.
5. Given a plane α and two distinct parallel lines l and m such that l is not parallel to α. From which theorem of this section may we conclude that m is not parallel to α?
6. See the proof of Theorem 8.13. Draw a figure like Figure 8-21. Add some segments and a label to the figure to suggest the plane δ.
7. Prove Theorem 8.15.

8.6 PARALLELISM AND PERPENDICULARITY

In this section we define measure of a dihedral angle, right dihedral angle, and perpendicular planes. This section includes, as you might have guessed, several theorems on parallelism and perpendicularity for lines and planes. We begin by repeating a definition from Chapter 4.

> ***Definition 4.20*** If two noncoplanar halfplanes have the same edge, then the union of these halfplanes and the line which is their common edge is a **dihedral angle.** The union of this common edge and either one of these two halfplanes is a **face** of the dihedral angle. The common edge is the **edge** of the dihedral angle.

Figure 8-24 suggests two dihedral angles, one of them appearing to be larger than the other. What does "larger" as used here mean? In the case of (plane) angles we defined larger in terms of measure. Since we do not have a measure for a dihedral angle, we develop one.

Figure 8-24

> ***Definition 8.5*** The intersection of a dihedral angle and a plane perpendicular to its edge is a **plane angle of the dihedral angle.**

Figure 8-25 suggests a dihedral angle $A–BC–D$, a plane α such that $\overline{BC} \perp \alpha$, and $\angle PQR$, a plane angle of the dihedral angle $A–BC–D$.

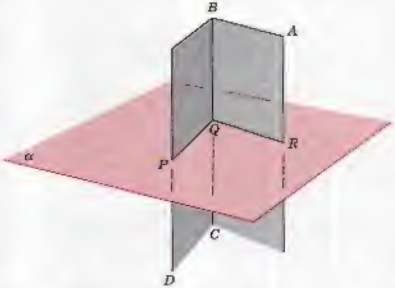


Figure 8-25

It seems plausible that all plane angles of a given dihedral angle should be congruent. This brings us to the next theorem.

THEOREM 8.18 Any two plane angles of a dihedral angle are congruent.

Proof: Suppose that $\angle ABC$ and $\angle DEF$ are two plane angles of dihedral angle $G–HI–J$, as in Figure 8-26. We may suppose that points have been picked in these plane angles and then labeled so that A and D lie in the plane GHI, C and F lie in plane HIJ, and

$$AB = BC = DE = EF.$$

It is easy to show that $BCFE$ and $BADE$ are parallelograms, that $ACFD$ is a parallelogram, that $\triangle ABC \cong \triangle DEF$, and hence that $\angle ABC \cong \angle DEF$.

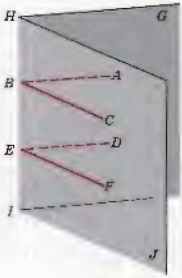


Figure 8-26

With Theorem 8.18 proved, it is now possible to define a measure for dihedral angles.

Definition 8.6 The measure of a dihedral angle is the measure of any one of its plane angles.

Definition 8.7 A right dihedral angle is a dihedral angle whose measure is 90.

Definition 8.8 Two planes are perpendicular if their union is the union of four right dihedral angles.

THEOREM 8.19 If a line is perpendicular to a plane, then any plane containing the given line is perpendicular to the given plane.

Proof: Assigned as an exercise.

THEOREM 8.20 If a line is perpendicular to one of two parallel planes, then it is perpendicular to the other plane also.

Proof: Assigned as an exercise.

THEOREM 8.21 If two planes are perpendicular, then any line in one of the planes and perpendicular to their line of intersection is perpendicular to the other plane.

Proof: Assigned as an exercise.

THEOREM 8.22 If two distinct intersecting planes are perpendicular to a third plane, then their line of intersection is perpendicular to the third plane.

Proof: Figure 8-27 suggests two distinct intersecting planes α and β perpendicular to plane γ. Let $l = \overleftrightarrow{AB}$ be the line of intersection of α and β; let $\overleftrightarrow{BC}$ and $\overleftrightarrow{BD}$ be lines in which α and β, respectively, intersect γ. Let l_1 be the line in α that is perpendicular to $\overleftrightarrow{BC}$ at B.

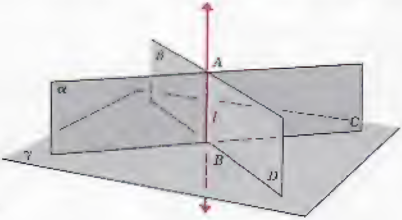


Figure 8-27

Let l_2 be the line in β that is perpendicular to $\overleftrightarrow{BD}$ at B. It follows from Theorem 8.21 that $l_1 \perp \gamma$ and $l_2 \perp \gamma$. The proof may be completed by showing that $l_1 = l_2 = l$, and hence that $l \perp \gamma$.

Definition 8.9 A segment, or ray, is **perpendicular** to a plane if the line which contains it is perpendicular to the plane. If a segment is perpendicular to a plane and one endpoint lies in the plane, then that segment is a **perpendicular** to the plane, and its endpoint in the plane is the **foot of the perpendicular.**

Definition 8.10 If α is a plane and S is a set of points, then the **projection** of S on α is the set of all points Q, each of which is the foot of the perpendicular from some point of S.

THEOREM 8.23 The projection of a line on a plane is either a line or a point.

Let l be a line and α a plane. There are three cases to consider.

1. l lies in α.
2. l is perpendicular to α.
3. l is not perpendicular to α and does not lie in α.

Proof of Case 1: If l lies in α, then every point of l is its own projection on α, and therefore l is its own projection on α.

Proof of Case 2: If l is perpendicular to α at the point Q, then the projection of every point of l on α is the point Q, and therefore the projection of l on α is the point Q.

Proof of Case 3: If l is not perpendicular to α and does not lie in α, let A, B, C be three distinct points of l that are not in α, and let A', B', C' be their respective projections on α. It follows from Definition 8.10 that $\overleftrightarrow{AA'} \perp \alpha$, $\overleftrightarrow{BB'} \perp \alpha$, $\overleftrightarrow{CC'} \perp \alpha$, and from Theorem 8.7 that $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, $\overleftrightarrow{BB'} \parallel \overleftrightarrow{CC'}$, $\overleftrightarrow{AA'} \parallel \overleftrightarrow{CC'}$. Since l is not perpendicular to α and since A, B, C are distinct points, it follows that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are distinct lines. Let β be the plane that contains $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. Then β contains the point C. Why? Since β is parallel to $\overleftrightarrow{BB'}$ (Why?) and $\overleftrightarrow{BB'}$ is parallel to $\overleftrightarrow{CC'}$, it follows from Theorem 8.11 that β is parallel to $\overleftrightarrow{CC'}$. Since β contains C and is parallel to $\overleftrightarrow{CC'}$, it follows that β contains $\overleftrightarrow{CC'}$. There-

fore C' lies in β. Let l' be the line in which β and α intersect. Since A', B', C' all lie in α and in β, it follows that they all lie on l'. If l intersects α, say in point D, then D is its own projection on α, and since D lies in both α and β, it follows that D lies on l'.

Note that l is determined by A and B, and that l' is determined by A' and B'. We have shown that if P is any point of l distinct from A and B (like C and D in the preceding paragraph), then its projection P' on α lies on l'.

Conversely, if Q' is any point of l', then Q' is the projection of some point Q on l. If l and α intersect in Q', then Q' is its own projection and $Q = Q'$. If Q' is a point of l' not on l, then there is a unique perpendicular to α at Q', and this perpendicular intersects l in the point Q which has the point Q' as its projection on α. It follows that every point of l has some point of l' as its projection, and that every point of l' is the projection of some point on l. Therefore the projection of l on α is the line l'.

COROLLARY 8.23.1 The projection of a segment on a plane is either a point or a segment.

Proof: Let s be a segment, α a plane, and s' the projection of s on α. If s is perpendicular to α, then s' is a point. If s lies in α, then $s' = s$ and hence s' is a segment. Let l be any line not in α and not perpendicular to α. Let A, B, C be three points of l such that A–B–C and let A', B', C' be their respective projections on α. Then it follows from Theorem 8.23 that A', B', C' are collinear and from Theorem 8.7 that A'–B'–C'. (If A'–B'–C' were not true, then we would have distinct parallel lines that intersect.) Since A–B–C on l implies A'–B'–C', we say that betweenness for points is preserved in the projection from l on α. If $s = \overline{AC}$ and $s' = \overline{A'C'}$, then B is between A and C if and only if B' is between A' and C'. Therefore s' is the projection of s.

THEOREM 8.24 Given a plane α and two distinct points A and B such that $\overleftrightarrow{AB}$ is parallel to α, if $\overline{A'B'}$ is the projection of $\overline{AB}$ on α, then $\overline{A'B'} \cong \overline{AB}$.

Proof: Assigned as an exercise.

THEOREM 8.25 Given parallel planes α and β and $\triangle ABC$ in α, if A', B', C' are the projections of A, B, C, respectively, on β,

$$\text{then } \triangle ABC \cong \triangle A'B'C'.$$

Proof: Assigned as an exercise.

THEOREM 8.26 The shortest segment joining a given point not in a given plane to a point in the given plane is the perpendicular that joins the given point to its projection in the given plane.

Proof: Assigned as an exercise.

THEOREM 8.27 All segments that are perpendicular to each of two distinct parallel planes and have their endpoints in these planes have the same length.

Proof: Assigned as an exercise.

Definition 8.11 The distance between a point and a plane not containing it is the length of the perpendicular segment joining the given point to the given plane.

Definition 8.12 The distance between two distinct parallel planes is the length of a segment that joins a point of one of the planes to a point of the other plane and is perpendicular to both of them.

EXERCISES 8.6

1. There is no definition of betweenness for halfplanes in this book. Write your own definition for it. Using your definition, prove that if $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are halfplanes with $\mathcal{H}_2$ between $\mathcal{H}_1$ and $\mathcal{H}_3$, then the measure of the dihedral angle formed by $\mathcal{H}_1$ and $\mathcal{H}_3$ and their common edge is the sum of the measures of the dihedral angles formed by $\mathcal{H}_1$ and $\mathcal{H}_2$ and their edge and by $\mathcal{H}_2$ and $\mathcal{H}_3$ and their edge.
2. There are no definitions in this book for the interior and the exterior of a dihedral angle. Write your own definitions for these terms.
3. Draw a picture of a cube and label its vertices. How many right dihedral angles are there each containing two faces of the cube?
4. Given the labeled cube of Exercise 3, select one of the right dihedral angles and, using the vertices of the cube, identify two of its plane angles.
5. See the proof of Theorem 8.18. Draw a figure like Figure 8-26 and modify it to show the quadrilateral *ACFD*. Prove that *ACFD* is a rectangle.

Exercises 6–16 are concerned with projections on a plane. In Exercises 6–14, sketch one or more figures and be prepared to defend your answer.

6. Is the projection of every triangle a triangle?
7. Is the projection of every ray a ray?
8. Is the projection of every point a point?
9. Is the projection of every segment a segment?
10. Is the projection of every line a line?
11. Is there an angle whose projection is a line? A ray? A segment? A point? An angle?
12. Is there an acute angle whose projection is an obtuse angle?
13. Is there a right angle whose projection is an acute angle? An obtuse angle? A right angle?
14. Is there a segment that is shorter than its projection? Longer?
15. If an edge of a cube is perpendicular to a plane, describe the projection of the cube on the plane. Draw a sketch.
16. CHALLENGE PROBLEM. If a diagonal of a cube is perpendicular to a plane, draw a sketch of the projection of the cube on the plane.

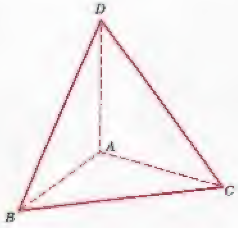
In Exercises 17–23, draw an appropriate figure and prove the theorem.

17. Theorem 8.19
18. Theorem 8.20
19. Theorem 8.21
20. Theorem 8.24
21. Theorem 8.25
22. Theorem 8.26
23. Theorem 8.27

24. In the proof of Theorem 8.22, show that $l_1 = l_2 = l$.
25. Let A–BC–D be a right dihedral angle with $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{CD}$, $AB = 3\sqrt{3}$, $BC = 5$, $CD = 12$. Prove that $AD = 14$.
26. The figure shows four noncoplanar points A, B, C, D such that

$$\overline{AB} \perp \overline{AC},\ \overline{AB} \perp \overline{AD},\ \overline{AC} \perp \overline{AD},\ AB = AC = AD.$$

Find the sum of the measures of $\angle BDA$, $\angle BDC$, $\angle CDA$.



In Exercises 27–34, determine if the given statement is true or false. (An if-then statement is false if there are one or more instances in which the if-part is true and the then-part false.)

27. If a line is perpendicular to two distinct intersecting lines at their point of intersection, then it is perpendicular to the plane that contains the intersecting lines.
28. If the intersection of a plane and a dihedral angle is an angle, then that angle is a plane angle of the dihedral angle.
29. If l and n are distinct intersecting lines and if l is parallel to plane α, then n is not parallel to plane α.
30. If l and n are distinct parallel lines intersecting (but not contained in) two distinct parallel planes, then the planes cut off segments of equal length on the two lines.
31. If two planes are perpendicular to the same line, they are parallel.
32. If two lines are parallel to the same plane, they are parallel.
33. Given a plane α, the set of all points each of which is at a distance of 5 from α is the union of two planes each parallel to α.
34. If α is a plane and P is a point, there is one and only one plane containing P and parallel to α.

CHAPTER SUMMARY

We began the chapter with some suggestions for drawing figures to represent geometrical figures that do not lie in a single plane. We used figures throughout to help you visualize the relationships of lines and planes in space.

Definitions of the following expressions were included.

A LINE AND A PLANE ARE PERPENDICULAR
THE PERPENDICULAR BISECTING PLANE OF A SEGMENT
PARALLEL PLANES
A LINE AND A PLANE ARE PARALLEL
A PLANE ANGLE OF A DIHEDRAL ANGLE
THE MEASURE OF A DIHEDRAL ANGLE
A RIGHT DIHEDRAL ANGLE
PERPENDICULAR PLANES
A SEGMENT (RAY) PERPENDICULAR TO A PLANE
THE PROJECTION OF A SET ON A PLANE
THE DISTANCE BETWEEN A POINT AND A PLANE
THE DISTANCE BETWEEN TWO PARALLEL PLANES

There are 27 theorems in this chapter. It is suggested that you write out these theorems and draw appropriate figures for them.

REVIEW EXERCISES

In Exercises 1–15, copy and complete the given statement to obtain a theorem of this chapter.

1. If a line is perpendicular to each of two distinct intersecting lines at their point of intersection, then [?].
2. Given a line and a point, there is a unique plane perpendicular [?].
3. The perpendicular bisecting plane of a segment is the set of all points each of which is [?].
4. If two lines are perpendicular to the same plane, they are [?].
5. If one of two distinct parallel lines is perpendicular to a plane, then [?].
6. Given a plane and a point, there is a unique line containing the given point and [?].
7. If a plane intersects one of two distinct parallel lines but does not contain it, then it [?] and does not contain it.
8. If a plane intersects each of two distinct parallel planes, the intersections are [?] lines.
9. If α, β, γ are three distinct planes such that α is parallel to β and such that α intersects γ, then [?].
10. If a line intersects one of two distinct parallel planes in a single point, then [?].
11. If a line is parallel to one of two distinct parallel planes, then [?].
12. Given a plane and a point, there is a unique plane that contains the given point and [?].
13. Any two plane angles of a dihedral angle are [?].
14. If a line is perpendicular to a plane, then every plane containing that line is [?].
15. If two distinct intersecting planes are perpendicular to a third plane, then their line of intersection is [?].

In Exercises 16–23, some lines or planes are described. In each exercise, state whether or not they must be parallel to each other.

16. Lines through a given point parallel to a given line.
17. Lines perpendicular to a given plane.
18. Lines perpendicular to a given line.
19. Lines parallel to a given plane.
20. Planes parallel to a given plane.
21. Planes perpendicular to a given plane.
22. Planes perpendicular to a given line.
23. Planes parallel to a given line.

In Exercises 24–30, write a plan for a proof of the given statement.

24. If $\overline{A'B'}$ is the projection of $\overline{AB}$ on a plane, then $A'B' \leq AB$.
25. If $\overline{A'B'}$ is the projection of $\overline{AB}$ on a plane α and if $A'B' = AB$, then $\overleftrightarrow{AB} \parallel \alpha$.
26. If α and β are distinct parallel planes, if parallel lines m and n are in β, and if lines m' and n' are the projections of m and n, respectively, on α, then m' is parallel to n'.
27. If α and β are distinct parallel planes, if s and t are parallel and congruent segments in β, and if s' and t' are the projections of s and t, respectively, on α, then s' and t' are congruent.
28. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$, and if A, B, C are noncollinear points, then plane ABC and plane $A'B'C'$ are parallel.
29. If α and β are distinct parallel planes and if l and m are distinct parallel lines that intersect α in points L_1 and M_1, respectively, and β in points L_2 and M_2, respectively, then $L_1M_1M_2L_2$ is a parallelogram.
30. If every plane containing a given line is perpendicular to a given plane, then the given line is perpendicular to the given plane.

Chapter 9

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Area and the Pythagorean Theorem

9.1 INTRODUCTION

You are familiar with the idea of area and have computed the areas enclosed by figures such as triangles, squares, and circles. As you know, areas are usually computed using numbers which are lengths or distances. You also know that a distance function is based on some segment as the unit of distance. In informal geometry, we usually combine a number and a word in expressing a distance, for example, 4.5 ft. In formal geometry, we frequently use the number by itself, omitting the unit when the distance function is understood. In our formal development of the area concept we shall suppose that a distance function is given. Then there will be just one area function.

In this chapter we adopt some postulates for area based on our experiences in computing areas and on our ideas about areas in informal geometry. These postulates include the formula for computing the area enclosed by a rectangle as well as statements of general properties that are useful in proving theorems about area. We also develop formulas for computing the areas enclosed by figures such as triangles and trapezoids, and we use area as a tool in proving the Pythagorean Theorem.

9.2 AREA IDEAS

Let us suppose that a unit segment for distance measure is given and that a square of side length 1 is given. We call this square the unit square. (See Figure 9-1.) Just as distances and lengths are numbers associated with sets of points (for example, the length AB is associated with $\overline{AB}$), areas are numbers associated with sets of points. The distance function matches a positive number with each segment. Similarly, the area function matches a positive number with each rectangle and also with many other simple figures.

Figure 9-1

We shall agree that the area of the unit square is 1. In practical applications there is a system of areas for each system of distances. If the side length of the unit square is 1 ft., then the area of the unit square is 1 sq. ft. If the side length of the unit square is 1 cm., then the area of the unit square is 1 sq. cm. (See Figure 9-2.)

Figure 9-2

When we speak of the area of the unit square as 1, we are thinking of 1 as the measure of the set that includes all of the points in the plane of the square that are inside of or on the square. (See Figure 9-3.) Strictly speaking, it would be better to speak of the area of the unit square *region*. We shall follow custom, however, and call it the area of the unit square. Similarly, we talk about areas of rectangles, triangles, and circles when we really mean the areas of the regions which are made up of these figures including their interiors. (See Figure 9-4.)

Figure 9-3

When we say the "area of a triangle," for example, we mean the "area of the triangular region."

Definition 9.1 A polygonal region is a triangular region, or it is the union of a finite number (two or more) of triangular regions such that the intersection of every two of them is the null set, or a vertex of each of them, or a side of each of them.

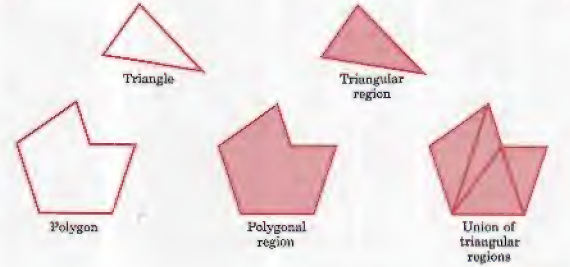


Figure 9-4

You may already know how to compute the area of a rectangle. If its sides are of length a and b, its area is ab. (See Figure 9-5.) In Section 9.3 we adopt this formula for the area of a rectangle as a postulate. Let us see if it is really a reasonable and basic assumption to make.

Figure 9-5

Figure 9-6 suggests our unit square and a rectangle with sides of lengths 3 and 4. The rectangular region is made up of $3 \cdot 4 = 12$ square regions each of the same size as the unit square region. So the area of the rectangle should be $3 \cdot 4$, or 12. Similarly if a and b are any two natural numbers (that is, positive whole numbers), then the area of a rectangle which is a by b should be ab.

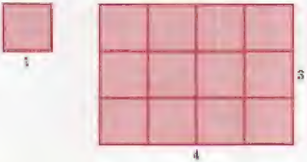


Figure 9-6

Figure 9-7 suggests our unit square and another square with sides of length $\frac{1}{3}$. Let x denote the area of the square with sides of length $\frac{1}{3}$. Since it takes 9 squares of this size (3 rows with 3 squares in each row as suggested in the figure) to fill up the unit square, you can see that the area of the unit square should be 9 times the area x of the $\frac{1}{3}$ by $\frac{1}{3}$ square. Therefore

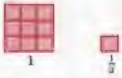


Figure 9-7

$$9x = 1 \qquad \text{and} \qquad x = \tfrac{1}{9}.$$

It appears, then, that the area formula,

$$S = ab,$$

is a reasonable formula for our $\frac{1}{3}$ by $\frac{1}{3}$ square. Taking $a = \frac{1}{3}$ and $b = \frac{1}{3}$, we have

$$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9},$$

the area of a $\frac{1}{3}$ by $\frac{1}{3}$ square. Similarly, if n is any natural number, then the area of a $\dfrac{1}{n}$ by $\dfrac{1}{n}$ square is $\dfrac{1}{n} \cdot \dfrac{1}{n}$, or $\dfrac{1}{n^2}$.

Figure 9-8 shows a rectangle which is $\frac{2}{3}$ by $\frac{5}{3}$. It is easy to see from the figure that the area of the rectangle should be $2 \cdot 5$, or 10, times the area of a $\frac{1}{3}$ by $\frac{1}{3}$ square; hence

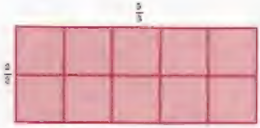


Figure 9-8

$$(2 \cdot 5)(\tfrac{1}{3} \cdot \tfrac{1}{3}) = \tfrac{2}{3} \cdot \tfrac{5}{3} = \tfrac{10}{9}.$$

Therefore the formula $S = ab$ is appropriate for a rectangle which is $\frac{2}{3}$ by $\frac{5}{3}$. Similarly, if a and b are any two positive rational numbers, then the area S of a rectangle which is a by b should be given by the formula $S = ab$. For example, if

$$a = 1\tfrac{7}{8} \qquad \text{and} \qquad b = \tfrac{3}{5},$$

then

$$a = \tfrac{15}{8} = \tfrac{75}{40}, \qquad b = \tfrac{3}{5} = \tfrac{24}{40},$$

and if R is a rectangle which is a by b, its area S is the sum of the areas of

$$75 \cdot 24 = 1800 \text{ little squares,}$$

each of which is $\frac{1}{40}$ by $\frac{1}{40}$. Hence

$$S = 1800\,(\tfrac{1}{40} \cdot \tfrac{1}{40}) = \tfrac{1800}{1600}.$$

Since

$$\tfrac{1800}{1600} = \tfrac{9}{8} = \tfrac{15}{8} \cdot \tfrac{3}{5} = (1\tfrac{7}{8})(\tfrac{3}{5}),$$

we see that the formula $S = ab$ holds, in this case.

Let R be a rectangle which is a by b and suppose that either a or b, perhaps both, is an irrational number. Then, using the idea that an irrational number can be squeezed between two rational numbers that are as close together as you desire, we can make it seem reasonable that the area S of R is again given by the formula $S = ab$. Figure 9-9 shows a rectangle R which is 1 by $\sqrt{2}$. (Recall that $\sqrt{2}$ is not rational and that $\sqrt{2}$ can be represented by the nonrepeating decimal 1.41421 . . . , where the three dots indicate an infinite sequence of digits.) Thus

$$1.4 < \sqrt{2} < 1.5,$$

where 1.4 and 1.5 are rational numbers. Now a rectangle 1 by 1.4 has area 1.4, a rectangle 1 by 1.5 has area 1.5, and therefore the area S of the 1 by $\sqrt{2}$ rectangle R is somewhere between 1.4 and 1.5, that is,

$$1.4 < S < 1.5.$$

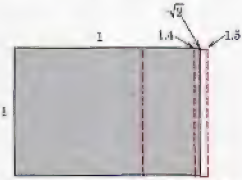


Figure 9-9

Continuing in this way, we can show that S is between 1.41 and 1.42, between 1.414 and 1.415, and so on. If you consider the sequence of areas of rectangles 1.4, 1.41, 1.414, . . . , you see that the sequence is increasing and the rational numbers in the sequence are getting closer and closer to $\sqrt{2}$ through numbers that are less than $\sqrt{2}$. Similarly, if you consider the sequence of areas of rectangles 1.5, 1.42, 1.415, . . . , you see that the sequence is decreasing and the rational numbers in the sequence are getting closer and closer to $\sqrt{2}$ through numbers that are greater than $\sqrt{2}$. It seems reasonable, then, to conclude that the area S of the 1 by $\sqrt{2}$ rectangle R is given by

$$S = 1 \cdot \sqrt{2} = \sqrt{2}.$$

The foregoing discussion may be summarized as follows. If the sides of a rectangle are rational numbers, the rectangular region can be either subdivided into a finite number of unit square regions and the area obtained by counting, or subdivided into a finite number of square regions each $\frac{1}{n}$ by $\frac{1}{n}$ for some integer $n \geq 2$ and then counting and

multiplying by $\frac{1}{n^2}$. In both cases the result S is the same as that obtained by using the formula $S = ab$, where a and b are the lengths of two adjacent sides of the rectangle. If the side lengths are not rational, then we cannot subdivide the rectangular region into square regions with a rational side length and find the area by counting, or by counting and multiplying by $\frac{1}{n^2}$ for some natural number n. We can, however, approximate the side lengths using rational numbers and consider the *limit* of the sequence of areas of the rectangles whose sides have lengths that are rational numbers. It can be proved that this limit exists and is equal to the product ab. In this book we adopt the formula $S = ab$ for the area of a rectangle as a postulate.

The ideas used to develop the formula $S = ab$ for the area of a rectangle are, of course, the same ideas with which you are familiar from your past experiences in informal geometry. They include the following: (1) every rectangle has an area, (2) congruent rectangles have the same area, and (3) area is additive in the sense that the area of a figure is the sum of the areas of the parts of the figure. Similar ideas may be used in developing area formulas for triangles, parallelograms, trapezoids, and so on. In Section 9.3 we state carefully our basic ideas about area. We call these statements the Area Postulates and use them to develop several area formulas.

9.3 AREA POSTULATES

Our formal development of area in this chapter is limited to areas of polygons and is based on the following postulates.

POSTULATE 27 (*Area Existence Postulate*) Every polygon has an area and that area is a (unique) positive number.

POSTULATE 28 (*Rectangle Area Postulate*) If two adjacent sides of a rectangle are of lengths a and b, then the area S of the rectangle is given by the formula $S = ab$.

POSTULATE 29 (*Area Congruence Postulate*) Congruent polygons have equal areas.

POSTULATE 30 (*Area Addition Postulate*) If a polygonal region is partitioned into a finite number of polygonal subregions by a finite number of segments (called boundary segments) such that no two subregions have points in common except for points on the boundary segments, then the area of the region is the sum of the areas of the subregions.

Notation. If $\mathcal{F}$ is a figure, we frequently use $|\mathcal{F}|$ to denote the area of $\mathcal{F}$. For example, $|\triangle ABC|$ denotes the area of $\triangle ABC$. The area of quadrilateral $ABCD$ may be denoted by $|\text{quadrilateral } ABCD|$ or by $|ABCD|$ if it is clear that $ABCD$ is a quadrilateral.

Example 1 Figure 9-10 suggests a polygonal region $ABCDEFG$. Let us call this region $\mathcal{P}$. Segments $\overline{CG}$ and $\overline{CF}$ partition, or separate, $\mathcal{P}$ into three subregions $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, where $\mathcal{P}_1$, is the union of quadrilateral $ABCG$ and its interior, $\mathcal{P}_2$ is the union of $\triangle CFG$ and its interior, and $\mathcal{P}_3$ is the union of quadrilateral $CDEF$ and its interior. According to Postulate 30, the Area Addition Postulate, the area of $\mathcal{P}$ is the sum of the areas of $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, that is,

$$|\mathcal{P}| = |\mathcal{P}_1| + |\mathcal{P}_2| + |\mathcal{P}_3|.$$

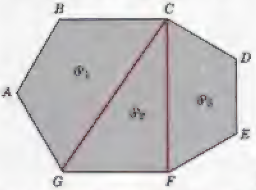


Figure 9-10

Example 2 Figure 9-11 suggests a figure consisting of a square $ABCD$ with its diagonals, $\overline{AC}$ and $\overline{DB}$, and an adjoining isosceles right triangle, $\triangle BFC$, with legs of length 1. Let S denote the area of $\triangle BFC$, S' the area of square $ABCD$, and S'' the area of pentagon $ABFCD$. Find S, S', S'' using the Area Postulates.

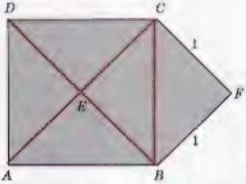


Figure 9-11

Solution: It is easy to show that $\triangle BFC \cong \triangle BEC$ so that the triangles have equal areas and that $BFCE$ is a rectangle, actually a unit square. Then

$$\begin{aligned}
1 &= |BFCE| && \text{by the Rectangle Area Postulate}\\
&= |\triangle BFC| + |\triangle BEC| && \text{by the Area Addition Postulate}\\
&= |\triangle BFC| + |\triangle BFC| && \text{by the Area Congruence Postulate}\\
&= S + S.
\end{aligned}$$

Therefore $2S = 1$ and $S = \frac{1}{2}$. Since square $ABCD$ is partitioned by its diagonals into four triangles, each congruent to $\triangle BFC$, it follows from the Area Addition and Congruence Postulates that $S' = 4S$ and hence that

$$S' = 4 \cdot \tfrac{1}{2} = 2.$$

From the Area Addition Postulate it follows that

$$S'' = S' + S = 2\tfrac{1}{2}.$$

Another way to find S' is as follows. Since $\triangle BFC \cong \triangle BEC$, we have

$$BE = EC = 1 \qquad \text{and} \qquad m\angle BEC = 90.$$

Then it follows from the Pythagorean Theorem, which is proved in Section 9.5, that

$$(BC)^2 = 1^2 + 1^2 = 2 \qquad \text{and} \qquad BC = \sqrt{2}.$$

Finally, it follows from the rectangle area formula that

$$S' = \sqrt{2} \cdot \sqrt{2} = 2.$$

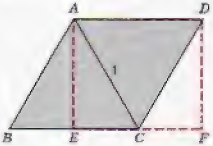
EXERCISES 9.3

In Exercises 1–8, given the lengths a and b of two adjacent sides of a rectangle, find the area of the rectangle.

1. $a = 2.5$, $b = 3.3$
2. $a = \frac{5}{3}$, $b = \frac{7}{3}$
3. $a = \frac{5}{3}$, $b = \frac{3}{4}$
4. $a = 2.32$, $b = 1.77$
5. $a = \frac{7}{3}$, $b = \frac{1}{2}$
6. $a = 7$, $b = 3\sqrt{2}$
7. $a = 4\sqrt{2}$, $b = 3\sqrt{2}$
8. $a = \sqrt{5}$, $b = 5\sqrt{5}$

9. (*An informal geometry exercise.*) The length of a side of a square is 10 yd. What is the area of the square in square yards? In square feet?
10. (*An informal geometry exercise.*) If the dimensions of a rectangular field are 40 rods by 80 rods, how many acres of land does it contain? (An acre contains 43,560 sq. ft. and 1 rod is $16\frac{1}{2}$ ft. long.)
11. (*An informal geometry exercise.*) How many feet of fence would it take to enclose the field in Exercise 10?
12. (*An informal geometry exercise.*) The dimensions of a football field (including the end zones) are $53\frac{1}{3}$ yd. wide by 120 yd. long. Would an acre of land be sufficient on which to lay out a football field? (See Exercise 10.)
13. If the perimeter of a square is 48, what is its area?
14. If the perimeter of a square is $64\sqrt{2}$, what is its area?

15. The length of one side of a rectangle is 3 times the length of an adjacent side. If the perimeter of the rectangle is 56, what is its area?
16. The length of one side of a rectangle is 5 more than twice the length of an adjacent side. If the perimeter of the rectangle is 70, what is its area?
17. CHALLENGE PROBLEM. The length of a rectangle is 4 more than twice its width. If the area of the rectangle is 48, what is its perimeter?
18. The width of a rectangle is $\frac{5}{8}$ of its length. If the area of the rectangle is 360, find its length and width.
19. *(An informal geometry exercise.)* How many squares 1 in. on a side are contained in a square 1 yd. on a side?
20. *(An informal geometry exercise.)* How many 9 in. $\times$ 9 in. tiles would it take to cover a $12\frac{1}{2}$ ft. $\times$ 9 ft. rectangular floor if we assume that there is no waste in cutting the tile?
21. If S is the area of an x by x square and S' is the area of a $2x$ by $2x$ square, prove that $S' = 4S$.
22. If S is the area of a right triangle with legs of lengths a and b, and S' is the area of an a by b rectangle, prove that $S' = 2S$.
23. If S is the area of a right triangle with legs of lengths a and b, and S' is the area of a right triangle with legs of lengths $2a$ and $2b$, prove $S' = 4S$.
24. The figure shows two coplanar equilateral triangles, $\triangle ABC$ and $\triangle ACD$, with $AC = 1$. The feet of the perpendiculars from A and D to $\overleftrightarrow{BC}$ are E and F, respectively. Prove that $ABCD$ is a parallelogram and that $AEFD$ is a rectangle.



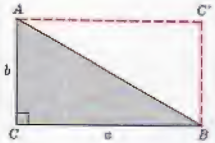
25. Complete the proof of the following theorem.

THEOREM The area of a right triangle is one-half the product of the lengths of its legs.

RESTATEMENT:

Given: A right triangle $\triangle ABC$ with the right angle at C, $AC=b$, and $BC = a$.

Prove: $|\triangle ABC| = \frac{1}{2}ab$



Proof: Let C' be the point of intersection of the line through A and parallel to $\overleftrightarrow{CB}$ and the line through B and parallel to $\overleftrightarrow{AC}$. (See the figure.) Why must these two lines intersect? Why is quadrilateral $ACBC'$ a rectangle?

Complete the proof by showing that $\triangle ABC \cong \triangle BAC'$ and that $|\triangle ABC| = \frac{1}{2}ab$.

9.4 AREA FORMULAS

Figure 9-12 shows a parallelogram $ABCD$ with points E and F the feet of the perpendiculars from D and C, respectively, to line $\overleftrightarrow{AB}$.

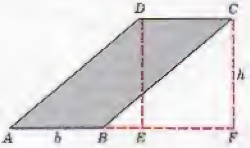


Figure 9-12

The following definition has two parts. In (1) we define base and altitude, thought of as segments. In (2) we define base and altitude, thought of as numbers (lengths of segments, or distances). (Compare the corresponding definitions for triangles on page 265.)

Definition 9.2
1. Any side of a parallelogram is a base of that parallelogram. Given a base of a parallelogram, a segment perpendicular to that base, with one endpoint on the line containing the base and the other endpoint on the line containing the opposite side, is an altitude of the parallelogram corresponding to that base.
2. The length of any side of a parallelogram is a base of that parallelogram. The distance between the parallel lines containing that side and the side that is opposite to it is the corresponding altitude, or height.

In Figure 9-12 we call h the altitude, or height, corresponding to the base AB. The number h is the distance between the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$. Similarly, the distance between the parallel lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ is the altitude which corresponds to the base AD.

Let S denote the area of parallelogram $ABCD$ in Figure 9-12, let $AB = b$, and let $CF = h$. We proceed to prove that the area S of the parallelogram is bh.

THEOREM 9.1 If b is a base of a parallelogram and if h is the corresponding height, then the area S of the parallelogram is given by the formula $S = bh$.

Proof:

Statement	Reason
1. $\angle DAE \cong \angle CBF$	1. Why?
2. $\angle AED \cong \angle BFC$	2. Why?
3. $\overline{DE} \cong \overline{CF}$	3. Why?
4. $\triangle AED \cong \triangle BFC$	4. S.A.A. Theorem
5. $\lvert\triangle AED\rvert = \lvert\triangle BFC\rvert$	5. Why?
6. $\lvert AFCD\rvert = S + \lvert\triangle BFC\rvert$	6. Why?
7. $\lvert AFCD\rvert = S + \lvert\triangle AED\rvert$	7. Why?
8. $S = \lvert AFCD\rvert - \lvert\triangle AED\rvert$	8. Addition Property of Equality
9. $\lvert EFCD\rvert + \lvert\triangle AED\rvert = \lvert AFCD\rvert$	9. Why?
10. $DC = b$	10. Why?
11. $\lvert EFCD\rvert = bh$	11. Why?
12. $bh + \lvert\triangle AED\rvert = \lvert AFCD\rvert$	12. Steps 9, 11; substitution
13. $bh = \lvert AFCD\rvert - \lvert\triangle AED\rvert$	13. Why?
14. $S = bh$	14. Steps 8, 13; substitution

In the remainder of this section we prove two more theorems regarding area formulas.

THEOREM 9.2 If b is a base of a triangle and if h is the corresponding altitude, then the area S of the triangle is given by the formula $S = \frac{1}{2}bh$.

Proof: (See Figure 9-13.) Let $\triangle ABC$ be given. Let $\overrightarrow{BD}$ be the ray with endpoint B and parallel to $\overrightarrow{AC}$. Let $\overrightarrow{CE}$ be the ray with endpoint C and parallel to $\overrightarrow{AB}$. Let F be the point of intersection of $\overrightarrow{BD}$ and $\overrightarrow{CE}$. Then $ABFC$ is a parallelogram and $\triangle ABC \cong \triangle FCB$.

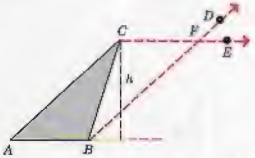


Figure 9-13

Now $b = AB$ and h is the distance from C to $\overleftrightarrow{AB}$, and so h is also the distance between $\overleftrightarrow{CF}$ and $\overleftrightarrow{AB}$. It follows from Theorem 9.1 that $|ABFC| = bh$. But

$$|\triangle ABC| = |\triangle FCB| \qquad \text{(Why?)}$$

and

$$|\triangle ABC| + |\triangle FCB| = |ABFC| \qquad \text{(Why?).}$$

Therefore

$$S + S = bh, \qquad 2S = bh, \qquad \text{and} \qquad S = \tfrac{1}{2}bh.$$

THEOREM 9.3 If b_1 and b_2 are the lengths of the parallel sides of a trapezoid and if h is the distance between the lines that contain these parallel sides, then the area S of the trapezoid is given by the formula

$$S = \tfrac{1}{2}(b_1 + b_2)h.$$

Proof: Let trapezoid $ABCD$, with parallel sides $\overline{AB}$ and $\overline{CD}$, be given as in Figure 9-14.

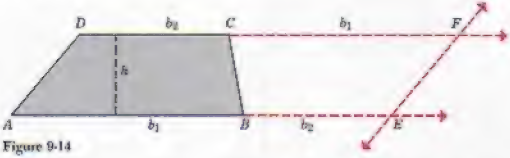


Figure 9-14

The lengths of these sides, or bases, are b_1 and b_2, and the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is h. The distance h is called the height. Let E be the point on *opp* $\overrightarrow{BA}$ such that $BE = b_2$. Let F be the point on *opp* $\overrightarrow{CD}$ such that $CF = b_1$. Then $AEFD$ is a parallelogram (Why?) and its area is $(b_1 + b_2)h$. We shall show that

$$ABCD \longleftrightarrow FCBE$$

is a congruence. Then it will follow that

$$\begin{aligned} |ABCD| + |FCBE| &= |AEFD| \\ S + S &= (b_1 + b_2)h \\ S &= \tfrac{1}{2}(b_1 + b_2)h. \end{aligned}$$

To see that $ABCD \longleftrightarrow FCBE$ is a congruence, note that

(1) $\angle DAB \cong \angle EFC$	(5) $\overline{AB} \cong \overline{FC}$
(2) $\angle ABC \cong \angle FCB$	(6) $\overline{BC} \cong \overline{CB}$
(3) $\angle BCD \cong \angle CBE$	(7) $\overline{CD} \cong \overline{BE}$
(4) $\angle CDA \cong \angle BEF$	(8) $\overline{DA} \cong \overline{EF}$

This completes the proof.

There are area formulas for other special polygons, but we shall not develop them here. In Section 9.5 we use area formulas as tools in proving the Pythagorean Theorem.

EXERCISES 9.4

1. If the area of a triangle is 40 and a base is 8, find the corresponding altitude.
2. If an altitude of a triangle is 12 and the area of the triangle is 62, find the base corresponding to the given altitude.
3. The lengths of the legs of a right triangle are 17 and 9. Find the area of the triangle.
4. Find the area of an isosceles right triangle if the length of one of its legs is 15.
5. Find the area of a rhombus if its height is $7\frac{1}{2}$ and the length of one of its sides is $15\frac{1}{3}$.

Exercises 6-8 are informal geometry exercises.

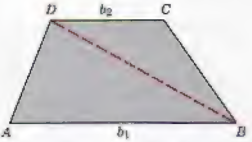
6. In the figure below, lines m and n are parallel and the distance between them is 5 ft. P and Q are points on m, and R and S are points on n such that $PQ = RS = 7$ ft. If the distance from P to R is 10 miles, find $|PQRS|$.

7. In the situation of Exercise 6, find $|PQRS|$ if $PR = 1$ mile.
8. In the situation of Exercise 6, find $|PQRS|$ if $PR = 10$ ft.

9. If the area of a triangle is S and its altitude is h, express the base b in terms of S and h.
10. The area of a parallelogram is 50.4. Find its base if the altitude is 4.2.

11. Find the area of a trapezoid if its height is 7 and the lengths of the parallel bases are 9 and 15.
12. Use the formula for the area of a trapezoid and express h in terms of S, b_1, and b_2.
13. If the area of a trapezoid is 128 and the lengths of the parallel bases are 7 and 9, find its height.
14. The area of a trapezoid is 147 and the length of one of its parallel bases is 18. If the height is 7, find the length of the other base.
15. $ABCD$ is a parallelogram with $AB = 25$. If $|ABCD| = 375$ and P is a point such that C-P-D, find $|\triangle APB|$.
16. (*An informal geometry exercise.*) A plot of land in the shape of a trapezoid has bases of lengths 121 yd. and 242 yd. If the distance between the bases is 300 yd., how many acres of land are contained in the plot? (One acre contains 43,560 sq. ft.)
17. Two triangles have the same base and their areas differ by 42. If the altitude of the larger triangle is 6 more than the altitude of the smaller one, find the length of the common base.
18. In the figure, $ABCD$ is a trapezoid with parallel bases $\overline{AB}$ and $\overline{CD}$. If $AB = b_1$, $CD = b_2$, and the altitude is h, draw $\overline{BD}$ and use the Area Postulates and the area formula for a triangle to prove that

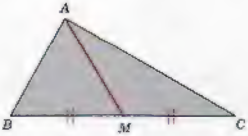
$$|ABCD| = \tfrac{1}{2}h(b_1 + b_2).$$



19. In the figure, $\overline{AM}$ is a median of $\triangle ABC$. Prove that

$$|\triangle ABM| = |\triangle ACM|.$$

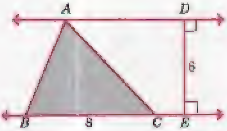
Is $\triangle ABM \cong \triangle ACM$ in every case? In any case?



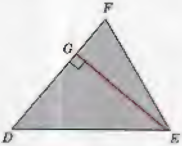
20. The area of a square is equal to the area of a parallelogram. If the base of the parallelogram is 32 and its height is 18, find the perimeter of the square.

In each of Exercises 21–29, find, on the basis of the given data, the area or areas indicated.

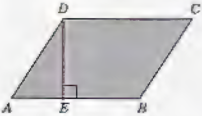
21. $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, $BC = 8$, $DE = 6$. $|\triangle ABC| = \boxed{?}$.



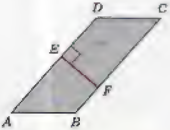
22. $D\text{-}G\text{-}F$, $DF = 6$, $DE = 6.5$, $GE = 5$. $|\triangle DEF| = \boxed{?}$.



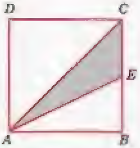
23. $AB = CD = 8$, $BC = DA = 6$, $DE = 5$. $|\triangle ABCD| = \boxed{?}$.



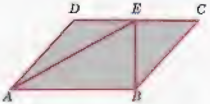
24. $ABCD$ is a parallelogram, $\overline{EF} \perp \overline{DA}$, $EF = 3$, $BC = 8$, $CD = 4$. $|\triangle ABCD| = \boxed{?}$.



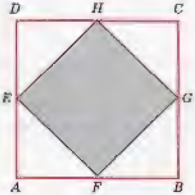
25. $ABCD$ is a square, E is the midpoint of $\overline{BC}$. The perimeter of $ABCD$ is 48. $|\triangle ACE| = \boxed{?}$.



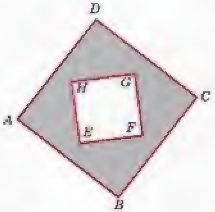
26. $ABCD$ is a parallelogram, E is the midpoint of $\overline{CD}$, and
$|ABCD| = 33$.
$|\triangle AED| = \boxed{?}$
$|\triangle ABE| = \boxed{?}$
$|\triangle BCE| = \boxed{?}$



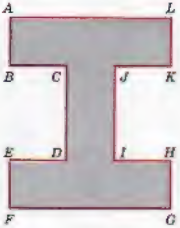
27. $ABCD$ is a square; E, F, G, H are midpoints of the sides; $|ABCD| = 196$. $|EFGH| = \boxed{?}$.



28. $EFGH$ is a square of side length 6; $ABCD$ is a square of side length 12; E, F, G, H are interior points of $ABCD$. The area of the shaded region is $\boxed{?}$.



29. B and E are collinear with A and F, K and H are collinear with L and G, C and J are collinear with B and K, D and I are collinear with E and H, $AFGL$ is a rectangle with area 46, $BCDE$ is a rectangle with area 8, $BCDE$ and $JKHI$ are congruent rectangles. $|ABCDEFGHIJKL| = \boxed{?}$.



30. Using the Area Postulates and the theorem of Exercise 25 in Exercises 9.3, derive the triangle area formula for any triangle.
31. (See Figure 9-12.) Let $b_1 = AD$ and let h_1 be the distance between $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. Is $b_1h_1 = bh$? Justify your answer.
32. Would any of the statements in the proof of Theorem 9.1 be different if point E were between A and B? Draw a figure for this case.
33. See the proof of Theorem 9.2. Prove that $\overleftrightarrow{CE}$ and $\overleftrightarrow{BD}$ are not parallel lines.
34. In the proof of Theorem 9.2, prove that $\triangle ABC \cong \triangle FCB$.
35. Recall that a parallelogram is a special trapezoid. Show that the formula for the area of a trapezoid simplifies to the formula for the area of a parallelogram if the trapezoid is a parallelogram.
36. A triangle might be thought of as a quadrilateral in which one of the bases has shrunk to a point. What number does $\frac{1}{2}(b_1 + b_2)h$ approach if b_1 and h are fixed and b_2 gets closer and closer to 0?
37. CHALLENGE PROBLEM. Prove that the area of the triangle determined by the midpoints of the sides of a given triangle is one-fourth the area of the given triangle.

9.5 PYTHAGOREAN THEOREM

In this section we state and prove the Pythagorean Theorem and its converse. Perhaps no other theorem in mathematics has inspired more mathematicians and nonmathematicians alike to find original proofs as has this one. There are over 370 known proofs of this theorem of which over 255 employ the use of areas. The first proof of this theorem has been attributed by many historians to Pythagoras (582–501 B.C.), a Greek mathematician and philosopher, although the practical application of the theorem was known many years before his time. It is not known which proof Pythagoras gave, but in the Exercises at the end of this section is an outline of a proof which appeared about 300 B.C. in the first of the thirteen books of Euclid's *Elements*.

Perhaps the most unique proof of the Pythagorean Theorem was devised by a 16-year-old schoolgirl, Miss Ann Condit, of South Bend, Indiana, in 1938. The proof devised by Miss Condit and an original proof devised in 1939 by Mr. Joseph Zelson, an 18-year-old junior in West Philadelphia, Pennsylvania High School, show that high school students are capable of original deductive reasoning. For a list of proofs of the Pythagorean Theorem, including the proofs by Miss Condit and Mr. Zelson, see *The Pythagorean Proposition* by Elisha Scott Loomis, published by the National Council of Teachers of Mathematics, 1968.

THEOREM 9.4 (***The Pythagorean Theorem***) If a, b, c are the lengths of the sides of a right triangle $\triangle ABC$, with $c = AB$, the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

Proof: Let $BC = a$, $CA = b$. Let $PQRS$ be a square of side length $a + b$ as in Figure 9-15. The points T, U, V, W are interior points of $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{SP}$, respectively, such that $PT = QU = RV = SW = a$.

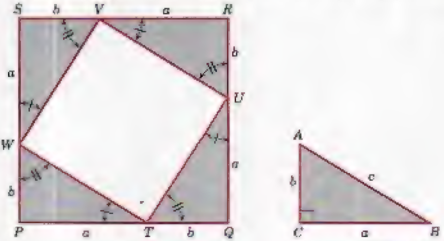


Figure 9-15

Then $TQ = UR = VS = WP = b$, and $\triangle TQU$, $\triangle URV$, $\triangle VSW$, $\triangle WPT$ are four congruent right triangles, each with area $\frac{1}{2}ab$. Hence $TUVW$ is an equilateral quadrilateral. Since the two acute angles in a right triangle are complementary angles, it follows that

$$\begin{aligned} m\angle WTU &= 180 - m\angle PTW - m\angle UTQ \\ &= 180 - m\angle PTW - m\angle PWT \\ &= 180 - (m\angle PTW + m\angle PWT) \\ &= 180 - 90 = 90. \end{aligned}$$

By a similar argument it can be shown that the other three angles of $TUVW$ are right angles. Therefore $TUVW$ is a square. Let x be its side length. Then

$$\begin{aligned} (a + b)^2 &= |PQRS| \\ (a + b)^2 &= 4|\triangle WPT| + |TUVW| \\ (a + b)^2 &= 4 \cdot \tfrac{1}{2}ab + x^2 \\ (a + b)^2 &= 2ab + x^2 \\ a^2 + 2ab + b^2 &= 2ab + x^2 \\ a^2 + b^2 &= x^2. \end{aligned}$$

It follows from the S.A.S. Congruence Postulate that $\triangle WPT \cong \triangle ABC$. Therefore

$$x = c \qquad \text{and} \qquad a^2 + b^2 = c^2.$$

THEOREM 9.5 ***(Converse of The Pythagorean Theorem)*** If a, b, c are the lengths of the sides of a triangle and if $a^2 + b^2 = c^2$, then the triangle is a right triangle and the right angle is opposite the side of length c.

Proof: Let there be given $\triangle ABC$ with $AB = c$, $BC = a$, $CA = b$, and $a^2 + b^2 = c^2$. Let $\triangle A'B'C'$ be a right triangle with $B'C' = a$, $C'A' = b$, $A'B' = x$, and the right angle at C' as shown in Figure 9-16.

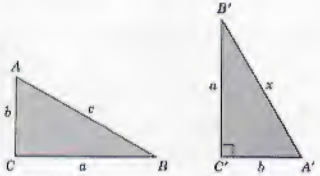


Figure 9-16

Then

$$a^2 + b^2 = x^2. \qquad \text{Why?}$$

By hypothesis, $a^2 + b^2 = c^2$. By substitution, $x^2 = c^2$ and, since $x > 0$ and $c > 0$, it follows that $x = c$. Therefore $\triangle ABC \cong \triangle A'B'C'$ by the S.S.S. Postulate and $\angle C \cong \angle C'$. It follows that $\angle C$ is a right angle and that $\triangle ABC$ is a right triangle.

When solving problems by use of the Pythagorean Theorem, it is often necessary to find the square root of a number that is not a perfect square. Recall that a rational number is a *perfect square* if it can be expressed as the square of a rational number. Thus 16 and $\frac{25}{9}$ are perfect squares because

$$16 = (4)^2 \qquad \text{and} \qquad \tfrac{25}{9} = (\tfrac{5}{3})^2,$$

whereas 32 and $\frac{27}{8}$ are not perfect squares. If we are required to find the square root of a number like 32 or $\frac{27}{8}$, we can approximate the square root by means of a rational number or we can leave our answer in what is called *simplest radical form*. If $x = \sqrt{A}$, where A is a positive rational number which is not a perfect square, we say that x is in simplest radical form when it is expressed as $B\sqrt{C}$, where B is a rational number and C is a positive integer which contains no factor (other than 1) that is a perfect square.

We make use of the following theorems from algebra when putting radicals in simplest form.

1. If a and b are positive numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

2. If a and b are positive numbers, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Example 1 Put (1) $\sqrt{32}$, (2) $\sqrt{\frac{27}{8}}$, and (3) $\sqrt{105}$ in simplest radical form.

Solution: 1. $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

2. $$\sqrt{\frac{27}{8}} = \sqrt{\frac{27}{8} \cdot \frac{2}{2}} = \frac{\sqrt{54}}{\sqrt{16}} = \frac{\sqrt{9 \cdot 6}}{4} = \frac{\sqrt{9} \cdot \sqrt{6}}{4}$$
$$= \frac{3}{4} \cdot \sqrt{6}$$

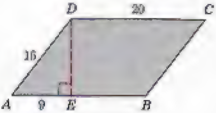
3. $\sqrt{105}$ is already in simplest radical form. Why?

EXERCISES 9.5

In Exercises 1–15, put the indicated radical in simplest radical form.

1. $\sqrt{8}$
2. $\sqrt{18}$
3. $\sqrt{27}$
4. $\sqrt{50}$
5. $\sqrt{72}$
6. $\sqrt{98}$
7. $\sqrt{300}$
8. $\sqrt{320}$
9. $\sqrt{2.88}$
10. $\sqrt{1.25}$
11. $\sqrt{\frac{2}{3}}$
12. $\sqrt{\frac{7}{9}}$
13. $\sqrt{\frac{36}{5}}$
14. $\sqrt{\frac{25}{18}}$
15. $\sqrt{\frac{125}{27}}$

16. If $a^2 + b^2 = c^2$, solve for a in terms of b and c.
17. If $a^2 + b^2 = c^2$, solve for b in terms of a and c.
18. If $a^2 + b^2 = c^2$, solve for c in terms of a and b.
19. If the length of a diagonal of a square is 15, find the area of the square. (*Hint:* Use the Pythagorean Theorem.)
20. In the figure, $ABCD$ is a parallelogram, $\overline{DE} \perp \overline{AB}$ at E, $AD = 15$, $CD = 20$, and $AE = 9$. Find the area of $ABCD$.



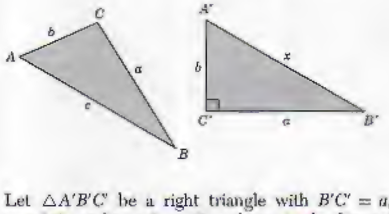
In Exercises 21–28, $\triangle ABC$ is a right triangle with right angle at C, $a = BC$, $b = CA$, $c = AB$. In each exercise, two of the three numbers a, b, c are given. Find the third one. Express each answer in decimal form correct to two decimal places.

21. $a = 1.0$, $b = 1.0$
22. $a = 1.0$, $b = 2.0$
23. $a = 10.0$, $b = 10.0$
24. $a = 3.0$, $b = 4.0$
25. $a = 9.0$, $b = 40.0$
26. $a = 3.1$, $b = 5.2$
27. $c = 10.0$, $b = 9.0$
28. $c = 10.0$, $a = 7.0$

In Exercises 29–36, $\triangle ABC$ is a right triangle with right angle at C, $a = BC$, $b = CA$, $c = AB$. In each exercise, two of the three numbers a, b, c are given. Find the third one. Express each answer exactly and in simplest form, using a radical if needed.

29. $a = 7, b = 40$
30. $a = 39, c = 89$
31. $a = \sqrt{2}, b = \sqrt{7}$
32. $a = \sqrt{31}, b = \sqrt{5}$
33. $a = \frac{2}{3}, b = \frac{1}{3}$
34. $a = 3x, b = 4x$ (Find c in terms of x.)
35. $a = x + y, b = x - y$ (Find c in terms of x and y.)
36. $c = u^2 + v^2, a = u^2 - v^2$ (Find b in terms of u and v.)

37. See the proof of Theorem 9.4. From which Triangle Congruence Postulate does the conclusion that $\triangle TQU \cong \triangle URV$ follow?
38. See the proof of Theorem 9.4. From which Area Postulate does it follow that the area of the large square is the sum of the areas of the four triangles and the smaller square?
39. See the proof of Theorem 9.4. From which Area Postulate does it follow that the four triangles have equal areas?
40. In the proof of Theorem 9.4, which Area Postulate supports the conclusion that the area of $TUVW$ is x^2?
41. Let $\triangle ABC$ with $BC = a$, $AC = b$, $AB = c$, $c \geq b$, $c \geq a$, and $c^2 > a^2 + b^2$ be given. Answer the "Whys?" in the following proof that $\angle C$ is an obtuse angle.



Proof: Let $\triangle A'B'C'$ be a right triangle with $B'C' = a$, $C'A' = b$, $A'B' = x$, and the right angle at C' as shown in the figure. Then

$$a^2 + b^2 = x^2$$

for $\triangle A'B'C'$. Why? By hypothesis, $a^2 + b^2 < c^2$ for $\triangle ABC$. By substitution, $x^2 < c^2$ and, since $x > 0$ and $c > 0$, it follows that $x < c$. Therefore $\angle C > \angle C'$. Why? Since $m\angle C' = 90$, it follows that $m\angle C > 90$. Then $\angle C$ is an obtuse angle. Why?

42. Let $\triangle ABC$ with $BC = a$, $AC = b$, $AB = c$, $c > b$, $c > a$, and $c^2 < a^2 + b^2$ be given. Prove that $\angle C$ is an acute angle. (See Exercise 41.)
43. Combine Theorem 9.5 and the statements proved in Exercises 41 and 42 into a single theorem.
44. CHALLENGE PROBLEM. The Pythagorean Theorem may be stated as follows.

> In a right triangle the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the legs.

RESTATEMENT: $\triangle ABC$ is a right triangle with the right angle at C, $BC = a$, $AC = b$, $AB = c$, squares $ABDE$, $BCLK$, and $AFGC$ as shown in Figure 9-17a. Prove that

$$|ABDE| = |BCLK| + |AFGC|$$

and hence that $c^2 = a^2 + b^2$. Complete the following proof which appeared in Euclid's *Elements* around 300 B.C.

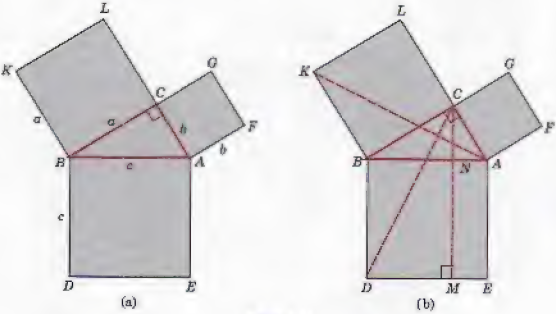


Figure 9-17

Proof: Draw $\overline{AK}$, $\overline{CD}$, and $\overline{CM} \perp \overline{DE}$ at M and intersecting $\overline{AB}$ at N as shown in Figure 9-17b.

(1) $\triangle ABK \cong \triangle DBC$ Why?
(2) $|\triangle ABK| = |\triangle DBC|$ Why?
(3) The altitude from A to $\overleftrightarrow{KB}$ of $\triangle ABK$ is equal to BC. Why?
(4) $|\triangle ABK| = \frac{1}{2} \cdot KB \cdot BC = \frac{1}{2} \cdot a^2$ Why?
(5) $|BCLK| = 2 \cdot |\triangle ABK|$ Why?
(6) The altitude from C to $\overleftrightarrow{DB}$ of $\triangle DBC$ is equal to BN. Why?
(7) $|\triangle CBD| = \frac{1}{2} \cdot DB \cdot BN$ Why?

(8) $|BDMN| = 2 \cdot |\triangle DBC|$ Why?
(9) $|BCLK| = |BDMN|$ Why?
(10) Draw $\overline{BF}$ and $\overline{CE}$. $\triangle BAF \cong \triangle EAC$ Why?
(11) $|AFGC| = 2 \cdot |\triangle BAF|$ Why?
(12) $|AEMN| = 2 \cdot |\triangle EAC|$ Why?
(13) $|AFGC| = |AEMN|$ Why?
(14) $|ABDE| = |BDMN| + |AEMN|$ Why?
$= |BCLK| + |AFGC|$ Why?
(15) Therefore $c^2 = a^2 + b^2$ Why?

45. CHALLENGE PROBLEM. If your high school library contains the book *The Pythagorean Proposition*, look up the proof devised by Miss Ann Condit and present it to your class.

CHAPTER SUMMARY

In this chapter there are four Area Postulates and several area formulas stated and proved as theorems. The last two theorems of the chapter are the Pythagorean Theorem and its converse.

POSTULATES
- Area Existence
- Rectangle Area
- Area Congruence
- Area Addition

FORMULAS

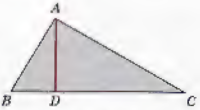
Rectangle Area Formula	$S = ab$
Parallelogram Area Formula	$S = bh$
Triangle Area Formula	$S = \frac{1}{2}bh$
Trapezoid Area Formula	$S = \frac{1}{2}(b_1 + b_2)h$

THE PYTHAGOREAN THEOREM. If a, b, c are the lengths of the sides of a right triangle $\triangle ABC$, with $c = AB$, the length of the hypotenuse, then $a^2 + b^2 = c^2$.

You should be able to state each of the postulates in your own words and to explain the meaning of each of the formulas. You should be able to use the postulates in developing the properties of area. You should understand the formulas so that you can recognize a situation in which a formula is applicable and then use it correctly.

REVIEW EXERCISES

1. Find the area of a trapezoid with two parallel sides of lengths 8 and 13 if the distance between the lines containing those sides is 24.
2. If the hypotenuse of a right triangle is 50 in. long and one of the legs is 40 in. long, how long is the other leg?
3. The figure shows a right triangle $\triangle ABC$ with right angle at A and with D the foot of the perpendicular from A to $\overleftrightarrow{BC}$. If $BD = 3$, $DC = 5\frac{1}{3}$, and $AB = 5$, find $|\triangle ABC|$.



4. Find the area of a right triangle if the length of the hypotenuse is $\sqrt{7}$ and the length of one leg is $\sqrt{3}$.
5. Find the base of a parallelogram if its area is 143 and its height is 7.
6. Find the altitude of a triangle corresponding to a base of length 12 if the area of the triangle is 62.
7. If $\triangle ABC$ is an equilateral triangle of side length 6, find the area of the triangle. (*Hint:* Use the Pythagorean Theorem to find an altitude of the triangle.)
8. See Exercise 7. Prove that the area S of an equilateral triangle of side length s is given by

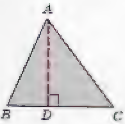
$$S = \frac{\sqrt{3}}{4} \cdot s^2.$$

9. Use the result of Exercise 8 to find the area of an equilateral triangle of side length 12. Compare the area of this triangle with that of Exercise 7. Is the ratio of the areas of the two triangles the same as the ratio of the lengths of their sides? If not, how do the areas compare?
10. If $ABCD$ is a parallelogram with $m\angle DAB = 45$, $DA = 12$, and $DC = 21$, find $|ABCD|$. (Hint: Draw $\overline{DE} \perp \overline{AB}$ at E. Which kind of triangle is $\triangle DAE$?)

In Exercises 11–13, $\triangle ABC$ is given with $a = BC$, $b = AC$, and $c = AB$. In each exercise the three numbers a, b, c are given. Is $\angle C$ a right, an obtuse, or an acute angle? (See Exercise 43 of Exercises 9.5.)

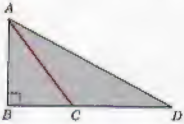
11. $a = 6$, $b = 9$, $c = 12$
12. $a = 8$, $b = 15$, $c = 17$
13. $a = 11$, $b = 13$, $c = 17$

14. The area of a trapezoid is 164 and the distance between the parallel bases is 10. If the length of one of the bases is 15, find the length of the other base.
15. (*An informal geometry exercise.*) The dimensions of a shower stall are 3 ft. by 3 ft. by 7 ft. How many 4 in. by 4 in. tiles are needed to cover three of the four rectangular walls if we assume that there is no waste in cutting the tile?
16. (*An informal geometry exercise.*) A rectangular plot of land is 100 yd. by 150 yd. A standard city lot for this particular plot of land is 75 ft. by 150 ft. and sells for \$3000. If a real estate agent can buy the plot of land for \$30,000, how much profit would he make if he divided it into standard lots and sold all of them? Disregard surveying and legal expenses.
17. An isosceles triangle has two sides of length 13. If the altitude to the base is 12, find the area of the triangle.
18. A rhombus has sides of length 10 and the measure of one of its angles is 45. Find its area.
19. The length of the hypotenuse of a right triangle is 25 and the length of one leg is 24.
    (a) Find the length of the other leg.
    (b) Find the area of the triangle.
    (c) Find the altitude to the hypotenuse.
20. A right triangle has legs of lengths 10 and 24 and hypotenuse of length 26.
    (a) Find the area of the triangle.
    (b) Find the altitude to the hypotenuse.
21. The length of one base of a trapezoid is 5 more than twice the length of the other base. If the area is 100 and the altitude is 10, find the lengths of the two bases.
22. In the figure, $\overline{AD}$ is an altitude of $\triangle ABC$. If $AD = 24$, $AB = 26$, and $AC = 30$, find BC and $|\triangle ABC|$.

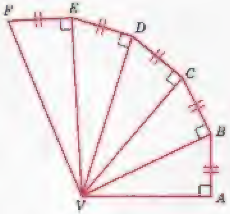


23. Recall that the diagonals of a rhombus are perpendicular to each other. Use this fact to find the area of a rhombus whose diagonals are of lengths 10 and 8. How is the area of the rhombus related to the lengths of its diagonals? State this relationship in the form of a theorem and prove it.

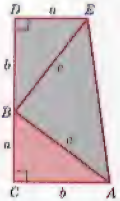
24. In the figure, $\overline{AB} \perp \overline{BD}$, B-C-D, $AC = 10$, $BC = 6$, and $AD = 17$. Find BD and $|\triangle ACD|$.



25. In the figure, with right angles and congruent segments as marked, if $VA = 2$ and $AB = 1$, find VF.

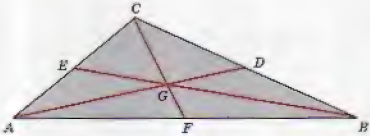


26. Taking the area of a given unit square as 1 and using any of the Area Postulates except the Rectangle Area Postulate, prove that the area of a 4 by 5 rectangle is 20.
27. Taking the area of a given unit square as 1 and using any of the Area Postulates except the Rectangle Area Postulate, prove that the area of a $\frac{1}{3}$ by $\frac{1}{2}$ rectangle is $\frac{1}{6}$.
28. The figure suggests an alternate proof of the Pythagorean Theorem which makes good use of the area formulas for a trapezoid and a triangle. Write out the main steps in this proof. (The proof suggested by this figure is attributed to President Garfield and was discovered by him around 1876.)



29. CHALLENGE PROBLEM. The figure shows a triangle and its three medians intersecting at the point G which is two-thirds of the way from any vertex to the midpoint of the opposite side. Prove that

$$|\triangle AGC| = 2 \cdot |\triangle GDC|.$$



30. CHALLENGE PROBLEM. Given the same situation as in Exercise 29, prove that the six small triangles all have the same area.

Chapter 10

Courtesy of Leo Castelli Gallery

Similarity

10.1 INTRODUCTION

Our experiences with objects of the same size and shape suggest the concept of congruence in formal geometry. The idea of same shape suggests the concept of similarity which you are about to study in this chapter. Consider a picture and an enlargement of it. In the enlargement each part has the same shape as it has in the original picture, but not the same size. The picture and its enlargement are examples of similar figures.

Consider a floor plan for a building. The floor plan is a drawing made up of segments labeled to show the lengths of the segments in the actual building which the segments on the drawing represent. The floor plan is similar to the actual floor of the building. Although the plan and the floor do not have the same size, they surely have the same shape. Each segment in this plan is much smaller than the segment it represents in the building. We express this fact by saying that the plan is a scale drawing of the floor. Each angle in the plan has the same size as the angle it represents on the actual floor.

In drawing a floor plan, lengths are reduced and angle measures are preserved. The lengths of segments in a floor plan are proportional to the lengths of the segments that they represent in the floor, and a statement of this fact is an example of a proportionality. In the next two sections we define a proportionality and develop some of its properties.

In the remaining sections we develop the concept of similar figures. The main theorems of this chapter have to do with triangle similarity. The chapter includes a proof of the Pythagorean Theorem based on similar triangles.

10.2 PROPORTIONALITY

Figure 10-1 shows two figures, one with side lengths labeled a, b, c, d, e and the other with a', b', c', d', e'. Note that $a = 2a'$, $b = 2b'$, $c = 2c'$, $d = 2d'$, $e = 2e'$, and hence that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'} = 2.$$

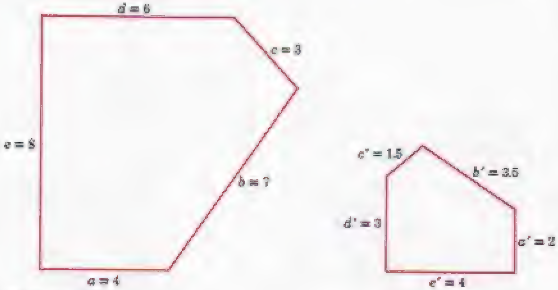


Figure 10-1

We can express the idea of these equations by saying that a, b, c, d, e are proportional to a', b', c', d', e', respectively.

> *Definition 10.1* Let a one-to-one correspondence between the real numbers $a, b, c, \ldots$ and the real numbers $a', b', c', \ldots$ in which a is matched with a', b is matched with b', c is matched with c', and so on, be given. Then the numbers $a, b, c, \ldots$ are said to be **proportional** to the numbers $a', b', c', \ldots$ if there is a nonzero number k such that $a = ka'$, $b = kb'$, $c = kc'$, …. The number k is called the **constant of proportionality.**

In Definition 10.1, why do you think k is required to be a nonzero number?

Notation. We use $\underset{p}{=}$ to mean "are proportional to." We use $(a, b, c, \ldots)$ to represent an ordered set of numbers. Then

$$(a, b, c, \ldots) \underset{p}{=} (a', b', c', \ldots)$$

means that the numbers $a, b, c, \ldots$ are proportional to the numbers $a', b', c', \ldots$, it being understood that a is matched with a', b with b', c with c', and so on. Note that the order of the numbers in the set $(a, b, c, \ldots)$ and in the set $(a', b', c', \ldots)$ is important only to the extent that corresponding numbers must appear in the same order in the proportionality. Thus if we have

$$(a, b, c, \ldots) \underset{p}{=} (a', b', c', \ldots),$$

we could also write $(b, c, a, \ldots) \underset{p}{=} (b', c', a', \ldots)$, and so on.

Example 1 (See Figure 10-1.) $(4, 7, 3, 6, 8) \underset{p}{=} (2, 3.5, 1.5, 3, 4)$.

Example 2 $(2, 3.5, 1.5, 3, 4) \underset{p}{=} (4, 7, 3, 6, 8)$.

Since

$$2 = \tfrac{1}{2} \cdot 4, \quad 3.5 = \tfrac{1}{2} \cdot 7, \quad 1.5 = \tfrac{1}{2} \cdot 3, \quad 3 = \tfrac{1}{2} \cdot 6, \quad 4 = \tfrac{1}{2} \cdot 8,$$

the constant of proportionality in Example 2 is $\frac{1}{2}$. What is the constant of proportionality in Example 1?

Example 3 If $(x, y, 9) \underset{p}{=} (7, 8, 6)$, find x and y.

Solution: There is a number k such that $x = k \cdot 7$, $y = k \cdot 8$, $9 = k \cdot 6$. From the last equation we have $k = \frac{3}{2}$, so

$$x = \tfrac{3}{2} \cdot 7 = 10.5 \quad \text{and} \quad y = \tfrac{3}{2} \cdot 8 = 12.$$

Example 4 Is the following statement true or false?

$$(5, 8, 10) \underset{p}{=} (8, 12.8, 16).$$

Solution: The statement is true if and only if there is a nonzero number k such that

$$\begin{aligned} 5 &= k \cdot 8 \\ 8 &= k \cdot 12.8 \\ 10 &= k \cdot 16 \end{aligned}$$

From the first of these three equations, we see that k must be $\frac{5}{8}$, or 0.625. We must now check to see if this k "works" in the other two equations.

Check. Is $8 = 0.625 \cdot 12.8$? Yes.
Is $10 = \frac{5}{8} \cdot 16$? Yes.
Therefore $(5, 8, 10) \underset{p}{=} (8, 12.8, 16)$ is true.

Example 5 Is the statement $(4, 6, 10) \underset{p}{=} (6, 9, 16)$ true?

Solution:
If $4 = k \cdot 6$, then $k = \frac{2}{3}$.
Is $6 = \frac{2}{3} \cdot 9$? Yes.
Is $10 = \frac{2}{3} \cdot 16$? No.
Is the statement $(4, 6, 10) \underset{p}{=} (6, 9, 16)$ true? No.

EXERCISES 10.2

In Exercises 1–12, determine whether the given statement is true or false. If it is true, find the constant of proportionality. The answer for Exercise 1 has been given as a sample.

1. $(1, 1, 5) \underset{p}{=} (3, 3, 15)$. True; $k = \frac{1}{3}$.
2. $(3, 6, 9) \underset{p}{=} (2, 4, 6)$
3. $(5, 1) \underset{p}{=} (8, 2)$
4. $(6, 10, 15, 21) \underset{p}{=} (7, 11, 16, 22)$
5. $(5, 15, 25, 35) \underset{p}{=} (1, 3, 5, 7)$
6. $(2, 6, 7, 15) \underset{p}{=} (1, 3, 3.5, 7.5)$
7. $(1, 3, 1 + 3) \underset{p}{=} (2, 6, 2 + 6)$
8. $(x, y, x + y) \underset{p}{=} (7x, 7y, 7x + 7y)$
9. $(u, v, w) \underset{p}{=} (-3u, -3v, -3w)$
10. $(0, 0, 0) \underset{p}{=} (5, 10, 15)$
11. $(5, 10, 15) \underset{p}{=} (0, 0, 0)$
12. $(0, 0, 3) \underset{p}{=} (0, 0, 4)$

In Exercises 13–20, copy and complete the given statement so that it will be true.

13. $(6, 8) \underset{p}{=} (\boxed{?}, 6)$
14. $(\boxed{?}, \boxed{?}, 100) \underset{p}{=} (5, 6, 10)$
15. $(5, \boxed{?}, 7) \underset{p}{=} (\boxed{?}, 8, 10.5)$
16. If $(3, x) \underset{p}{=} (8, 10)$, then $x = \boxed{?}$.
17. If $(7, 9) \underset{p}{=} (7x, 18)$, then $x = \boxed{?}$.
18. If $(7, x) \underset{p}{=} (14, 21)$, then $x = \boxed{?}$.
19. If $(5, x) \underset{p}{=} (x, 125)$ and $x > 0$, then $x = \boxed{?}$.
20. If $(3, x) \underset{p}{=} (8, y)$ and $y \neq 0$, then $(3, \boxed{?}) \underset{p}{=} (x, y)$.

In Exercises 21–30, x is a positive number and $(3, x) \underset{p}{=} (5, 15)$. In each exercise, determine whether the given statement is true.

21. $\frac{3}{5} = \frac{x}{15}$
22. $\frac{3}{x} = \frac{5}{15}$
23. $(3, x, 3 + x) \underset{p}{=} (5, 15, 20)$
24. $(x, 3 - x) \underset{p}{=} (15, 10)$
25. $(x, 3) \underset{p}{=} (15, 5)$
26. $(3, 5) \underset{p}{=} (x, 15)$
27. $(3, 15) \underset{p}{=} (x, 5)$
28. $(3, 5) \underset{p}{=} (15, x)$
29. $3 \cdot 15 = x \cdot 5$
30. $\frac{3}{x + 3} = \frac{5}{5 + 15}$
31. Does $(a, b, c) \longleftrightarrow (d, e, f)$ indicate the same one-to-one correspondence as $(a, c, b) \longleftrightarrow (d, f, e)$? As $(c, a, b) \longleftrightarrow (f, d, e)$? As $(c, b, a) \longleftrightarrow (f, e, d)$?
32. If a, b, c, d are nonzero numbers such that $(a, b) \underset{p}{=} (c, d)$ with proportionality constant 2, is it true that $(b, a) \underset{p}{=} (d, c)$ with proportionality constant 2? Give your reasoning.
33. If a, b, c, d are nonzero numbers such that $(a, b) \underset{p}{=} (c, d)$ with proportionality constant 2, is it true that $(c, d) \underset{p}{=} (a, b)$ with proportionality constant 2? Give your reasoning.
34. If a, b, c, d are nonzero numbers such that $(a, b) \underset{p}{=} (c, d)$, prove that $(a, c) \underset{p}{=} (b, d)$.
35. If $(a, b) \underset{p}{=} (c, d)$, prove that $ad = bc$.
36. If $ad = bc$, prove that $(a, b) \underset{p}{=} (c, d)$.

In Exercises 37–40, copy and complete the given statement so that it will be a proportionality.

37. $(5, 7, 10, 12) \underset{p}{=} (10, \boxed{?}, \boxed{?}, \boxed{?})$
38. $(\boxed{?}, \boxed{?}, 1) \underset{p}{=} (15, 37, 100)$
39. $(0.37, 0.67, 0.93) \underset{p}{=} (\boxed{?}, 67, \boxed{?})$
40. $(\sqrt{2}, \sqrt{3}, \sqrt{5}) \underset{p}{=} (\sqrt{8}, \boxed{?}, \boxed{?})$

10.3 PROPERTIES OF PROPORTIONALITIES

As you might expect from your study of algebra and from some of Exercises 10.2, proportionalities have some interesting properties. In this section we show that the proportionality relation is reflexive, symmetric, and transitive, and therefore it is an equivalence relation. Hence the relation denoted by "$\underset{p}{=}$" has some properties in common with the relations denoted by "$=$" and "$\cong$." We shall also state and prove addition and multiplication properties. At the end of this section we consider some special proportionalites called proportions.

We shall prove the next two theorems for proportionalities involving triples of numbers. It is easy to see how the statements and proofs can be modified for proportionalities involving more than or fewer than three numbers.

THEOREM 10.1 The proportionality relation is an equivalence relation.

Proof:

1. **The Reflexive Property.** Let a, b, c be any real numbers. Observe that $(a, b, c) \underset{p}{=} (a, b, c)$ with proportionality constant 1.

2. **The Symmetric Property.** Suppose that $(a, b, c) \underset{p}{=} (d, e, f)$. Then there is a nonzero number k such that $a = kd$, $b = ke$, $c = kf$. Why? Then $d = k'a$, $e = k'b$, $f = k'c$, where $k' = \dfrac{1}{k}$ and $k' \neq 0$. Therefore $(d, e, f) \underset{p}{=} (a, b, c)$. Thus if $(a, b, c) \underset{p}{=} (d, e, f)$ with constant of proportionality k, then $(d, e, f) \underset{p}{=} (a, b, c)$ with constant of proportionality $\dfrac{1}{k}$.

3. **The Transitive Property.** Suppose $(a, b, c) \underset{p}{=} (d, e, f)$ and $(d, e, f) \underset{p}{=} (g, h, i)$. Then there are nonzero numbers k_1 and k_2 such that

$$a = k_1 d, \qquad b = k_1 e, \qquad c = k_1 f,$$
$$d = k_2 g, \qquad e = k_2 h, \qquad f = k_2 i.$$

Then

$$a = k_1 d = k_1(k_2 g) = (k_1 k_2)g,$$
$$b = k_1 e = k_1(k_2 h) = (k_1 k_2)h,$$
$$c = k_1 f = k_1(k_2 i) = (k_1 k_2)i.$$

Since $k_1 \neq 0$ and $k_2 \neq 0$, it follows that $k_1k_2 \neq 0$. Therefore, if a, b, c are proportional to d, e, f with constant of proportionality k_1, and d, e, f are proportional to g, h, i with constant of proportionality k_2, then a, b, c are proportional to g, h, i with constant of proportionality k_1k_2.

In the next theorem note that (1) is an "addition" property and that (2) is a "multiplication" property.

THEOREM 10.2 If $(a, b, c) \underset{p}{=} (d, e, f)$, then

$$(1) \qquad (a, b, c, a + b + c) \underset{p}{=} (d, e, f, d + e + f),$$

and, if $h \neq 0$,

$$(2) \qquad (ha, b, c) \underset{p}{=} (hd, e, f).$$

Proof: Let it be given that $(a, b, c) \underset{p}{=} (d, e, f)$. Then there is a nonzero number k such that

$$a = kd, \qquad b = ke, \qquad c = kf.$$

Adding, we get

$$a + b + c = kd + ke + kf,$$

and, by the Distributive Property,

$$a + b + c = k(d + e + f).$$

Multiplying both sides of $a = kd$ by h, we get $ha = h(kd)$, and hence

$$ha = k(hd).$$

We have shown that if a, b, c are proportional to d, e, f with constant of proportionality k, then (1) and (2) hold with the same constant of proportionality.

Example 1 Figure 10-2 shows a triangle $\triangle ABC$ and a segment $\overline{DE}$ joining a point D of $\overline{AB}$ to a point E of $\overline{AC}$. Suppose we know that $(AD, DB) \underset{p}{=} (AE, EC)$.

Figure 10-2

We may conclude from the addition property of proportionality that

$$(AD, DB, AD + DB) \underset{p}{=} (AE, EC, AE + EC)$$

and hence that

$$(AD, DB, AB) \underset{p}{=} (AE, EC, AC),$$

since $AB = AD + DB$ and $AC = AE + EC$. Of course, we may also conclude that

$$(AD, AB) \underset{p}{=} AE, AC).$$

This may not be the "whole truth," but it is certainly the "truth." It is like concluding that if $x = 3u$, $y = 3v$, and $z = 3w$, then $x = 3u$ and $y = 3v$. Similarly, we may conclude that

$$(DB, AB) \underset{p}{=} (EC, AC).$$

Example 2 Let $\triangle ABC$ with D an interior point of $\overline{BC}$ be given as suggested in Figure 10-3. Let h denote the distance from A to $\overleftrightarrow{BC}$, let $BD = b_1$, and let $DC = b_2$. Then

$$|\triangle ABD| = \tfrac{1}{2}hb_1, \qquad |\triangle ADC| = \tfrac{1}{2}hb_2.$$

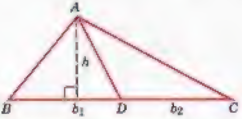


Figure 10-3

Therefore $(|\triangle ABD|, |\triangle ADC|) \underset{p}{=} (b_1, b_2)$, and the constant of proportionality is $\frac{1}{2}h$. Thus the areas of the two triangles formed from $\triangle ABC$ by inserting the segment $\overline{AD}$ are proportional to the lengths of the segments formed from $\overline{BC}$ by inserting the point D. In this connection some of you will recall Challenge Problem 29 in the Review Exercises of Chapter 9.

We frequently work in geometry with proportionalities in which two numbers are proportional to two numbers. Such proportionalities are generally called **proportions.** Following a formal definition is a list of some of their special properties.

> ***Definition 10.2*** If a, b, c, d are numbers such that $(a, b) \underset{p}{=} (c, d)$ is a proportionality, then that proportionality is a **proportion.**

In other words, a proportion is a proportionality with two numbers on each side of the "$\underset{p}{=}$" symbol.

The following theorem includes four named properties of proportions.

THEOREM 10.3 Proportions involving nonzero numbers a, b, c, d have the following properties:

1. **Alternation Property.** If $(a, b) \underset{p}{=} (c, d)$, then
$$(a, c) \underset{p}{=} (b, d) \qquad \text{and} \qquad (d, b) \underset{p}{=} (c, a).$$
2. **Inversion Property.** If $(a, b) \underset{p}{=} (c, d)$, then
$$(b, a) \underset{p}{=} (d, c).$$
3. **Product Property.** $(a, b) \underset{p}{=} (c, d)$ if and only if $ad = bc$.
4. **Ratio Property.** $(a, b) \underset{p}{=} (c, d)$ if and only if $\frac{a}{b} = \frac{c}{d}$.

Proof: Assigned as exercises.

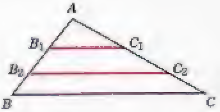
If you think of a and d as the "outside" numbers and b and c as the "inside" numbers of a proportion
$$(a, b) \underset{p}{=} (c, d),$$
the Alternation Property says that if you interchange either the outside numbers or the inside numbers in a proportion, then the result is a proportion.

If you combine the Inversion Property with the Ratio Property, the Inversion Property amounts to saying that if two ratios are equal, then their "inversions" (reciprocals) are equal.

In Theorem 10.3, the numbers in the proportions are required to be nonzero numbers. What would be the situation if zeros were permitted? Note that $(0, 5) \underset{p}{=} (0, 10)$ is a true statement, whereas $(0, 0) \underset{p}{=} (5, 10)$ is a false statement. Therefore the proportion
$$(0, 5) \underset{p}{=} (0, 10)$$
does not have the Alternation Property. Observe that $0 \cdot 7 = 0 \cdot 8$ is a true statement, whereas $(0, 0) \underset{p}{=} (8, 7)$ is a false statement. Therefore the Product Property does not apply to $0 \cdot 7 = 0 \cdot 8$. Observe that
$$(1, 0) \underset{p}{=} (2, 0)$$
is a true statement, whereas $\frac{1}{0} = \frac{2}{0}$ is a false statement. Therefore the Ratio Property does not apply to $(1, 0) \underset{p}{=} (2, 0)$.

EXERCISES 10.3

1. If a, b, c, d are all nonzero numbers, then $(a, b, c, d) \underset{p}{=} (a, b, c, d)$. What is the constant of proportionality in this proportion? Which property of an equivalence relation does this illustrate?
2. Given $(a, b, c, d) \underset{p}{=} (e, f, g, h)$, prove that
$$(e, f, g, h) \underset{p}{=} (a, b, c\ d).$$
Which property of an equivalence relation does this illustrate? What is the relation between the constants of proportionality for these two proportionalities?
3. Given $(a, b) \underset{p}{=} (c, d)$ with proportionality constant k_1 and $(c, d) \underset{p}{=} (e, f)$ with proportionality constant k_2, then $(a, b) \underset{p}{=} (e, f)$ with some proportionality constant, say k_3. How are k_1, k_2, k_3 related? Which property of an equivalence relation do we have illustrated here?
4. In the figure A, B, C are noncollinear points; A, B_1, B_2, B are collinear and arranged in the order named; A, C_1, C_2, C are collinear and arranged in the order named. It is given that
$$(AB_1, B_1B_2, B_2B) \underset{p}{=} (AC_1, C_1C_2, C_2C)$$
with constant of proportionality 0.8. Prove, using the properties of proportionalities, that
(a) $(AB_1, AB) \underset{p}{=} (AC_1, AC)$.
(b) $(16AB_1, AB) \underset{p}{=} (16AC_1, AC)$.



In Exercises 5–11, complete the given statement and name the property which it illustrates. Assume that none of the numbers in these exercises is zero.

5. If $(a, b) \underset{p}{=} (3, 5)$, then $(a, 3) \underset{p}{=} (b, \boxed{?})$.
6. If $(x, y) \underset{p}{=} (6, 7)$, then $(y, x) \underset{p}{=} (7, \boxed{?})$.
7. If $(u, v) \underset{p}{=} (5, 6)$ and $(5, 6) \underset{p}{=} (x, y)$, then $(u, v) \underset{p}{=} (x, \boxed{?})$.
8. If $(a, b) \underset{p}{=} (c, \boxed{?})$, then $(a, a + b) \underset{p}{=} (c, c + d)$.
9. If $(x, y) \underset{p}{=} (1, 2)$, then $(3x, y) \underset{p}{=} (\boxed{?}, 2)$.
10. If $(x, y) \underset{p}{=} (\boxed{?}, 7)$, then $7x = 4y$.
11. If $(x, y) \underset{p}{=} (5, 8)$, then $\dfrac{x}{y} = \boxed{?}$.

In Exercises 12–21, write a proportion, or complete the given one, so that it will be equivalent to the given information. Starting with the given information, you should be able to prove that the proportion is true. Starting with the proportion, you should be able to prove that the given equation (or equations) is true.

12. $\frac{3}{4} = \frac{x}{y}$; $(x, y) \underset{p}{=} (\boxed{?}, \boxed{?})$

13. $\frac{3}{4} = \frac{x}{y}$; $(x, 3) \underset{p}{=} (\boxed{?}, \boxed{?})$

14. $\frac{3}{4} = \frac{x}{y}$; $(y, x) \underset{p}{=} (\boxed{?}, \boxed{?})$

15. $\frac{3}{4} = \frac{x}{y}$; $(y, 4) \underset{p}{=} (\boxed{?}, \boxed{?})$

16. $a = 3x,\ b = 3y$

17. $x = \frac{2}{3}y,\ y = \frac{2}{3}z$

18. $\frac{x}{7} = \frac{5}{6}$

19. $\frac{x}{10} = \frac{y}{30}$

20. $\frac{a}{7} = \frac{y}{14}$

21. $\frac{3}{5} = \frac{7}{t}$

Exercises 22–25 refer to Theorem 10.3. Prove that proportions involving nonzero numbers have the indicated properties.

22. The Alternation Property.
23. The Inversion Property.
24. The Product Property.
25. The Ratio Property.

10.4 SIMILARITIES BETWEEN POLYGONS

In this section we define what is meant by a similarity between two polygons. In developing our formal geometry we use similarities as tools; in most cases we consider similarities between triangles.

Definition 10.3

1. A one-to-one correspondence $ABC \ldots \longleftrightarrow A'B'C' \ldots$ between the vertices of polygon $ABC \ldots$ and polygon $A'B'C' \ldots$ is a **similarity** between the polygons if and only if corresponding angles are congruent and lengths of corresponding sides are proportional.
2. If $ABC \ldots \longleftrightarrow A'B'C' \ldots$ is a similarity, then polygon $ABC \ldots$ and polygon $A'B'C' \ldots$ are **similar polygons** and each is **similar** to the other.
3. If $ABC \ldots \longleftrightarrow A'B'C' \ldots$ is a similarity with $AB = kA'B'$, $BC = kB'C'$, and so on, then k is the **constant of proportionality**, or the **proportionality constant**, for that similarity.

Notation. The symbol "$\sim$" is read "is similar to"; hence $ABC\ldots \sim A'B'C'\ldots$ is read "$ABC\ldots$ is similar to $A'B'C'\ldots$." This means that

$$ABC\ldots \longleftrightarrow A'B'C'\ldots$$

is a similarity.

Example 1 Figure 10-4 shows two quadrilaterals $ABCD$ and $A'B'C'D'$ with segment lengths and angle measures as indicated. It appears that $ABCD \longleftrightarrow A'B'C'D'$ is a similarity and hence that $ABCD \sim A'B'C'D'$. Let us check this conjecture. There are eight pairs of corresponding parts including four pairs of corresponding angles and four pairs of corresponding sides:

$\angle A$	and	$\angle A'$;	$\overline{AB}$	and	$\overline{A'B'}$;
$\angle B$	and	$\angle B'$;	$\overline{BC}$	and	$\overline{B'C'}$;
$\angle C$	and	$\angle C'$;	$\overline{CD}$	and	$\overline{C'D'}$;
$\angle D$	and	$\angle D'$;	$\overline{DA}$	and	$\overline{D'A'}$.

Figure 10-4

First we check the angles.

$$\begin{aligned} m\angle A &= m\angle A' = 90, \\ m\angle B &= m\angle B' = 45, \\ m\angle C &= m\angle C' = 135, \\ m\angle D &= m\angle D' = 90. \end{aligned}$$

Therefore, corresponding angles are congruent. Next we check to see if the lengths of corresponding sides are proportional. We want

$$(AB, BC, CD, DA) \underset{p}{=} (A'B', B'C', C'D', D'A').$$

Substituting, we get

$$(7, 3\sqrt{2}, 4, 3) \underset{p}{=} (3.5, 1.5\sqrt{2}, 2, 1.5).$$

Since this is indeed a proportionality, with constant of proportionality 2, the lengths of corresponding sides are proportional. Therefore, it follows directly from the definition of a similarity that $ABCD \sim A'B'C'D'$.

In Example 1 the constant of proportionality between the lengths of the sides of quadrilateral $ABCD$ and the lengths of the corresponding sides of quadrilateral $A'B'C'D'$ is 2. Is it possible that in another example the constant of proportionality might be 1? Of course it is. In this special case of a similarity, corresponding angles are congruent and corresponding sides are congruent. Hence for this special case you should see that the similarity is a congruence. In other words, a **congruence** between polygons is a similarity between polygons for which the constant of proportionality is 1.

Just as congruence for triangles is an equivalence relation, so also is similarity for polygons. We state this fact as our next theorem.

THEOREM 10.4 The relation of similarity between polygons is reflexive, symmetric, and transitive.

Proof: We shall prove the theorem for triangles. It is easy to modify this proof to get a proof for quadrilaterals, pentagons, and so on.

Reflexive. Let $\triangle ABC$ be given. Since $\triangle ABC \cong \triangle ABC$, it follows that $\triangle ABC \sim \triangle ABC$. What is the constant of proportionality for

$$(AB, AC, BC) \underset{p}{=} (AB, AC, BC)?$$

Symmetric. Suppose that $\triangle ABC \sim \triangle DEF$; then $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, and

$$(AB, BC, CA) \underset{p}{=} (DE, EF, DF).$$

We want to prove that $\triangle DEF \sim \triangle ABC$, which means that $\angle D \cong \angle A$, $\angle E \cong \angle B$, $\angle F \cong \angle C$, and that

$$(DE, EF, DF) \underset{p}{=} (AB, BC, CA).$$

Now $\angle A \cong \angle D$ implies $\angle D \cong \angle A$, $\angle B \cong \angle E$ implies $\angle E \cong \angle B$, and $\angle C \cong \angle F$ implies $\angle F \cong \angle C$. (Which property of congruence for angles supports this deduction?) Also,

$$(AB, BC, CA) \underset{p}{=} (DE, EF, DF)$$

implies that

$$(DE, EF, DF) \underset{p}{=} (AB, BC, CA).$$

(Which property of proportionality supports this deduction?) Therefore

$$\triangle ABC \sim \triangle DEF$$

implies that

$$\triangle DEF \sim \triangle ABC,$$

and similarity for triangles is a symmetric relation.

Transitive. Let it be given that $\triangle ABC \sim \triangle DEF$ and that $\triangle DEF \sim \triangle GHI$. The proof that $\triangle ABC \sim \triangle GHI$ is assigned as an exercise. From this it follows that similarity for triangles is a transitive relation.

THEOREM 10.5 The perimeters of two similar polygons are proportional to the lengths of any two corresponding sides.

Proof: As in the case for Theorem 10.4, we shall prove Theorem 10.5 for triangles. It is easy to modify this proof to get a proof for quadrilaterals, pentagons, and so on.

Let $\triangle ABC \sim \triangle A'B'C'$ be given and let p be the perimeter of $\triangle ABC$ and p' the perimeter of $\triangle A'B'C'$. By the definition of similarity we have

$$(AB, BC, CA) \underset{p}{=} (A'B', B'C', C'A').$$

By Theorem 10.2,

$$(AB, BC, AC, AB + BC + CA) \underset{p}{=} (A'B', B'C', A'C', A'B' + B'C' + C'A').$$

But

$$p = AB + BC + CA \qquad \text{and} \qquad p' = A'B' + B'C' + C'A'.$$

Therefore

$$(AB, BC, CA, p) \underset{p}{=} (A'B', B'C', C'A', p').$$

It follows that

$$(AB, p) \underset{p}{=} (A'B', p')$$

and, from the Alternation and Inversion Properties that

$$(p, p') \underset{p}{=} (AB, A'B').$$

In a similar way, it can be shown that

$$(p, p') \underset{p}{=} (BC, B'C') \qquad \text{and} \qquad (p, p') \underset{p}{=} (CA, C'A').$$

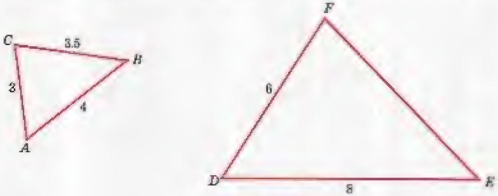
This completes the proof of Theorem 10.5 for triangles.

EXERCISES 10.4

1. Complete the proof of Theorem 10.4 by proving that $\triangle ABC \sim \triangle GHI$.
2. Given $\triangle ABC \sim \triangle DEF$, $m\angle A = 30$, $m\angle B = 60$, $AB = 20$, $BC = 10$, $CA = 10\sqrt{3}$, $DE = 100$, find $m\angle D$, $m\angle E$, $m\angle F$, EF, and DF.

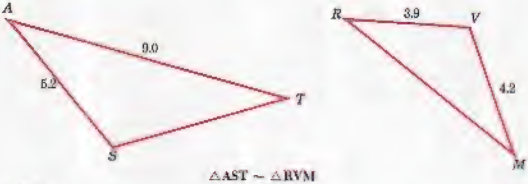
In Exercises 3–10, there are figures showing two similar triangles with some side lengths indicated and a similarity statement given. In each case, find the proportionality constant of the given similarity statement and the lengths of any sides whose lengths are not given.

3.



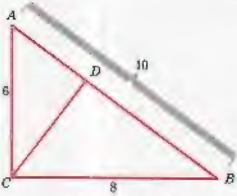
△ABC ~ △DEF

4.



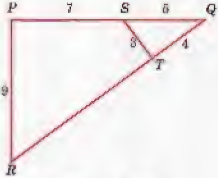
△AST ~ △RVM

5.



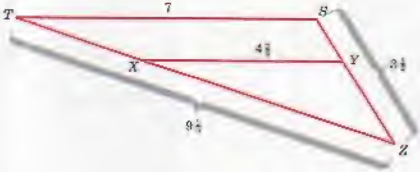
△ADC ~ △ACB

6.



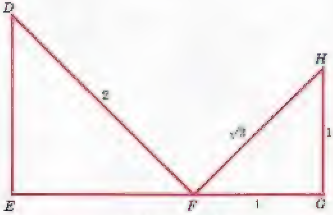
△PQR ~ △TQS

7.



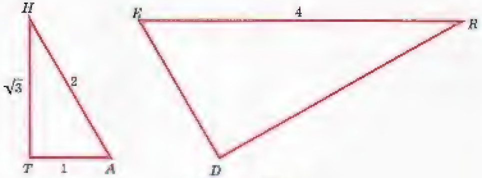
△TSZ ~ △XYZ

8.



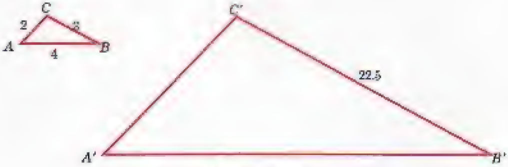
$\triangle DEF \sim \triangle HGF$

9.



$\triangle RED \sim \triangle HAT$

10.



$\triangle ABC \sim \triangle A'B'C'$

11. *Given:* $\triangle ABC \sim \triangle A'B'C'$
$\triangle A'B'C' \sim \triangle A''B''C''$
$AB = \frac{1}{5}A'B'$
$A'B' = 5A''B''$
Prove: $\triangle ABC \cong \triangle A''B''C''$

12. Let $\triangle ANT \sim \triangle MUD$ with

$$\frac{AN}{MU} = \frac{AT}{MD} = \frac{NT}{UD} = 1000.$$

If the shortest side of $\triangle MUD$ is of length 1000, find the length of the shortest side of $\triangle ANT$.

13. If in Exercise 12, p and p' denote the perimeters of the smaller triangle and the larger triangle, respectively, find $\frac{p'}{p}$.
14. $ABCDE$ and $A'B'C'D'E'$ are pentagons such that

$$(AB, BC, CD, DE, EA) \underset{p}{=} (A'B', B'C', C'D', D'E', E'A'),$$
$$\angle A \cong \angle A', \qquad \angle B \cong \angle B', \qquad \angle C \cong \angle C',$$
$$\angle D \cong \angle D', \qquad \angle E \cong \angle E', \qquad \text{and} \qquad AB = 13A'B'.$$

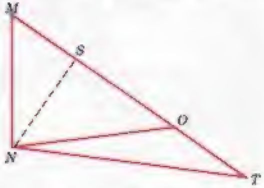
Prove that the perimeter of $ABCDE$ is 13 times the perimeter of $A'B'C'D'E'$.
15. If $\triangle ABC \sim \triangle A'B'C'$ and $AB = 10$, $BC = 8$, $A'B' = 25$, $A'C' = 35$, find $B'C'$ and AC.
16. If $\triangle PQR \sim \triangle STV$ and $PQ = 24$, $ST = 16$, $PR = 18$, $TV = 10$, find QR and SV.
17. If $\triangle ABC \sim \triangle RST$ and $7 \cdot AB = 4 \cdot RS$, what is the ratio of the perimeter of $\triangle ABC$ to the perimeter of $\triangle RST$?
18. If $\triangle ABC \sim \triangle DEF$, $AB = 5$, $BC = 7$, $AC = 8$, and the perimeter of $\triangle DEF$ is 60, find DE, EF, and DF.
19. In the figure, $A\text{–}K\text{–}T\text{–}S$, $\overline{RK} \perp \overline{AS}$, $RK = 3$, $AT = 8$, $TS = 4$.

(a) Find $|\triangle RAT|$, the area of $\triangle RAT$.
(b) Find $|\triangle RTS|$.
(c) Solve for x: $(|\triangle RAT|, |\triangle RTS|) \underset{p}{=} (x, 4)$

20. In the figure, $A\text{–}K\text{–}T\text{–}S$, $\overline{RK} \perp \overline{AS}$, $RK = 3$, $AT = 12$, $TS = 2$.

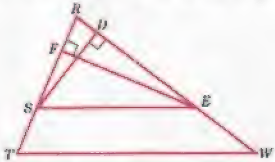
(a) Find $|\triangle RAT|$.
(b) Find $|\triangle RTS|$.
(c) Solve for x: $(x, |\triangle RTS|) \underset{p}{=} (12, 2)$.

21. In the figure, M-S-O-T, $\overline{NS} \perp \overline{MT}$, $MO = 12$, $OT = 5$.



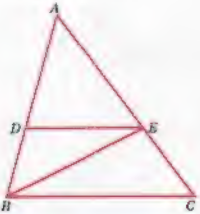
(a) Solve for x: $(|\triangle NMO|, |\triangle NOT|) \underset{p}{=} (12, x)$.
(b) Solve for y: $(|\triangle NMO|, y) \underset{p}{=} (|\triangle NOT|, 5)$.
(c) Solve for z: $\dfrac{|\triangle NMT|}{|\triangle NOT|} = \dfrac{z}{5}$

22. In the figure, R–F–S–T, R–D–E–W, $\overline{SE} \parallel \overline{TW}$, $\overline{FE} \perp \overline{RT}$, $\overline{DS} \perp \overline{RW}$, $RS = 20$, $ST = 10$, $RE = 30$, $EW = 15$, $FE = 29$.



(a) Find $|\triangle RSE|$. (b) Find $|\triangle STE|$.
(c) Find SD. (d) Find $|\triangle SEW|$.

23. In the figure, A-D-B, A-E-C, $\overline{DE} \parallel \overline{BC}$, $BD = 12$, $DA = 20$, $|\triangle BDE| = 120$.
(a) Find $|\triangle ADE|$.
(b) Find $|\triangle DEC|$.
(c) Find the ratio of AE to EC.
(d) Find the ratio of AD to DB.
(e) Compare the ratios in (c) and (d).



10.5 SOME LENGTH PROPORTIONALITIES

In this section there are several theorems regarding the lengths of segments formed by lines parallel to one side of a triangle intersecting the other two sides, and there are other theorems extending these ideas to lengths of segments formed when three or more parallel lines are cut by two transversals. These theorems are useful in proving the similarity theorems of Section 10.6.

THEOREM 10.6 (*Triangle Proportionality Theorem*) If a line parallel to one side of a triangle intersects a second side in an interior point, then it intersects the third side in an interior point, and the lengths of the segments formed on the second side are proportional to the lengths of the segments formed on the third side.

Proof: Let there be given a triangle $\triangle ABC$ and a line l such that l intersects $\overline{AB}$ in an interior point D and such that l is parallel to $\overline{BC}$ as suggested in Figure 10-5. We know from Theorem 2.6 that l intersects $\overline{AC}$ in an interior point. Call it E. We shall prove that

$$(BD, AD, BA) \underset{p}{=} (CE, AE, CA).$$

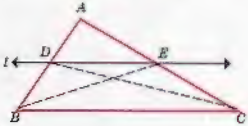


Figure 10-5

Let

h_1 be the distance from E to $\overleftrightarrow{AB}$,
h_2 be the distance from D to $\overleftrightarrow{AC}$,
h_3 be the distance between $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$.

Following are the main steps in the deductive reasoning which completes the proof of the theorem. You are asked in the Exercises to supply the reasons for steps 1 through 7 (definitions, theorems, preceding steps, etc.).

1. $\dfrac{BD}{AD} = \dfrac{\frac{1}{2}h_1 \cdot BD}{\frac{1}{2}h_1 \cdot AD} = \dfrac{|\triangle BED|}{|\triangle AED|}$
2. $\dfrac{CE}{AE} = \dfrac{\frac{1}{2}h_2 \cdot CE}{\frac{1}{2}h_2 \cdot AE} = \dfrac{|\triangle CED|}{|\triangle AED|}$
3. $|\triangle BED| = \frac{1}{2}h_3 \cdot DE = |\triangle CED|$

4. $\dfrac{BD}{AD} = \dfrac{CE}{AE}$

5. $BD \cdot AE = AD \cdot CE$

6. $(BD, AD) \underset{p}{=} (CE, AE)$

7. $(BD, AD, BA) \underset{p}{=} (CE, AE, CA)$

Therefore, the lengths of $\overline{AB}$ and the segments formed by l cutting $\overline{AB}$ are proportional to the lengths of $\overline{AC}$ and the segments formed by l cutting $\overline{AC}$. From step 7 we get step 8.

8. $(BD, BA) \underset{p}{=} (CE, CA)$ and $(AD, BA) \underset{p}{=} (AE, CA)$

From step 8 we get step 9 using the Alternation Property.

9. $(BD, CE) \underset{p}{=} (BA, CA)$ and $(AD, AE) \underset{p}{=} (AB, AC)$

Therefore, the lengths of AB and the segments formed by l cutting other two sides in interior points, then it cuts off segments whose lengths are proportional to the lengths of those sides.

THEOREM 10.7 ***(Converse of the Triangle Proportionality Theorem)*** Let $\triangle ABC$ with points D and E such that A–D–B and A–E–C be given. If

$$(AD, AB) \underset{p}{=} (AE, AC),$$

then $\overline{DE} \parallel \overline{BC}$.

Proof: Let $\triangle ABC$ with points D and E such that A–D–B, A–E–C, $(AD, AB) \underset{p}{=} (AE, AC)$, as suggested in Figure 10-6, be given.

Figure 10-6

Let l be the unique line through D and parallel to $\overline{BC}$. Let E' be the point in which l intersects $\overline{AC}$. In the figure, E and E' appear to be different points. We shall show that they are actually the same point. Since l is parallel to $\overleftrightarrow{BC}$, it follows from Theorem 10.6 that

$$(AD, AB) \underset{p}{=} (AE', AC).$$

But it is given that

$$(AD, AB) \underset{p}{=} (AE, AC).$$

Therefore

$$(AE', AC) \underset{p}{=} (AE, AC). \qquad \text{Why?}$$

Then

$$AE' \cdot AC = AE \cdot AC \qquad \text{(Why?)}$$

and $AE' = AE$. Since E and E' are points of $\overleftrightarrow{AC}$ that are on the same side of A and at the same distance from A, it follows that $E = E'$. Since

$$l = \overleftrightarrow{DE'} = \overleftrightarrow{DE},$$

it follows that $\overline{DE} \parallel \overline{BC}$.

The word "cut" is used frequently as a synonym for "intersect," particularly in situations involving several lines and a transversal, as in our next theorem.

THEOREM 10.8 If two distinct transversals cut three or more distinct lines that are coplanar and parallel, then the lengths of the segments formed on one transversal are proportional to the lengths of the segments formed on the other transversal.

Proof: We shall prove the theorem for three parallel lines. It is easy to modify the proof for more than three parallel lines.

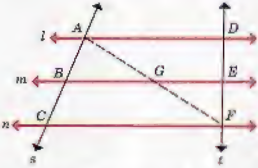


Figure 10-7

Let s and t be two distinct transversals of three distinct lines l, m, n that are coplanar and parallel as suggested in Figure 10-7. Label the points of intersection as in the figure. Draw $\overline{AF}$. Let G be the point of intersection of m with $\overline{AF}$. It follows from the Triangle Proportionality Theorem, applied to $\triangle ACF$, that

(1) $$(AB, AC) \underset{p}{=} (AG, AF)$$

and from the same theorem, applied to $\triangle AFD$, that

(2) $$(GA, FA) \underset{p}{=} (ED, FD).$$

From the Transitive Property of Proportionality it follows that

$$(3) \qquad (AB, AC) \underset{p}{=} (DE, DF).$$

Similarly, we may show that

$$(4) \qquad (BC, AC) \underset{p}{=} (EF, DF).$$

Then from (3) and (4) we get

$$(5) \qquad (AB, BC, AC) \underset{p}{=} (DE, EF, DF).$$

COROLLARY 10.8.1 If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side.

Proof: Assigned as an exercise.

EXERCISES 10.5

In Exercises 1–5, there is a triangle, $\triangle ABC$, with points D and E such that A-D-B and A-E-C, $\overline{DE} \parallel \overline{BC}$, and with lengths denoted as follows: $p = AD$, $q = DB$, $r = AE$, $s = EC$. In each case, given three of the four numbers p, q, r, s, find the missing one. Draw and label a figure for each of these exercises.

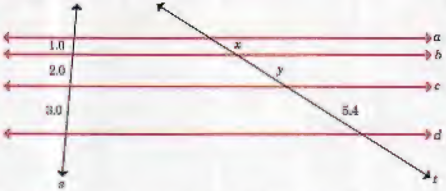
1. $p = 5, \quad q = 6, \quad r = 7.5, \quad s = \boxed{?}$
2. $p = 8, \quad q = 6, \quad r = \boxed{?}, \quad s = 9$
3. $p = 18, \quad q = \boxed{?}, \quad r = 10, \quad s = 13\frac{1}{3}$
4. $p = \boxed{?}, \quad q = 6\frac{2}{3}, \quad r = 7, \quad s = 20$
5. $p = \sqrt{2}, \quad q = \sqrt{3}, \quad r = 4 \quad s = \boxed{?}$

In Exercises 6–10, there is a triangle, $\triangle ABC$, with points D and E such that A-D-B and A-E-C, and with lengths denoted as follows: $x = AB$, $y = AC$, $p = AD$, $q = DB$, $r = AE$, $s = EC$. In each case, given some of these lengths, determine whether or not the lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel. Draw a figure for each exercise.

6. $x = 10, \quad y = 15, \quad p = 5, \quad r = 8$
7. $x = 10, \quad y = 15, \quad p = 5, \quad r = 7.5$
8. $p = 25, \quad q = 15, \quad r = 60, \quad s = 36$
9. $p = 0.9, \quad s = 0.88, \quad r = 0.81, \quad q = 0.81$
10. $p = \sqrt{2}, \quad x = 2, \quad r = 2, \quad y = 4$

11. Write a "reason" for each of steps 1 through 7 in the proof of Theorem 10.6.

12. Coplanar and parallel lines a, b, c, d are cut by transversals s and t as suggested in the figure. Given lengths of segments as labeled in the figure, find x and y.



13. If three distinct coplanar and parallel lines are cut by two distinct parallel transversals, then the lengths of the segments formed on one transversal are proportional to the lengths of the segments formed on the other transversal. What is the constant of proportionality in this case?
14. In the proof of Theorem 10.8, Figure 10-7 suggests that s and t do not intersect in the portion of the plane between lines l and n. Draw a figure for this theorem which shows s and t intersecting at a point between lines l and n. Is the proof for this theorem, as given, applicable for the case suggested by your figure?
15. Consider again Figure 10-7. Suppose that the figure is modified to show s and t intersecting at a point P on the opposite side of line l from n. Using Theorem 10.6 and Figure 10-7 suitably modified and labeled, obtain some proportionalities involving lengths of segments with P as one endpoint. Use these proportionalities to prove that

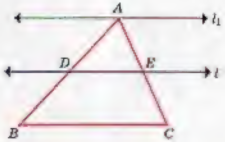
$$(AB, AC) \underset{p}{=} (DE, DF).$$

16. Complete the following proof of Corollary 10.8.1. Let $\triangle ABC$ be given with D the midpoint of $\overline{AB}$ and let l be the line through D, parallel to $\overleftrightarrow{BC}$, and intersecting $\overline{AC}$ at E as shown in the figure. Let l_1 be the line through A and parallel to l. Then $l_1 \parallel \overleftrightarrow{BC}$. Why? Then

$$(AD, DB) \underset{p}{=} (AE, EC). \qquad \text{Why?}$$

It follows that $\quad \dfrac{AD}{DB} = \dfrac{AE}{EC} = 1. \qquad$ Why?

Complete the proof by showing that l bisects $\overline{AC}$.



17. (*An informal geometry exercise.*) With the aid of a ruler and a pair of compasses, draw two triangles, $\triangle ABC$ and $\triangle A'B'C'$, so that the length of the sides (in centimeters) are as listed in the table. Using a protractor, measure the angles A, B, C, A', B', C' to the nearest degree. Copy the table and complete it by recording the angle measures.

$\triangle ABC$	$\triangle A'B'C'$
$AB = 9.0$	$A'B' = 13.5$
$BC = 7.0$	$B'C' = 10.5$
$AC = 6.0$	$A'C' = 9.0$
$m\angle A = \boxed{?}$	$m\angle A' = \boxed{?}$
$m\angle B = \boxed{?}$	$m\angle B' = \boxed{?}$
$m\angle C = \boxed{?}$	$m\angle C' = \boxed{?}$

18. (*An informal geometry exercise.*) With the aid of a ruler and protractor, draw two triangles, $\triangle ABC$ and $\triangle A'B'C'$, with side lengths (in centimeters) and angle measures as indicated in the table. Measure the remaining parts of the two triangles and record the results. Measure lengths to the nearest 0.1 cm. and angles to the nearest degree.

$\triangle ABC$	$\triangle A'B'C'$
$AB = 8.0$	$A'B' = 10.0$
$BC = 10.0$	$B'C' = 12.5$
$AC = \boxed{?}$	$A'C' = \boxed{?}$
$m\angle A = \boxed{?}$	$m\angle A' = \boxed{?}$
$m\angle B = 46$	$m\angle B' = 46$
$m\angle C = \boxed{?}$	$m\angle C' = \boxed{?}$

19. (*An informal geometry exercise.*) With the aid of a ruler and a protractor, draw two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with side lengths (in centimeters) and angle measures as indicated in the table. Measure the remaining parts of the two triangles and record the results. Measure lengths to the nearest 0.1 cm. and angles to the nearest degree.

$\triangle ABC$	$\triangle A'B'C'$
$AB = 10$	$A'B' = 6$
$BC = \boxed{?}$	$B'C' = \boxed{?}$
$AC = \boxed{?}$	$A'C' = \boxed{?}$
$m\angle A = 32$	$m\angle A' = 32$
$m\angle B = 51$	$m\angle B' = 51$
$m\angle C = \boxed{?}$	$m\angle C' = \boxed{?}$

10.6 TRIANGLE SIMILARITY THEOREMS

In our study of congruence for triangles we first defined congruence so that, by definition, all six parts of one triangle must be congruent to the corresponding parts of a second triangle in order for the triangles to be congruent to each other. On the basis of our experiences with triangles it seemed reasonable to expect all six of the required congruences involving sides and angles to be satisfied if certain sets of three of them are satisfied. So we adopted the well-known Triangle Congruence Postulates, referred to as S.A.S., A.S.A., and S.S.S. Similarly, our experiences with triangles, especially the triangles of Exercises 17, 18, 19 of Exercises 10.5, suggest that, if certain combinations of some of the definitional requirements for a triangle similarity are verified, then all of the requirements for a similarity are satisfied. Since we adopted Congruence Postulates, it would seem reasonable to adopt Similarity Postulates. It turns out, however, that it is not difficult to prove what we want to know about similarity; hence in this instance postulates are not necessary. First, we prove a theorem that is useful in proving the main Similarity Theorems.

THEOREM 10.9 If $\triangle ABC$ is any triangle and k is any positive number, then there is a triangle $\triangle A'B'C'$ such that $\triangle A'B'C' \sim \triangle ABC$ with constant of proportionality k.

Proof: Let triangle $\triangle ABC$ and a positive number k be given. We consider three cases.

Case 1. $k < 1$.
Case 2. $k = 1$.
Case 3. $k > 1$.

We shall prove the theorem for Case 1 and assign the other two Cases in the Exercises.

Suppose that $k < 1$; then there is a point D on $\overline{AB}$ such that $AD = k \cdot AB$ and a point E on $\overline{AC}$ such that $AE = k \cdot AC$. In Figure 10-8, $\overline{DE}$ is drawn so that k appears to be about 0.6.

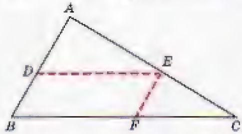


Figure 10-8

It follows from the converse of the Triangle Proportionality Theorem that $\overline{DE} \parallel \overline{BC}$. It follows from theorems regarding parallel lines that

$$\angle ADE \cong \angle B \qquad \text{and} \qquad \angle AED \cong \angle C.$$

Of course, $\angle A \cong \angle A$. As you might expect, it is $\triangle ADE$ that qualifies as a suitable $\triangle A'B'C'$, that is, $ADE \longleftrightarrow ABC$ is a similarity. So far we have shown that corresponding angles are congruent and, from the way we have chosen D and E, we know that

$$(AD, AE) \underset{p}{=} (AB, AC).$$

We need to show that

$$(AD, AE, DE) \underset{p}{=} (AB, AC, BC).$$

Let F be the point of $\overline{BC}$ in which the line through E and parallel to $\overleftrightarrow{AB}$ intersects $\overline{BC}$. Then $DE = BF$ (Why?) and

$$(AE, BF) \underset{p}{=} (AC, BC). \qquad \text{Why?}$$

Substituting, we get

$$(AE, DE) \underset{p}{=} (AC, BC).$$

From this proportion and the preceding proportion,

$$(AD, AE) \underset{p}{=} (AB, AC),$$

it follows that

$$(AD, DE, AE) \underset{p}{=} (AB, BC, AC),$$

which completes the proof for Case 1 in which $k < 1$.

THEOREM 10.10 ***(S.S.S. Similarity Theorem)*** Given $\triangle ABC$ and $\triangle DEF$, if

$$(AB, BC, CA) \underset{p}{=} (DE, EF, FD),$$

then $\triangle ABC \sim \triangle DEF$.

Proof: Let $\triangle ABC$ and $\triangle DEF$ such that

$$(AB, BC, CA) \underset{p}{=} (DE, EF, FD)$$

be given. (See Figure 10-9.) Suppose the constant of proportionality is k. From Theorem 10.9 it follows that there is a triangle $\triangle D'E'F'$ such that $\triangle D'E'F' \sim \triangle DEF$ with proportionality constant k. Then

$(AB, BC, CA) \underset{p}{=} (DE, EF, FD)$	with proportionality constant k,
$(D'E', E'F', F'D') \underset{p}{=} (DE, EF, FD)$	with proportionality constant k,
$(AB, BC, CA) \underset{p}{=} (D'E', E'F', F'D')$	with proportionality constant 1. Why?

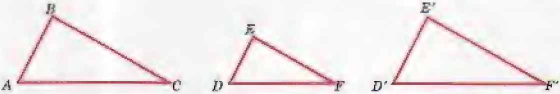


Figure 10-9

From this we conclude that $AB = D'E'$, $BC = E'F'$, and $CA = F'D'$. It follows from the S.S.S. Congruence Postulate that

$$\triangle ABC \cong \triangle D'E'F'.$$

Recall now that triangle congruence is a special case of triangle similarity and that triangle similarity is an equivalence relation. Therefore $\triangle ABC \sim \triangle D'E'F'$. But $\triangle D'E'F' \sim \triangle DEF$. It follows that $\triangle ABC \sim \triangle DEF$.

THEOREM 10.11 (*S.A.S. Similarity Theorem*) Given $\triangle ABC$ and $\triangle DEF$, if

$$\angle A \cong \angle D \qquad \text{and} \qquad (AB, AC) \underset{p}{=} (DE, DF),$$

then $\triangle ABC \sim \triangle DEF$.

Proof: Let $\triangle ABC$ and $\triangle DEF$ be given with $\angle A \cong \angle D$ and

$$(AB, AC) \underset{p}{=} (DE, DF).$$

(Use Figure 10-9 again.) Suppose the constant of proportionality is k. Let $\triangle D'E'F'$ be a triangle such that $\triangle D'E'F' \sim \triangle DEF$ with proportionality constant k. Then

$$AB = k \cdot DE, \qquad D'E' = k \cdot DE, \qquad AB = D'E',$$

and

$$AC = k \cdot DF, \qquad D'F' = k \cdot DF, \qquad AC = D'F'.$$

It follows from the S.A.S. Congruence Postulate that

$$\triangle ABC \cong \triangle D'E'F'.$$

Then $\triangle ABC \sim \triangle D'E'F'$ and $\triangle D'E'F' \sim \triangle DEF$, and we may conclude that $\triangle ABC \sim \triangle DEF$.

COROLLARY 10.11.1 A segment which joins the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.

Proof: Assigned as an exercise.

THEOREM 10.12 **(*A.A. Similarity Theorem*)** Given $\triangle ABC$ and $\triangle DEF$, if $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$.

Proof: Let $\triangle ABC$ and $\triangle DEF$ such that $\angle A \cong \angle D$ and $\angle B \cong \angle E$ be given. (Use Figure 10-9 once more.) Let

$$\frac{AB}{DE} = k.$$

Let $\triangle D'E'F'$ be a triangle such that $\triangle D'E'F' \sim \triangle DEF$ with proportionality constant k. Then

$$\begin{array}{llll} \angle A \cong \angle D, & \angle D \cong \angle D', & \text{and} & \angle A \cong \angle D'; \\ \angle B \cong \angle E, & \angle E \cong \angle E', & \text{and} & \angle B \cong \angle E'; \\ AB = k \cdot DE, & D'E' = k \cdot DE, & \text{and} & AB = D'E'. \end{array}$$

It follows from the A.S.A. Congruence Postulate that

$$\triangle ABC \cong \triangle D'E'F'.$$

Then $\triangle ABC \sim \triangle D'E'F'$ and $\triangle D'E'F' \sim \triangle DEF$, and we conclude that $\triangle ABC \sim \triangle DEF$.

Note that we have an S.S.S. Congruence Postulate and an S.S.S. Similarity Theorem, and that we have an S.A.S. Congruence Postulate and an S.A.S. Similarity Theorem, but that we do *not* have an A.S.A. Similarity Theorem to match our A.S.A. Congruence Postulate. Of course, we could, if we wished, call our A.A. Similarity Theorem the A.S.A. Similarity Theorem. But if $\angle A \cong \angle D$ and $\angle B \cong \angle E$, we do not need to be concerned about whether "AB is proportional to DE." Indeed, if AB and DE are any two positive numbers whatsoever, there is a number k such that

$$AB = k \cdot DE.$$

Look at the tables you prepared for Exercises 17, 18, 19 of Exercises 10.5. Do the measurement data recorded in the tables illustrate the triangle Similarity Theorems? They should. Which theorem does Exercise 17 illustrate? Which theorem does Exercise 18 illustrate? Which theorem does Exercise 19 illustrate?

We have written the triangle Similarity Theorems using quite a few symbols. Is it possible to state them in a more relaxed form without symbols? In the following versions of the theorems we use the word "corresponding" without "pinning it down." It should be understood in each case that a correspondence between the vertices of one triangle and the vertices of the other triangle is fixed so that there are corresponding parts.

THEOREM 10.10 (*S.S.S. Similarity Theorem—Alternate Form*) If the lengths of the sides of one triangle are proportional to the lengths of the corresponding sides of the other triangle, then the triangles are similar.

THEOREM 10.11 (*S.A.S. Similarity Theorem—Alternate Form*) If an angle of one triangle is congruent to an angle of another triangle and if the lengths of the including sides are proportional to the lengths of the corresponding sides in the other triangle, then the triangles are similar.

THEOREM 10.12 (*A.A. Similarity Theorem—Alternate Form*) If two angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar.

The following theorem points out that if the sides of one triangle are parallel to the sides of a second triangle, then the two triangles are similar. The property of parallel sides is a sufficient condition to ensure similarity, but, of course, it is not a necessary condition.

THEOREM 10.13 If triangles $\triangle ABC$ and $\triangle DEF$ are such that $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CA} \parallel \overline{FD}$, then $\triangle ABC \sim \triangle DEF$.

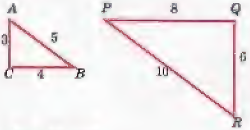
Proof: Assigned as an exercise.

EXERCISES 10.6

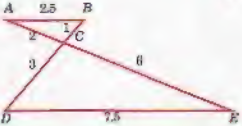
1. Prove Theorem 10.9 for the case in which $k = 1$.
2. Prove Theorem 10.9 for the case in which $k > 1$.

In Exercises 3 and 4, two triangles and the lengths of their sides are given by means of a labeled figure.

3. Is $\triangle ABC \sim \triangle PQR$?
   Is $\triangle ABC \sim \triangle QPR$?
   Is $\triangle ABC \sim \triangle PRQ$?
   Is $\triangle ABC \sim \triangle RPQ$?
   Is $\triangle ACB \sim \triangle RQP$?
   Is $\triangle CBA \sim \triangle QPR$?

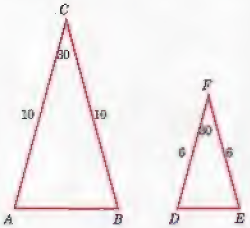


4. Is $\triangle ABC \sim \triangle CDE$?
   Is $\triangle ABC \sim \triangle DEC$?
   Is $\triangle ABC \sim \triangle EDC$?
   Is $\triangle CAB \sim \triangle DEC$?
   Is $\triangle CBA \sim \triangle DCE$?
   Is $\triangle BAC \sim \triangle DEC$?

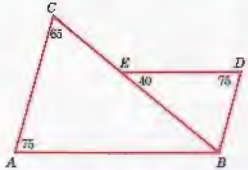


In Exercises 5 and 6, two triangles are given in a figure with some segment lengths and angle measures.

5. Is $\triangle ABC \sim \triangle DEF$?
Is $\triangle ABC \sim \triangle EFD$?
Is $\triangle ABC \sim \triangle DFE$?
Is $\triangle ABC \sim \triangle EDF$?
Is $\triangle ABC \sim \triangle FED$?
Is $\triangle ABC \sim \triangle FDE$?



6. Is $\triangle ABC \sim \triangle DEB$?
Is $\triangle ABC \sim \triangle DBE$?
Is $\triangle ABC \sim \triangle BED$?
Is $\triangle ABC \sim \triangle BDE$?
Is $\triangle ABC \sim \triangle EBD$?
Is $\triangle ABC \sim \triangle EDB$?

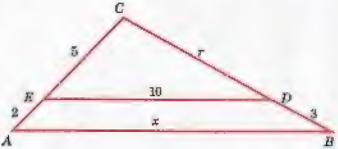


7. Given isosceles $\triangle ABC$ with $AB = AC$ and with points D, E, F such that A–D–B, B–E–C, C–F–A, $\overline{DE} \perp \overline{AB}$, $\overline{FE} \perp \overline{AC}$, prove that $\triangle BDE \sim \triangle CFE$.

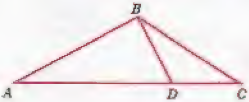
8. If at a certain time, in a certain place, a certain tree casts a shadow 40 ft. long and a 6-ft. man casts a shadow 2 ft. and 3 in. long, find the height of the tree.

Exercises 9–11 refer to the figure with A–E–C, B–D–C, $\overline{ED} \parallel \overline{AB}$, and segment lengths as marked.

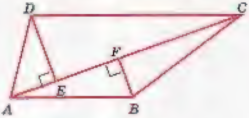
9. Name a pair of similar triangles and explain why they are similar.
10. Find r.
11. Find x.



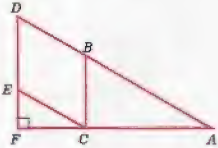
12. In the figure, A–D–C and $\angle ABC \cong \angle BDC$. Name a pair of similar triangles and explain why they are similar.



13. In the figure, $\overline{AB} \parallel \overline{CD}$, $\overline{DE} \perp \overline{AC}$, $\overline{BF} \perp \overline{AC}$, A–E–F, and E–F–C. Name a pair of similar triangles and explain why they are similar.

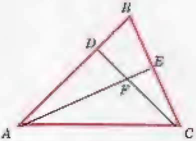


Exercises 14–20 refer to the figure with $\overline{BC} \parallel \overline{DE}$, $\overline{DB} \parallel \overline{EC}$, A–B–D, D–E–F, A–C–F, and $\overline{AF} \perp \overline{FD}$.

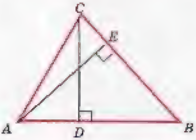


14. Prove $\triangle AFD \sim \triangle CFE$.
15. Prove $\triangle CFE \sim \triangle ACB$.
16. Prove $\triangle AFD \sim \triangle ACB$.
17. If $CF = 2$, $BD = 3$, and $AC = 8$, find AB.
18. If $AB = 12$, $BD = 3$, and $BC = 8$, find DF.
19. If $AD = 18$, $AF = 9$, and $AC = 7$, find EC.
20. If $CF = 3$, $EF = 4$, and $AC = 7$, find DF.

21. In the figure, $\overline{AE}$ and $\overline{CD}$ are altitudes of $\triangle ABC$, A–F–E, and C–F–D.
(a) Prove $\triangle BEA \sim \triangle BDC$.
(b) Prove $\triangle ADF \sim \triangle CEF$.

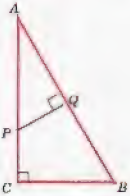


22. Given $\triangle ABC$ with points D and E such that A–D–B, B–E–C, $\overline{AE} \perp \overline{BC}$, $\overline{CD} \perp \overline{AB}$, prove that $(AE, EB) \underset{p}{=} (CD, DB)$.



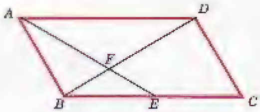
23. Given two right triangles, $\triangle ABC$ and $\triangle APQ$, as in the figure, copy and complete the following proportionality involving lengths of the sides of these triangles:

$$(AB, BC, CA) \underset{p}{=} (AP, \boxed{?}, \boxed{?}).$$



24. Given $\triangle ABC \sim \triangle DEF$, $AB = 5$, $BC = 7$, $AC = 10$, $DE = 7$, find EF and DF.

25. Given $\triangle ABC \sim \triangle PQR$ prove that if $\triangle ABC$ is a right triangle, then $\triangle PQR$ is also a right triangle.
26. Prove Corollary 10.11.1.
27. CHALLENGE PROBLEM. Given parallelogram $ABCD$ with B–E–C, $\overline{AE}$ and $\overline{BD}$ intersecting at F, and $BF = \frac{1}{3} \cdot BD$, prove that $BE = \frac{1}{2} \cdot BC$.



28. Prove Theorem 10.13. Consider two cases: (a) $\overrightarrow{AB}$ and $\overrightarrow{DE}$ are parallel, $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are parallel, $\overrightarrow{CA}$ and $\overrightarrow{FD}$ are parallel, and (b) $\overrightarrow{AB}$ and $\overrightarrow{DE}$ are antiparallel, $\overrightarrow{BC}$ and $\overrightarrow{EF}$ are antiparallel, $\overrightarrow{CA}$ and $\overrightarrow{FD}$ are antiparallel. Use Theorems 7.26 and 7.28.

10.7 SIMILARITIES IN RIGHT TRIANGLES

Sometimes base and altitude are interpreted as segments and sometimes as numbers (lengths of segments). In our next theorem altitudes are segments. If $\triangle ABC \sim \triangle A'B'C'$, then we have agreed that A and A' are corresponding vertices, $\overline{AB}$ and $\overline{A'B'}$ are corresponding sides, and so on. It is natural to extend this idea to include corresponding altitudes, that is, altitudes from corresponding vertices.

THEOREM 10.14 If two triangles are similar, then the lengths of any two corresponding altitudes are proportional to the lengths of any two corresponding sides.

Proof: Given $\triangle ABC \sim \triangle A'B'C'$, let D and D' be the feet of the altitudes from A to $\overline{BC}$ and from A' to $\overline{B'C'}$, respectively. Let $a = BC$, $b = CA$, $c = AB$, $h = AD$, $a' = B'C'$, $b' = C'A'$, $c' = A'B'$, $h' = A'D'$.

We shall prove the theorem for the case in which B–D–C, as shown in Figure 10-10. The remainder of the proof is assigned in the Exercises.

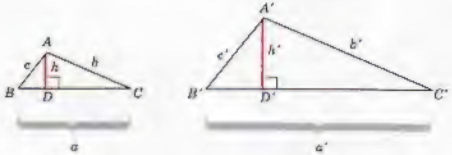


Figure 10-10

Since B–D–C, $\angle B$ and $\angle C$ are acute angles. Why? Then $\angle B'$ and $\angle C'$ are acute angles. Why? Then it is impossible for D' to be either the point B' or the point C'. Why? Also, it is impossible that D'–B'–C' or that B'–C'–D'. If D'–B'–C', then $\triangle B'D'A'$ is a right triangle with an acute exterior angle contrary to the Exterior Angle Theorem. Therefore B'–D'–C' as indicated in Figure 10.10.

Now $\angle B \cong \angle B'$ (Why?) and $\angle BDA \cong \angle B'D'A'$ (Why?). It follows from the A.A. Similarity Theorem that $\triangle ADB \sim \triangle A'D'B'$. Therefore

$$(c, h) \underset{p}{=} (c', h').$$

But

$$(a, b, c) \underset{p}{=} (a', b', c').$$

Therefore

$$(a, c) \underset{p}{=} (a', c') \qquad \text{and} \qquad (b, c) \underset{p}{=} (b', c').$$

It follows from the Alternation and Inversion Properties of Proportions that

$$(h, h') \underset{p}{=} (c, c'), \qquad (c, c') \underset{p}{=} (a, a'), \qquad \text{and} \qquad (c, c') \underset{p}{=} (b, b').$$

It follows from the Equivalence Properties of Proportionalities that h and h' are proportional to the lengths of any two corresponding sides.

THEOREM 10.15 If two triangles are similar, then their areas are proportional to the squares of the lengths of any two corresponding sides.

Proof: Given $\triangle ABC \sim \triangle A'B'C'$, let D and D' be the feet of the altitudes from A to $\overline{BC}$ and from A' to $\overline{B'C'}$, respectively. Let $a = BC$, $b = CA$, $c = AB$, $h = AD$, $a' = B'C'$, $b' = C'A'$, $c' = A'B'$, $h' = A'D'$. (See Figure 10-11.) It follows from Theorem 10.14 that

$$(h, h') \underset{p}{=} (a, a').$$

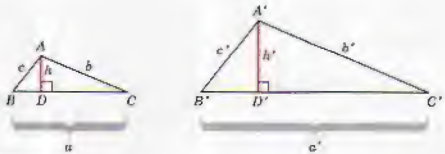


Figure 10-11

Suppose that $h = ka$; then $h' = ka'$ and

$$|\triangle ABC| = \tfrac{1}{2}ah = \tfrac{1}{2}a(ka) = (\tfrac{1}{2}k)a^2,$$
$$|\triangle A'B'C'| = \tfrac{1}{2}a'h' = \tfrac{1}{2}a'(ka') = (\tfrac{1}{2}k)(a')^2.$$

Therefore the areas of $\triangle ABC$ and $\triangle A'B'C'$ are proportional to a^2 and $(a')^2$. Similarly, it may be shown that the areas are proportional to b^2 and $(b')^2$ and to c^2 and $(c')^2$.

THEOREM 10.16 In any right triangle the altitude to the hypotenuse separates the right triangle into two triangles each similar to the original triangle, and hence also to each other.

Proof: Let $\triangle ABC$ be a right triangle with the right angle at C and with D the foot of the altitude to the hypotenuse. Then $D \neq A$ and $D \neq B$. (Which theorem is the basis for this assertion?) Also, it is impossible to have D–A–B or A–B–D. (If either of these betweenness relations is true, there is a triangle with D as one vertex with one interior angle a right angle and one exterior angle an acute angle. Which theorem does this contradict?) Therefore D is an interior point of $\overline{AB}$ as suggested in Figure 10-12.

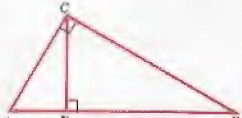


Figure 10-12

You are asked in the Exercises at the end of this section to complete the proof by showing that $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$ are all similar to each other.

Next we have two corollaries that follow easily from Theorem 10.16, but first we need some definitions.

Definition 10.4 If P is a point and l is a line, the projection of P on l is (1) the point P if P is on l and (2) the foot of the perpendicular from P to l if P is not on l.

Definition 10.5 The projection of a set S on a line l is the set of all points Q on l such that each Q is the projection on l of some P in S.

Compare Definitions 10.4 and 10.5 with Definitions 8.9 and 8.10 in which we defined a projection on a plane.

Note in Figure 10-12 that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overleftrightarrow{AB}$. Indeed, A is the projection of A, D is the projection of C, and every point Q such that A–Q–D is the projection of a point P such that A–P–C. Conversely, every point P such that A–P–C has as its projection on $\overleftrightarrow{AB}$ a point Q such that A–Q–D.

Since the projection of $\overline{AC}$ on $\overleftrightarrow{AB}$ is a part of the hypotenuse in the situation of Theorem 10.16, we may say that $\overline{AD}$ is the projection of $\overline{AC}$ on the hypotenuse. Similarly, $\overline{DB}$ is the projection of $\overline{CB}$ on the hypotenuse.

COROLLARY 10.16.1 The square of the length of the altitude to the hypotenuse of a right triangle is equal to the product of the lengths of the projections of the legs on the hypotenuse.

Proof: In Figure 10-12, $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{AB}$, and $\overline{DB}$ is the projection of $\overline{BC}$ on $\overline{AB}$. In the notation of the figure, we must prove that

$$(CD)^2 = AD \cdot DB.$$

Using Theorem 10.16 and some properties of proportionalities, we have

$$\begin{gathered} \triangle ACD \sim \triangle CBD \\ (AC, CD, AD) \underset{p}{=} (CB, BD, CD) \\ (CD, AD) \underset{p}{=} (BD, CD) \\ (CD)^2 = AD \cdot DB \end{gathered}$$

COROLLARY 10.16.2 The square of the length of a leg of a right triangle is equal to the product of the lengths of the hypotenuse and the projection of that leg on the hypotenuse.

Proof: Assigned as an exercise.

Definition 10.6 If a and b are positive numbers such that

$$(a, x) \underset{p}{=} (x, b)$$

or that

$$(x, a) \underset{p}{=} (b, x),$$

then x is called a geometric mean of a and b.

Note that if

$$(a, x) \underset{p}{=} (x, b)$$

or if

$$(x, a) \underset{p}{=} (b, x),$$

then $x^2 = ab$ (Why?) and $x = \sqrt{ab}$ or $x = -\sqrt{ab}$. We often call $\sqrt{ab}$ *the* geometric mean of a and b. In view of Definition 10.6, Corollary 10.16.1 and Corollary 10.16.2 can be restated as follows.

COROLLARY 10.16.1 (***Alternate Form***) The length of the altitude to the hypotenuse of a right triangle is the geometric mean of the lengths of the projections of the legs on the hypotenuse.

COROLLARY 10.16.2 (***Alternate Form***) The length of a leg of a right triangle is the geometric mean of the lengths of the hypotenuse and the projection of that leg on the hypotenuse.

In Chapter 9 we proved the Pythagorean Theorem using properties of areas and suggested two other area proofs in the Exercises. One of the shortest proofs of the Pythagorean Theorem is an algebraic proof that follows easily from Corollary 10.16.2. We state the Pythagorean Theorem again and outline a proof that employs Corollary 10.16.2. We also proved the Converse of the Pythagorean Theorem in Chapter 9. We state the converse again; we shall not prove it again.

THEOREM 10.17 (***The Pythagorean Theorem***) In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

Proof: Let $\triangle ABC$ with a right angle at C be given. (See Figure 10-13.) Let D be the foot of the altitude to the hypotenuse $\overline{AB}$. Let $AB = c$, $BC = a$, $CA = b$, $AD = x$, and $DB = c - x$. Then it follows from Corollary 10.16.2, with a the length of a leg and $c - x$ the length of its projection on the hypotenuse that $a^2 = (c - x)c$, and with b the length of a leg and x the length of its projection on the hypotenuse that $b^2 = xc$. The proof may be completed by showing that $a^2 + b^2 = c^2$.

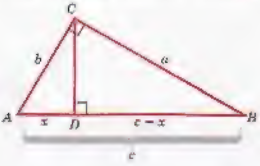


Figure 10-13

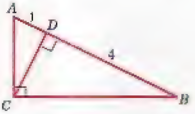
THEOREM 10.18 (***Converse of the Pythagorean Theorem***) If $a^2 + b^2 = c^2$, where a, b, c are the lengths of the sides of a triangle, then the triangle is a right triangle with c the length of the hypotenuse.

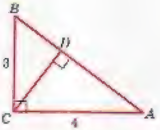
EXERCISES 10.7

1. Given right triangle $\triangle ABC$ with hypotenuse $\overline{AB}$, let D be the foot of the altitude to $\overline{BC}$, E the foot of the altitude to $\overline{AC}$, and F the foot of the altitude to $\overline{AB}$. How many distinct points are there in the set

$$\{A, B, C, D, E, F\}?$$

In Exercises 2–7, there is a figure showing a right triangle with hypotenuse $\overline{AB}$ and with D the foot of the altitude to $\overline{AB}$. In each case, given the lengths of some of the six segments, find the lengths of the other segments. Express your answers in exact form using radicals if necessary.

2.


3.


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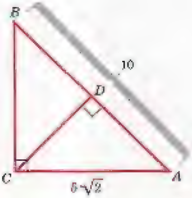

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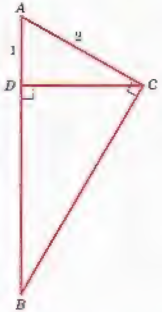

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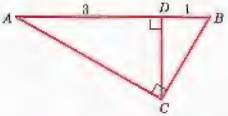
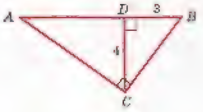


7.


8. Find the perimeter of an equilateral triangle if the length of each of its altitudes is 10.
9. Find the length of the diagonal of a rectangular floor to the nearest foot if the floor is 21 ft. wide and 28 ft. long.
10. A ladder 12 ft. long reaches to a window sill on the side of a house. If the window sill is 9 ft. above the (level) ground, how far is the foot of the ladder from the side of the house?
11. Find the length of the hypotenuse of a right triangle if its legs each have length 1. Express the answer exactly using a radical.
12. Find the length of the hypotenuse of a right triangle if its legs each have length 100.
13. Find the lengths of the legs of an isosceles right triangle whose hypotenuse has length 1.
14. Find the lengths of the legs of an isosceles right triangle whose hypotenuse has length 2.
15. Find the lengths of the legs of an isosceles right triangle if the length of its hypotenuse is $\sqrt{2}$.
16. If one leg and the hypotenuse of a right triangle have lengths 1 and 2, respectively, find the length of the other leg.
17. Find the length of a leg of a right triangle if the other leg and the hypotenuse have lengths 1 and $\sqrt{3}$, respectively.
18. Find the length of the leg of a right triangle if the other leg and the hypotenuse have lengths 100 and $100\sqrt{3}$, respectively.
19. Given $\triangle ABC$ with $m\angle C = 90$ and with D the midpoint of $\overline{AB}$ and E the midpoint of $\overline{BC}$, prove that $\triangle CED \cong \triangle BED$.
20. For $\triangle ABC$, $m\angle C = 90$ and D is the midpoint of $\overline{AB}$. If $AC = \sqrt{7}$ and $BC = 3$, find CD.
21. Complete the proof of Theorem 10.16 by showing that

$$ABC \longleftrightarrow ACD$$

and

$$ABC \longleftrightarrow CBD$$

are similarities. (See Figure 10-12.) It will then follow from the equivalence properties of similarity for triangles that $ACD \longleftrightarrow CBD$ is also a similarity.
22. Prove Corollary 10.16.2 for the leg $\overline{AC}$ in Figure 10-12.
23. Prove Corollary 10.16.2 for the leg $\overline{BC}$ in Figure 10-12.
24. See the proof of Theorem 10.17. Show that $a^2 + b^2 = c^2$. (See Figure 10-13.)
25. If $\triangle ABC \sim \triangle DEF$ and $5 \cdot AB = 3 \cdot DE$, what is the ratio of the length of an altitude of the smaller triangle to the length of the corresponding altitude of the larger triangle? Which theorem justifies your answer?
26. In Exercise 25, what is the ratio of the area of the smaller triangle to the area of the larger triangle? Which theorem justifies your answer?

27. If $16 \cdot |\triangle PQR| = 25 \cdot |\triangle ABC|$ and if $\triangle PQR \sim \triangle ABC$, what is the ratio of PR to AC? Which theorem justifies your answer?
28. Prove Theorem 10.14 for the case in which $D = B$ or $D = C$.
29. Prove Theorem 10.14 for the case in which $D\text{-}B\text{-}C$ (the proof for the case in which $B\text{-}C\text{-}D$ is similar to the proof for the case in which $D\text{-}B\text{-}C$).

10.8 SOME RIGHT TRIANGLE THEOREMS

Following are some theorems regarding right triangles. Although they are not profound, they are useful theorems that every mathematics student who has studied formal geometry ought to know. These theorems should not surprise you. If you worked the exercises in Exercises 10.7, you will recognize them as "old stuff."

THEOREM 10.19 The median to the hypotenuse of a right triangle is one-half as long as the hypotenuse.

Proof: Let $\triangle ABC$ be a right triangle with D the midpoint of the hypotenuse. (See Figure 10-14.) We want to prove that $CD = \frac{1}{2} \cdot AB$, or equivalently, that $CD = DB$. Let point E be the midpoint of $\overline{CB}$. Then

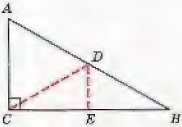


Figure 10-14

$$(BD, BE) \underset{p}{=} (BA, BC) \qquad \text{(Why?)} \qquad \text{and} \qquad \angle B \cong \angle B.$$

It follows from the S.A.S. Similarity Theorem that $\triangle ABC \sim \triangle DBE$. Then

$$m\angle BED = m\angle BCA = 90 = m\angle CED.$$

It follows from the S.A.S. Congruence Postulate that $\triangle CDE \cong \triangle BDE$ and therefore $CD = DB$.

There are some special right triangles that are referred to in special ways. First we mention the 3, 4, 5 triangles. A triangle whose sides have lengths 3, 4, 5 is a right triangle. We know this since $3^2 + 4^2 = 5^2$. (Are we using the Pythagorean Theorem when we make this conclusion, or are we using its converse?)

Given a distance function, there are infinitely many 3, 4, 5 right triangles. Indeed, if A is any point in space (infinitely many choices here) and if B is any point such that $AB = 5$ (infinitely many choices here), there are infinitely many possible points C so that $\triangle ABC$ is a right triangle with $AB = 5$, $BC = 3$, and $CA = 4$, and infinitely many

possible points C such that $AB = 5$, $BC = 4$, and $CA = 3$. But there are many, many more, not included among these, that are also frequently referred to as 3, 4, 5 right triangles as our next theorem suggests.

THEOREM 10.20 (***The 3, 4, 5 Theorem***) If x is any positive number, then every triangle with side lengths $3x$, $4x$, $5x$ is a right triangle.

Proof: Let $\triangle ABC$ be a triangle with $BC = 3$, $CA = 4$, $AB = 5$. Let x be any positive number. Let $\triangle A'B'C'$ be any triangle with $B'C' = 3x$, $C'A' = 4x$, $A'B' = 5x$. Then $\triangle A'B'C' \sim \triangle ABC$. (Which triangle Similarity Theorem do we use in making this deduction?) Since $\triangle ABC$ is a right triangle, it follows that $\triangle A'B'C'$ is a right triangle, and this completes the proof.

A triangle is called a **3, 4, 5 triangle** if its sides are of lengths 3, 4, 5 or if the lengths of its sides are proportional to 3, 4, 5. All 3, 4, 5 triangles are right triangles.

THEOREM 10.21 (***The 5, 12, 13 Theorem***) If x is a positive number and if the lengths of the sides of a triangle are $5x$, $12x$, and $13x$, then the triangle is a right triangle.

Proof: Assigned as an exercise.

A triangle is called a **5, 12, 13 triangle** if the lengths of its sides are proportional to 5, 12, 13.

THEOREM 10.22 (***The 1, 1, $\sqrt{2}$ Theorem***) If the lengths of the sides of a triangle are proportional to 1, 1, $\sqrt{2}$, then the triangle is an isosceles right triangle.

Proof: Let the lengths of the sides of a triangle be a, b, c and suppose

$$(a, b, c) \underset{p}{=} (1, 1, \sqrt{2}).$$

Then there is a positive number k such that $a = k \cdot 1$, $b = k \cdot 1$, and $c = k \cdot \sqrt{2}$. Then

$$a = b,$$
$$a^2 + b^2 = k^2 + k^2 = 2k^2,$$
$$c^2 = (k\sqrt{2})^2 = 2k^2,$$
$$a^2 + b^2 = c^2.$$

Therefore the triangle is an isosceles right triangle.

A triangle is called a **1, 1, $\sqrt{2}$ triangle,** or a **45, 45, 90 triangle,** if the lengths of its sides are proportional to 1, 1, $\sqrt{2}$.

THEOREM 10.23 (*The 1, $\sqrt{3}$, 2 Theorem*) If the lengths of the sides of a triangle are proportional to 1, $\sqrt{3}$, 2, then it is a right triangle with its shortest side half as long as its hypotenuse.

Proof: Assigned as an exercise.

A triangle is called a **1, $\sqrt{3}$, 2 triangle** if the lengths of its sides are proportional to 1, $\sqrt{3}$, 2.

A triangle is called a **30, 60, 90 triangle** if the measures of its acute angles are 30 and 60.

THEOREM 10.24 A triangle is a 30, 60, 90 triangle if and only if it is a 1, $\sqrt{3}$, 2 triangle with the shortest side opposite the 30 degree angle.

Proof: Let $\triangle ABC$ be a 1, $\sqrt{3}$, 2 triangle and k a positive number such that $AC = k$, $BC = \sqrt{3}k$, and $AB = 2k$. (See Figure 10-15.)

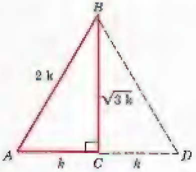


Figure 10-15

Let D be the point on *opp* $\overrightarrow{CA}$ such that $CD = k$. Then

1. $\triangle ABC \cong \triangle DBC$	1. Why?
2. $AB = DB = DA$	2. Why?
3. $m\angle ABD + m\angle BDA + m\angle DAB = 180$	3. Why?
4. $m\angle ABD = m\angle BDA = m\angle DAB = 60$	4. Why?
5. $m\angle ABC = m\angle DBC$	5. Why?
6. $m\angle ABC + m\angle DBC = 60$	6. Why?
7. $m\angle ABC = 30$	7. Why?
8. $m\angle BAC = 60$	8. Why?
9. $m\angle ACB = 90$	9. Why?

Since $\angle ABC$ is opposite the shortest side of $\triangle ABC$, this completes the "if" part of the proof.

Suppose next that $\triangle ABC$ is a 30, 60, 90 triangle. (See Figure 10-16.) Let D be the unique point on *opp* $\overrightarrow{CB}$ such that $CB = CD$.

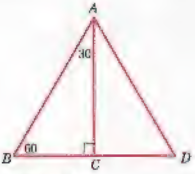


Figure 10-16

Draw $\overline{DA}$. Then

10. $\triangle ABC \cong \triangle ADC$	10. Why?
11. $m\angle CAB = m\angle CAD = 30$	11. Why?
12. $m\angle BAD = m\angle ADB = m\angle DBA = 60$	12. Why?
13. $BA = AD = DB$	13. Why?
14. $BC = CD$	14. Why?
15. $BC + CD = BD$	15. Why?
16. $2BC = AB$	16. Why?

Let $BC = k$. Then

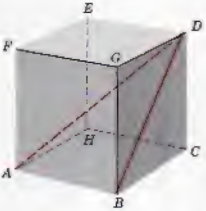
17. $AB = 2k$	17. Why?
18. $(AC)^2 + (BC)^2 = (AB)^2$	18. Why?
19. $(AC)^2 + k^2 = 4k^2$	19. Why?
20. $(AC)^2 = 3k^2$	20. Why?
21. $AC = \sqrt{3}k$	21. Why?
22. $(BC, CA, AB) \underset{p}{=} (1, \sqrt{3}, 2)$	22. Why?

This shows that $\triangle ABC$ is a 1, $\sqrt{3}$, 2 triangle and hence the "only if" part of the proof is completed.

Note that in some of these names for special triangles the numbers are side lengths (or numbers proportional to them), whereas in others they are angle measures. There should be no confusion in regard to the 30, 60, 90 name and the 45, 45, 90 name. Because 30, 60, 90 are not the lengths of the sides of any triangle, and 45, 45, 90 are not the lengths of the sides of any triangle. Which postulate justifies this statement?

EXERCISES 10.8

1. If $\triangle ABC$ is a right triangle with $m\angle C = 90$, $AC = 60$, $BC = 80$, and with D the midpoint of $\overline{AB}$, find CD.
2. (See Figure 10-14.) $\angle ACB$ and $\angle DEB$ are congruent angles in the situation represented by this figure. Find several other pairs of congruent angles. (Six more pairs would be rather good.)
3. In the proof of Theorem 10.19 we asserted that $\triangle CDE \cong \triangle BDE$. Write a two-column proof for this deduction.
4. In a book on the history of mathematics find something about the rope stretchers in ancient Egypt. Explain the connection between rope stretchers and right triangles.
5. A baseball diamond is a square whose sides are 90 ft. long. What is the distance (to the nearest foot) between first and third bases?
6. The figure represents a cube whose six faces are 1 in. by 1 in. squares. Using the Pythagorean Theorem twice, once on $\triangle BCD$ and once on $\triangle ABD$, find AD. Express the answer exactly using a radical if necessary. (Why is $\angle ABD$ a right angle?)



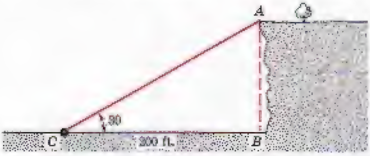
7. A room in the shape of a rectangular box is 15 ft. wide, 18 ft. long, and 8 ft. high. Find the distance to the nearest foot between one corner of the floor and a diagonally opposite corner of the ceiling.

In Exercises 8–16, the lengths of the hypotenuse and one leg of a right triangle are given. In each exercise, the triangle is a 1, 1, $\sqrt{2}$ triangle, or a 1, $\sqrt{3}$, 2 triangle, or a 3, 4, 5 triangle, or a 5, 12, 13 triangle. Determine which one.

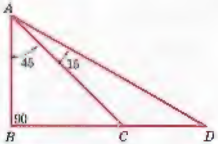
8. 100, 50
9. 100, 60
10. 100, $50\sqrt{2}$
11. 100, 80
12. 100, $50\sqrt{3}$
13. 100, $92\frac{4}{13}$
14. $\sqrt{2}$, $\frac{1}{2}\sqrt{6}$
15. 145, 116
16. 65, 25

17. Let $\triangle ABC$ be a right triangle with the right angle at C. The midpoint of $\overline{AB}$ is the center of a circle which lies in the plane of the triangle and which contains the points A and B. Does the point C lie inside of the circle, on the circle, or outside of the circle? Why?

18. The figure suggests a point A on a high bluff above a level plane. If the angle of elevation of the point A from the point C is a 30 degree angle and if it is 200 ft. from C to B, what is the height BA to the nearest 10 ft.? Assume that $\angle CBA$ is a right angle.



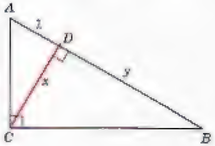
19. The figure represents an observer A in an airplane 5000 ft. directly above a point B on the ground. If B, C, D are three collinear points on the ground and if $m\angle ABC = 90$, $m\angle BAC = 45$, $m\angle CAD = 15$, find to the nearest 100 ft. the distance from C to D.



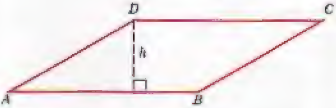
20. Find three positive integers, a, b, c such that $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are the lengths of the sides of a right triangle. How many such triples of positive integers are there?

21. If x and y are any positive integers, distinct or not, show that $\sqrt{x}$, $\sqrt{y}$, $\sqrt{x+y}$ are the lengths of the sides of a right triangle.

22. In the figure is shown a right triangle $\triangle ABC$ with $\overline{CD}$ the altitude to the hypotenuse. If $AD = 1$, $DC = x$, $BD = y$, show that $x = \sqrt{y}$.



23. In the figure below, $ABCD$ is a parallelogram with $AB = 76$, $AD = 50$, $m\angle A = 30$, and h the length of the altitude from D to $\overline{AB}$. Find $|ABCD|$.



24. A parallelogram has adjacent sides of lengths 22 and 14. If the measure of one of its angles is 30, find the area of the parallelogram.
25. Find the area of a rhombus of side length 12 if one of its angles has a measure of 60.
26. The measure of each base angle of an isosceles triangle is 30 and each of the two congruent sides has length 24.
    (a) How long is the base?
    (b) What is the area of the triangle?
27. In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$, $AD = BC = 20$, $CD = 28$, and $m\angle A = m\angle B = 60$. Find $|ABCD|$.

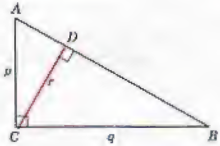
28. Use the figure to complete the proof that in a 30, 60 right triangle the side opposite the 30 degree angle is one-half as long as the hypotenuse.

Proof: In the figure, $m\angle A = 30$, $m\angle B = 60$, and D is the midpoint of $\overline{AB}$. Show that $\triangle BCD$ is equilateral and that

$$BC = CD = \tfrac{1}{2} \cdot AB.$$

29. In a 30, 60 right triangle, the length of the hypotenuse is $8\sqrt{3}$.
    (a) Find the length of the shorter leg.
    (b) Find the length of the longer leg.
    (c) Find the area of the triangle.
30. Prove Theorem 10.21.

31. Prove Theorem 10.23.
32. If u and v are positive integers such that $u > v$, and if A, B, C are points such that $AC = 2uv$, $BC = u^2 - v^2$, $AB = u^2 + v^2$, prove that $\triangle ABC$ is a right triangle. (This exercise also appears in Chapter 9, but it is good for a repeat appearance here.)
33. See Exercise 32. If u and v are relatively prime positive integers (this means that no positive integer except 1 divides both of them), if u and v are not both odd, and if $u > v$, then it can be shown that the three integers, $u^2 - v^2$, $2uv$, $u^2 + v^2$, are relatively prime. If the lengths of the sides of a right triangle are relatively prime positive integers, the triangle is called a primitive Pythagorean triangle and the triple of its side lengths is called a primitive Pythagorean triple. Two examples of primitive Pythagorean triples are (3, 4, 5) and (5, 12, 13). Find five more primitive Pythagorean triples.
34. CHALLENGE PROBLEM. The figure shows a right triangle $\triangle ABC$ with $\overline{CD}$ the altitude to the hypotenuse.



If $AC = p$, $BC = q$, $CD = r$, prove that $\frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{r^2}$.

CHAPTER SUMMARY

The central theme of this chapter is SIMILARITY. The concept of similarity is based on our experiences with objects which have the same shape. The relationship of lengths in one figure to the corresponding lengths in a similar figure suggests the idea of a PROPORTIONALITY. In this chapter we studied the properties of proportionalities and we used them in developing the geometry of similar polygons.

The key theorems of this chapter include THE TRIANGLE PROPORTIONALITY THEOREM, THE CONVERSE OF THE TRIANGLE PROPORTIONALITY THEOREM, THE S.S.S. SIMILARITY THEOREM, THE S.A.S. SIMILARITY THEOREM, THE A.A. SIMILARITY THEOREM, THE PYTHAGOREAN THEOREM, and THE CONVERSE OF THE PYTHAGOREAN THEOREM.

The chapter concludes with a study of special right triangles. A knowledge of these triangles will prove useful as you continue your study of mathematics.

REVIEW EXERCISES

In Exercises 1–10, complete the statement so that it will be a proportionality.

1. $(5, 12) \underset{p}{=} (35, \boxed{?})$
2. $(1, 2, 3, 4, 5) \underset{p}{=} (\boxed{?}, \boxed{?}, \boxed{?}, \boxed{?}, 15)$
3. $(25, 60, 65) \underset{p}{=} (\boxed{?}, 12, \boxed{?})$
4. $(4, 10, 21) \underset{p}{=} (6, \boxed{?}, \boxed{?})$
5. $(4, 10, 21) \underset{p}{=} (\boxed{?}, 6, \boxed{?})$
6. $(4, 10, 21) \underset{p}{=} (\boxed{?}, \boxed{?}, 6)$
7. $(5000, 3000, 1500) \underset{p}{=} (200, \boxed{?}, \boxed{?})$
8. $(100, 400, 500) \underset{p}{=} (\boxed{?}, \boxed{?}, 10)$
9. $(27, 27, 81) \underset{p}{=} (\boxed{?}, \boxed{?}, 3)$
10. $(357, 1309, 833) \underset{p}{=} (\boxed{?}, 11, \boxed{?})$

In Exercises 11–20, determine if the given statement is true or if it is false.

11. If $x = y$, then $(5, x) \underset{p}{=} (5, y)$.
12. If $x \neq y$, then $(5, x) \underset{p}{=} (5, y)$.
13. If $\dfrac{3}{x} = \dfrac{5}{y}$, then $(3, x) \underset{p}{=} (5, y)$.
14. If $\dfrac{3}{5} = \dfrac{x}{y}$, then $(3, 5) \underset{p}{=} (x, y)$.
15. If $x = \dfrac{a+b}{2}$, then $(a, x) \underset{p}{=} (x, b)$.
16. If $x^2 = ab$, then $(a, x) \underset{p}{=} (x, b)$.
17. If $x^2 = ab$, then $(x, a) \underset{p}{=} (b, x)$.
18. If $(a, b) \underset{p}{=} (c, d)$, then $(a, b) \underset{p}{=} (d, c)$.
19. If $(a, b) \underset{p}{=} (c, d)$, then $(a, c) \underset{p}{=} (b, d)$.
20. If $ad = bc$, then $(a, b) \underset{p}{=} (c, d)$.

21. If $\triangle ABC$ is any triangle, then $\triangle ABC \sim \triangle ABC$. Which property of an equivalence relation does this illustrate: the Reflexive Property, the Symmetric Property, or the Transitive Property?
22. If $\triangle ABC \sim \triangle DEF$, then $\triangle DEF \sim \triangle ABC$. Which property of an equivalence relation does this illustrate?
23. If $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHI$, then $\triangle ABC \sim \triangle GHI$. Which property of an equivalence relation does this illustrate?
24. State the Triangle Proportionality Theorem and show how it may be used to prove the theorem regarding the lengths of segments formed by two transversals cutting three or more coplanar and parallel lines.
25. State the three triangle Similarity Theorems.

26. According to Theorem 10.9, if $\triangle ABC$ is any triangle and k is any positive number, then there is a triangle $\triangle DEF$ such that $\triangle DEF \overset{=}{p} \triangle ABC$ with proportionality constant k. Explain how this theorem was used in proving the triangle Similarity Theorems.
27. Explain why there is an A.A. Similarity Theorem but no A.S.A. Similarity Theorem.
28. Prove that the altitude to the hypotenuse of a right triangle determines two triangles that are similar to each other.
29. Using similar triangles, prove the Pythagorean Theorem.
30. If A, B, C, D are points such that A–D–B, $AD = BD = 25$, $AC = 30$, $BC = 40$, find CD.
31. If the lengths of two sides of a right triangle are 10 and 15, find the length of the third side. (Two possibilities.)
32. Find the measures of the angles of a triangle if the lengths of its sides are $\sqrt{2}$, $\sqrt{2}$, 2.
33. Find the measures of the angles of a triangle if the lengths of its sides are $\sqrt{3}$, $2\sqrt{3}$, 3.
34. We know that 30, 60, 90 cannot be the lengths of the sides of a triangle. Which postulate justifies this statement?

In Exercises 35–43, refer to Figure 10-17 in which $\triangle ABC$ is a right triangle with the right angle at C, $\overline{CD}$ is the altitude from C to the hypotenuse, $AD = x$, $DB = y$, $AB = c$, $BC = a$, $AC = b$, and $CD = h$.

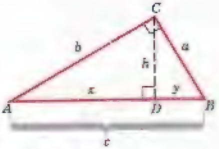


Figure 10-17

35. $\triangle ACB \sim \triangle ADC$ and $\triangle ACB \sim \triangle\boxed{?}$. By the symmetric and $\boxed{?}$ properties of similarity for triangles, $\triangle\boxed{?} \sim \triangle\boxed{?}$.
36. h^2 equals the product of x and $\boxed{?}$ (a, b, or y).
37. b^2 equals the product of x and $\boxed{?}$ (a, c, or y).
38. a^2 equals the product of y and $\boxed{?}$ (b, c, or x).
39. If $x = 16$ and $y = 9$, find h, a, and b.
40. If D is the midpoint of $\overline{AB}$, then $CD = \boxed{?}$ (in terms of c).
41. If $m\angle A = 30$ and $c = 15$, then $a = \boxed{?}$.
42. If $a = 10$ and $c = 20$, then $b = \boxed{?}$ and $m\angle A = \boxed{?}$.
43. If $m\angle A = 45$ and $a = 12$, then $b = \boxed{?}$ and $c = \boxed{?}$.

44. Copy and complete: In a 30, 60 right triangle, the side opposite the 30 degree angle is $\boxed{?}$ (in terms of the hypotenuse).

45. If a boy 5 ft. tall casts a shadow 2 ft. long, how high (to the nearest 10 ft.) is a tree if its shadow is 73 ft. long? What assumptions did you make in working this exercise?

46. Find the distance from C to $\overleftrightarrow{AB}$ if $AC = 10$, $BC = 10\sqrt{3}$, $AB = 20$.

47. If the hypotenuse and a leg of a right triangle have lengths 241 and 220, respectively, find the length of the other leg.

48. CHALLENGE PROBLEM. Given rectangle $ADEH$ and points B and C on $\overline{AD}$ such that $HA = AB = BC = CD = DE = 1$, prove that $m\angle EAD = m\angle EBD = m\angle ECD$.

Chapter 11

Joyce R. Wilson/Photo Researchers

Coordinates in a Plane

11.1 INTRODUCTION

In Chapter 3 we introduced the idea of a coordinate system on a line. Recall that if P and Q are any two distinct points, then there is a unique coordinate system on $\overleftrightarrow{PQ}$ with P as origin and Q as unit point. Thus a coordinate system on a line is *determined* by choosing any two distinct points on the line, one of them the origin and the other the unit point. If, on line l, P is the origin and Q is the unit point, then $\overline{PQ}$ is called the unit segment for the coordinate system on l determined by P and Q. A coordinate system on l is a one-to-one correspondence between the set of all points of l and the set of all real numbers. The numbers associated with the points of l are called coordinates, and they can be used to determine distances (in the system based on $\overline{PQ}$ as the unit segment) between points on l.

In this chapter we introduce the idea of a coordinate system in a *plane*. In a plane, each point is associated with a *pair* of numbers, rather than a single number. After proving some basic theorems concerning a coordinate system in a plane, we develop some equations for a line. We then show how coordinates can be used to provide simpler proofs of some geometric theorems.

11.2 A COORDINATE SYSTEM IN A PLANE

Suppose that a plane is given and, unless we specify otherwise, that all sets of points under consideration are subsets of this plane. Suppose further than a unit segment is given and that all distances are relative to this unit segment unless otherwise indicated.

Let $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ be perpendicular lines in the plane intersecting in the point O as shown in Figure 11-1. Let I and J be points on $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$, respectively, such that

$$OI = OJ = 1.$$

There is a unique coordinate system on $\overleftrightarrow{OX}$ with origin O and unit point I. This is called the x-coordinate system, and the coordinate of a point R of $\overleftrightarrow{OX}$ in this system is called the x-coordinate or abscissa of R.

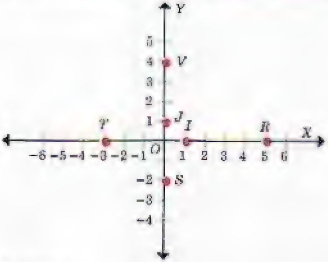


Figure 11-1

In Figure 11-1, the abscissa of R is 5. There is a unique coordinate system on $\overleftrightarrow{OY}$ with origin O and unit point J. This is called the y-coordinate system, and the coordinate of a point S of $\overleftrightarrow{OY}$ in this system is called the y-coordinate or ordinate of S. In Figure 11-1, the ordinate of S is -2. Name the coordinates of T and V in Figure 11-1. Is it necessary to specify the coordinate system in each case? Why?

The line $\overleftrightarrow{OX}$ is called the x-axis and $\overleftrightarrow{OY}$ is called the y-axis. Together they are called the coordinate axes. Their point of intersection, O, is called the origin and the plane is called the xy-plane. Although we usually represent a line with a segment and an arrowhead at each end, it is common practice to represent an axis with a segment having an arrowhead only on the end that "points in the positive direction." In many of the figures in this book axes are represented in this way.

The projection of a point P on a line l is (1) P itself if P is on l, and (2) the foot of the perpendicular from P to l if P is not on l. (See Definition 10.4.) Since the perpendicular segment from an external point to a line is unique, each point in a plane has a unique projection on a given line in that plane.

In Figure 11-2, $\overline{PA}$ is perpendicular to the x-axis at A and $\overline{PB}$ is perpendicular to the y-axis at B. Therefore A is the projection of P on the x-axis and B is the projection of P on the y-axis. If 4 and 3 are the coordinates of A and B, respectively, then we call the ordered pair of numbers (4, 3) the coordinates of P.

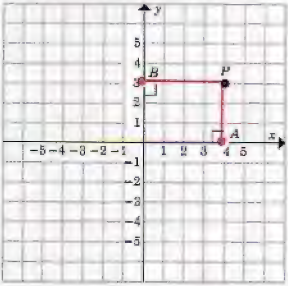


Figure 11-2

More generally, if P is any point in the xy-plane, the *x-coordinate* (abscissa) of P is the x-coordinate of the projection of P on the x-axis. The *y-coordinate* (ordinate) of P is the y-coordinate of the projection of P on the y-axis. We call the x-coordinate of P and the y-coordinate of P *the coordinates of P*. The xy-coordinates or, simply, the coordinates of P are an ordered pair of real numbers in which the abscissa is the first number of the pair and the ordinate is the second. Thus if the abscissa of P is a and the ordinate of P is b, then the xy-coordinates of P are written as (a, b).

THEOREM 11.1 The correspondence which matches each point in the xy-plane with its xy-coordinates is a one-to-one correspondence between the set of all ordered pairs of real numbers and the set of all points in the xy-plane.

Proof: In the xy-plane, let an x-coordinate system on the x-axis and a y-coordinate system on the y-axis be given. Let P be any point in the given xy-plane. If A and B are the projections of P on the x-axis and the

y-axis, respectively, let the abscissa of A be a and the ordinate of B be b. Since each point on the x-axis has a unique x-coordinate and each point on the y-axis has a unique y-coordinate, and since the projections of P on the x- and y-axes are unique, it follows that there is exactly one ordered pair of real numbers (a, b) that corresponds to the point P.

Conversely, let (a, b) be any ordered pair of real numbers; then there is a unique point A on the x-axis with abscissa a and a unique point B on the y-axis with ordinate b. Also, there is a unique line l_1 through A and perpendicular to the x-axis and a unique line l_2 through B and perpendicular to the y-axis. These two lines intersect (Why?) in a unique point P. Hence every ordered pair of real numbers corresponds to exactly one point in the given xy-plane and the proof is complete.

Definition 11.1 The one-to-one correspondence between the set of all points in an xy-plane and the set of all ordered pairs of real numbers in which each point P in the plane corresponds to the ordered pair (a, b), in which a is the x-coordinate of P and b is the y-coordinate of P, is an **xy-coordinate system.**

Since there are many pairs of perpendicular lines in a plane and since any pair of such lines may serve as axes, it follows that there are many xy-coordinate systems in a given plane. In a given problem situation, we are free to choose whichever coordinate system seems most appropriate.

In view of the one-to-one correspondence between the set of all ordered pairs of real numbers and the set of all points in a given xy-plane, it is clear that symbols used to denote the ordered pairs may be used to denote the corresponding points. Thus, if P is the point whose coordinates are the ordered pair $(4, -3)$, we may speak of the point $(4, -3)$ or we may write $P = (4, -3)$. Sometimes we simply write $P(4, -3)$.

It should be noted that the numbers in an ordered pair need not be distinct. Thus $(3, 3)$ is an ordered pair of real numbers. Of course, the ordered pair $(3, 5)$ is not the same as the ordered pair $(5, 3)$. Indeed, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

As shown in Figure 11-3, it is customary to think of the unit point I as lying to the "right" of the origin (that is, on ray $\overrightarrow{OX}$ in the figure) and of unit point J as lying "above" the origin (that is, on ray $\overrightarrow{OY}$). This means, then, that the points on the x-axis with positive abscissas lie to the right of the origin and those points on the x-axis with negative

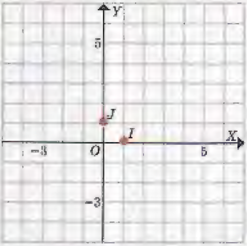


Figure 11-3

abscissas lie to the "left" of the origin (that is, on *opp* $\overrightarrow{OX}$). Where do the points on the y-axis with positive ordinates lie? Where do the points on the y-axis with negative ordinates lie?

There are situations in which it is convenient to think of the positive part of the x-axis as extending to the left, or the positive part of the y-axis as extending downward, or some other variation. However, it will not be necessary to do this.

A line in a plane separates the points of the plane not on the line into two halfplanes, the line being the edge of each halfplane. Similarly, the coordinate axes separate the points of an xy-plane not on the axes into four "quarter-planes," or quadrants, the union of whose edges is the axes. For convenience, these quadrants are numbered I, II, III, IV as indicated in Figure 11-4.

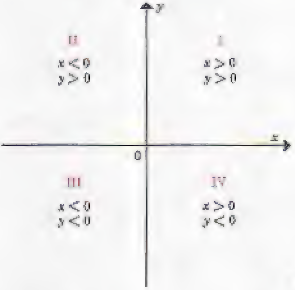


Figure 11-4

Quadrant I is the set of all points (x, y) such that $x > 0$ and $y > 0$.
Quadrant II is the set of all points (x, y) such that $x < 0$ and $y > 0$.
Quadrant III is the set of all points (x, y) such that $x < 0$ and $y < 0$.
Quadrant IV is the set of all points (x, y) such that $x > 0$ and $y < 0$.

We can describe the coordinates of those points (x, y) on $\overrightarrow{OX}$ by $x \geq 0$ and $y = 0$, and the coordinates of those points on *opp* $\overrightarrow{OX}$ by $x \leq 0$ and $y = 0$. Describe, in a similar way, the coordinates of those points (x, y) on $\overrightarrow{OY}$; on *opp* $\overrightarrow{OY}$.

Since we usually think of an xy-coordinate system oriented as we have shown in Figures 11-1 through 11-4, it is customary to call all lines parallel to $\overleftrightarrow{OY}$ **vertical** lines. Similarly, we call all lines parallel to $\overleftrightarrow{OX}$ **horizontal** lines.

It is often convenient to use "above," "below," "right," "left" to describe the position of a point. However, we can get along without these words if challenged to do so. For example, we could describe the position of the point $P = (2, -5)$ by saying that P is 5 units "below" the x-axis and 2 units to the "right" of the y-axis in an xy-plane. Or we could say that P is in the fourth quadrant, that it is on a vertical line which intersects the x-axis in a point 2 units from the origin, and that it is on a horizontal line which intersects the y-axis in a point 5 units from the origin.

EXERCISES 11.2

In Exercises 1–8, name the quadrant in which the point lies.

1. $(-2, 4)$
2. $(7, -3)$
3. $(-\sqrt{2}, -5)$
4. $(1.2, 6)$
5. $(\pi, -2)$
6. $(-7.3, -1)$
7. $(-\sqrt{5}, \sqrt{3})$
8. $(\frac{1}{2}, -\frac{2}{3})$

In Exercises 9–18, describe the set of all points (x, y) which satisfy the given conditions.

9. $x < 0, y \leq 0$
10. $x > 0, y \leq 0$
11. $x \leq 0, y = 0$
12. $x = 0, y > 0$
13. $x = 0, y \leq 0$
14. x is any real number, $y = 4$
15. $x = -2$, y is any real number
16. $x \geq 3, y = -5$
17. $x = 2, y \leq 1$
18. $xy = 0$

19. If $R = (-3, 7)$, and
    (a) if S is the point where the vertical line through R intersects the x-axis, what is the abscissa of S? The ordinate of S? What are the coordinates of S?
    (b) if T is the point where the horizontal line through R intersects the y-axis, what is the abscissa of T? The ordinate of T? What are the coordinates of T?

20. Of the following points, find three that are collinear: $(3, -5)$, $(5, 7)$, $(-5, -5)$, $(5, 2)$, $(\pi, -5)$.
21. Describe the set of all points in the xy-plane for which the abscissa is -2; for which the ordinate is 6. Describe the intersection of these two sets.
22. Describe the set of all points in the xy-plane for which the abscissa is zero; the ordinate is zero. Describe the intersection of these two sets. Describe the union of these two sets.

11.3 GRAPHS IN A PLANE

A graph is a set of points. To draw a graph or to plot a graph is to draw a picture that suggests which points belong to the graph. The picture of a graph shows the axes, but they are not usually a part of the graph. Of course, a subset of the axes is often a part of a graph.

It is customary to label the x- and y-axes as shown in Figure 11-5. It is usually desirable to label at least one point (other than the origin) on the x-axis with its x-coordinate and at least one point (other than the origin) on the y-axis with its y-coordinate.

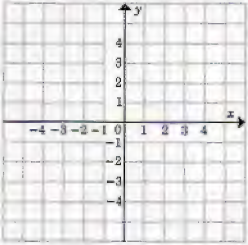


Figure 11-5

In setting up an xy-coordinate system we start with three distinct points O, I, J such that $\overleftrightarrow{OI} \perp \overleftrightarrow{OJ}$ and $OI = OJ = 1$, as in Figure 11-1. Although it is understood that $\overline{OI}$ and $\overline{OJ}$ are congruent segments (based on lengths in the distance system that we consider to be fixed), it is sometimes helpful, particularly in applied problems, to take points I and J so that $\overline{OI}$ and $\overline{OJ}$ "appear" to be different in length. If this is done, a picture of the xy-plane may be described by saying that the "scale" on the x-axis is different from the "scale" on the y-axis. The word *scale*, as used here, is not part of our formal geometry. As far as our formal geometry is concerned the distance from O to I is the same as the distance from O to J regardless of appearance.

If the scale on the x-axis is different from the scale on the y-axis, a graph may appear distorted. An answer to the question "When is a square not a square?" might be "When different scales for the x- and y-axes are used in graphing its length and width." For example, Figure 11-6 shows a picture of a quadrilateral $ABCD$ all of whose angles are right angles. Since $AB = CD = 4$ and $AD = BC = 4$ in our formal geometry, it is true that $ABCD$ is a square in our formal geometry. In physical (informal) geometry, if we were to measure the sides of quadrilateral $ABCD$ with a ruler, we would find that they are of unequal length and conclude that $ABCD$ is not a square.

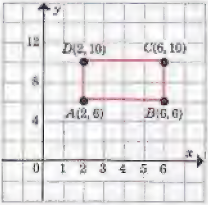


Figure 11-6

If a graph contains only a few points, it may be desirable to write the coordinates of each point beside the dot that represents it.

Example 1 Plot the points $A(-4, 3)$, $B(0, -5)$, $C(5, -2)$, $D(4, 0)$. (See Figure 11-7.)

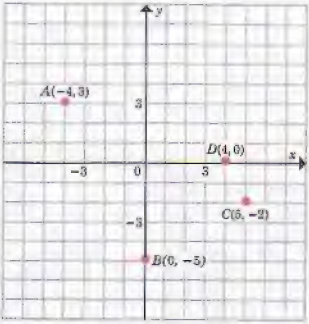


Figure 11-7

Sometimes a small open circle is used to indicate that the endpoint of a segment or of a ray does not belong to the graph. In this connection recall the symbol for a halfline introduced in Chapter 2.

Example 2 Draw the graph of $\{(x, y) : x < 2 \text{ and } y = 3\}$. (See Figure 11-8.)

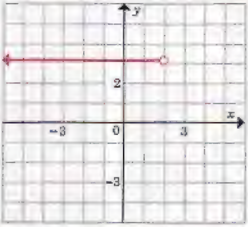


Figure 11-8

If there are infinitely many points in the graph, the picture may contain segments or curves, and sometimes shaded regions or arrows, to indicate which points belong to the graph.

Example 3 Draw the graph of $\{(x, y) : 1 \leq x \leq 3 \text{ or } 2 < y < 4\}$. (See Figure 11-9.)

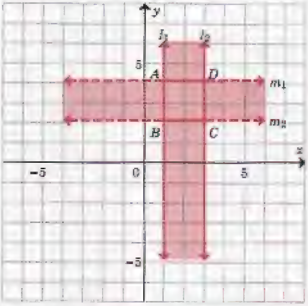


Figure 11-9

Note that in Figure 11-9 we have shown l_1 and l_2 as solid lines to indicate that they are a part of the graph. We have shown parts of m_1 and m_2 as dashed lines to indicate that these parts do not belong to the graph. Of course, segment $\overline{AD}$ on m_1 and segment $\overline{BC}$ on m_2 are part of the graph. It is desirable to indicate all of lines m_1 and m_2 in some manner since they are a part of the "boundary" of the graph. Let us agree, then, that if a line, a segment, or a ray is not part of a graph, but serves as a boundary to a graph, it will be shown in the graph as a dashed line, segment, or ray.

Write the coordinates of the point of intersection of lines l_2 and m_1 in Figure 11-9. Is this point a point of the graph?

Example 4 Draw the graph of $\{(x, y) : x < -2\}$. (See Figure 11-10.)

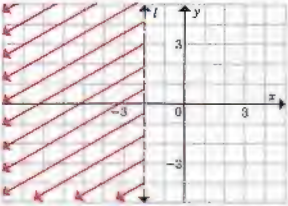


Figure 11-10

As Figure 11-10 suggests, the graph of $\{(x, y) : x < -2\}$ is the halfplane on the left of the vertical line l. The interiors of the rays shown in the graph are intended to indicate this halfplane. Why does line l appear as a dashed line in this graph? Since line l is a vertical line, it is parallel to the y-axis. Therefore all the points of line l are on the same side of the y-axis in the xy-plane. Another way of describing the graph in Figure 11-10, then, is as the halfplane with edge l on the opposite side of line l from the y-axis. Since line l represents the set of all points in the xy-plane whose abscissa is -2, another way of describing line l is the line with an equation $x = -2$. The halfplane pictured in Figure 11-10 is the set of all points (x, y) such that $x < -2$; or we may describe it as all of the xy-plane which lies to the left of the line $x = -2$.

EXERCISES 11.3

1. If $A = (-2, 2)$, $B = (3, 3)$, $C = (4, -2)$, $D = (-3, -3)$, draw the graph of the set of all points which belong to the polygon $ABCD$.
2. Draw the graph of the set of points that belong to the interior of the polygon in Exercise 1.
3. In your graph for Exercise 2, should the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ appear as dashed lines or as solid lines?
4. Draw the graph of

$$\{(x, y) : x = 3 \text{ and } -1 \leq y < 2\}.$$

5. Is the point $(3, 2)$ part of the graph for Exercise 4? Is the point $(3, -1)$ part of the graph?

In Exercises 6–13, graph the set of all points (x, y) in an xy-plane satisfying the given conditions. Then describe the graph of the set in words. Exercise 6 has been worked as a sample.

6. $2 \leq x \leq 6$ and $y = 3$.

Graph:

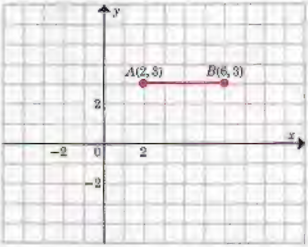


Figure 11-11

Description: The graph is the segment whose endpoints are $A(2, 3)$ and $B(6, 3)$.

7. $x \geq -2$ and $y = 2$
8. $x = 5$ and $y \geq 0$
9. $x \geq 3$
10. $y < 4$
11. $x = -3$ and $-2 < y < 5$
12. $-5 \leq x < -1$ or $2 < y \leq 5$
13. $-5 \leq x < -1$ and $2 < y \leq 5$
14. Which, if any, of the following points are not part of the graph for Exercise 12: $(-5, 2)$, $(-1, 2)$, $(-1, 5)$, $(-5, 5)$? Which of these points are not part of the graph for Exercise 13?
15. If $P = (2, 0)$ and $Q = (9, 0)$, what is the length of segment $\overline{PQ}$? Justify your conclusion.
16. If $A = (2, 5)$ and $B = (8, 5)$, what is the length of $\overline{AB}$? Why?
17. If $C = (-3, -4)$ and $D = (-3, 6)$, what is the length of $\overline{CD}$? Why?
18. Give the coordinates of the midpoints of the segments whose endpoints are the following:
    (a) $(3, 2)$, $(3, 12)$
    (b) $(1, 4)$, $(9, 4)$
    (c) $(-3, 1)$, $(7, 1)$
19. CHALLENGE PROBLEM. Give the coordinates of the midpoint of the segment whose endpoints are $(3, 5)$ and $(8, 7)$.
20. CHALLENGE PROBLEM. If $P = (5, 1)$ and $Q = (8, 6)$, what is the exact length of $\overline{PQ}$?

11.4 DISTANCE FORMULAS

Consider an xy-coordinate system as shown in Figure 11-12. Let I be the unit point on the x-axis and let P and Q be any two points with abscissas x_1 and x_2, respectively, on the x-axis.

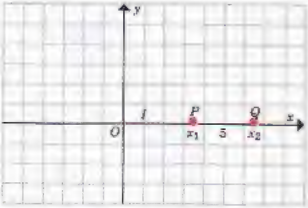


Figure 11-12

We know, by the definition of a coordinate system on $\overleftrightarrow{OX}$ relative to unit segment $\overline{OI}$ (Definition 3.3), that

$$PQ \text{ (in } \overline{OI} \text{ units)} = |x_1 - x_2| = |x_2 - x_1|.$$

For example, if $P = (3, 0)$ and $Q = (7, 0)$, then $x_1 = 3$, $x_2 = 7$, and

$$PQ = |3 - 7| = |7 - 3| = 4.$$

Now suppose that l is any line in the xy-plane and parallel to the x-axis as in Figure 11-13 (that is, l is any horizontal line). Let P and Q be any two points on l and let $P_1(x_1, 0)$ and $Q_1(x_2, 0)$ be the projections of P and Q, respectively, on the x-axis.

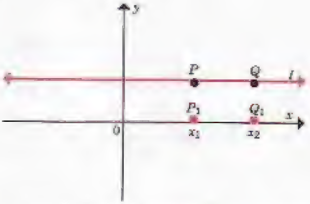


Figure 11-13

By our definition of the abscissa of a point in an xy-plane, we know that the abscissa of P is x_1 and that the abscissa of Q is x_2. If $P = P_1$ and $Q = Q_1$, then

$$PQ = P_1Q_1 = |x_1 - x_2|.$$

If $P \neq P_1$, then $Q \neq Q_1$ and the quadrilateral PP_1Q_1Q is a parallelogram. Therefore we again have

$$PQ = P_1Q_1 = |x_1 - x_2|.$$

Note that if l is a horizontal line, then l is perpendicular to the y-axis (Why?) and every point of l projects onto the same point in the y-axis. Thus all points on a horizontal line have the same ordinate. We have proved the following theorem.

THEOREM 11.2 If $P(x_1, y_1)$ and $Q(x_2, y_1)$ are points on the same horizontal line in an xy-plane, then

$$PQ = |x_1 - x_2|.$$

It should be noted that if $P = Q$ in Theorem 11.2, then

$$x_1 = x_2 \qquad \text{and} \qquad PQ = |0| = 0.$$

THEOREM 11.3 If $P(x_1, y_1)$ and $Q(x_1, y_2)$ are points on the same vertical line in an xy-plane, then

$$PQ = |y_1 - y_2|.$$

Proof: The proof is similar to the one given for Theorem 11.2 and is assigned as an exercise.

Example 1 If $A = (3, 5)$ and $B = (-6, 5)$, find AB.

Solution: $AB = |3 - (-6)| = 9.$

Note that if two points have the same ordinate, then they lie on the same horizontal line. Thus $\overleftrightarrow{AB}$ in Example 1 is a horizontal line. Hence AB is the absolute value of the difference of the abscissas of the points A and B, as shown in the solution to Example 1.

Example 2 If $C = (-2, -3)$ and $D = (-2, 4)$, find CD.

Solution: $CD = |-3 - 4| = |-7| = 7.$

Note that if two points have the same abscissa, then they lie on the same vertical line. Thus $\overleftrightarrow{CD}$ in Example 2 is a vertical line. Hence CD is the absolute value of the difference of the ordinates of the points C and D, as shown in the solution to Example 2.

You have seen how to find the distance between any two points in an xy-coordinate system if the two points lie on the same horizontal line or if the two points lie on the same vertical line. Next we show how to find the distance between two points if they lie on the same oblique line, that is, a line that is neither horizontal nor vertical. Before proceeding to the general case, let us consider an example.

Example 3 If $P = (-2, -3)$ and $Q = (4, 5)$, find PQ.

Solution: (See Figure 11-14.) Let l_1 be the line through Q and parallel to the y-axis and let l_2 be the line through P and parallel to the x-axis. These two lines intersect in a point R such that $l_1 \perp l_2$ at R. Why? Therefore $\overline{PQ}$ is the hypotenuse of a right triangle, $\triangle PQR$, with the right angle at R. Since l_2 is a horizontal line and l_1 is a vertical line, we have

$$PR = |4 - (-2)| = 6$$
$$QR = |5 - (-3)| = 8$$

by Theorems 11.2 and 11.3. By the Pythagorean Theorem,

$$\begin{aligned}(PQ)^2 &= (PR)^2 + (QR)^2,\\ &= 6^2 + 8^2,\\ &= 36 + 64,\\ &= 100,\end{aligned}$$

and hence

$$PQ = 10.$$

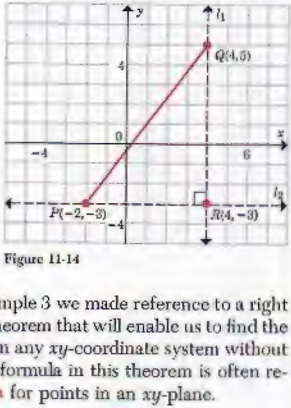


Figure 11-14

Note that in finding PQ in Example 3 we made reference to a right triangle. We proceed now to the theorem that will enable us to find the distance between any two points in any xy-coordinate system without reference to a right triangle. The formula in this theorem is often referred to as the Distance Formula for points in an xy-plane.

THEOREM 11.4 If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are any two points in an xy-plane, then

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Proof: There are four cases to consider.

Case 1. $P_1 = P_2$.
Case 2. P_1 and P_2 are distinct points on a horizontal line.
Case 3. P_1 and P_2 are distinct points on a vertical line.
Case 4. P_1 and P_2 are distinct points on an oblique line.

Proof of Case 1: If $P_1 = P_2$, then $x_1 = x_2$, $y_1 = y_2$, and, by the formula of Theorem 11.4, we have

$$P_1P_2 = \sqrt{0} = 0,$$

as it should since the distance between a point and itself is defined to be zero.

Proof of Case 2: If P_1 and P_2 are distinct points on the same horizontal line, then $y_1 = y_2$ and, by the formula of Theorem 11.4, we have

$$P_1P_2 = \sqrt{(x_1 - x_2)^2} = |x_1 - x_2|.$$

This result agrees with the statement of Theorem 11.2 for two points on the same horizontal line.

Proof of Case 3: If P_1 and P_2 are distinct points on the same vertical line, then $x_1 = x_2$ and, by the formula of Theorem 11.4, we have

$$P_1P_2 = \sqrt{(y_1 - y_2)^2} = |y_1 - y_2|,$$

which agrees with the statement of Theorem 11.3.

Proof of Case 4: If P_1 and P_2 are distinct points on an oblique line as shown in Figure 11-15, then the line through P_1 and parallel to the y-axis intersects the line through P_2 and parallel to the x-axis in a point $R(x_1, y_2)$ such that $\triangle P_1P_2R$ is a right triangle.

$$P_2R = |x_1 - x_2|$$

by Theorem 11.2 and

$$P_1R = |y_1 - y_2|$$

by Theorem 11.3. We have

$$(P_2R)^2 = |x_1 - x_2|^2 = (x_1 - x_2)^2$$

and

$$(P_1R)^2 = |y_1 - y_2|^2 = (y_1 - y_2)^2.$$

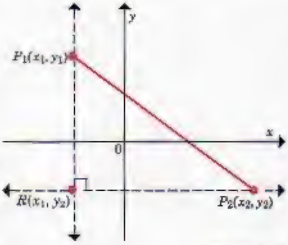


Figure 11-15

Since $\overline{P_1P_2}$ is the hypotenuse of $\triangle P_1P_2R$, we have, by the Pythagorean Theorem, that

$$(P_1P_2)^2 = (P_2R)^2 + (P_1R)^2$$

or that

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

and the proof is complete.

Since $(x_1 - x_2)^2 = (x_2 - x_1)^2$ and $(y_1 - y_2)^2 = (y_2 - y_1)^2$, in using the Distance Formula of Theorem 11.4, it does not matter which point is designated $P_1(x_1, y_1)$ and which point is designated $P_2(x_2, y_2)$.

Example 4 If $A = (2, -4)$ and $B = (-5, 3)$, find AB.

Solution: Substituting the coordinates of the given points in the Distance Formula, we have

$$\begin{aligned} AB &= \sqrt{[2 - (-5)]^2 + (-4 - 3)^2} \\ &= \sqrt{7^2 + (-7)^2} \\ &= \sqrt{49 + 49} \\ &= \sqrt{49 \cdot 2} \\ &= 7\sqrt{2} \end{aligned}$$

Note that in working Example 4 we considered $A(2, -4)$ as the point $P_1(x_1, y_1)$ and $B(-5, 3)$ as the point $P_2(x_2, y_2)$ when we substituted these coordinates into the distance formula. Show, by working Example 4 again, that we would have obtained the same result for AB had we considered $A(2, -4)$ as the point $P_2(x_2, y_2)$ and $B(-5, 3)$ as the point $P_1(x_1, y_1)$.

EXERCISES 11.4

1. Prove Theorem 11.3.

In Exercises 2–10, use Theorem 11.2, Theorem 11.3, or the distance formula to find the distance between the given points.

2. $(-3, 2)$ and $(-3, 11)$
3. $(\frac{2}{3}, 1\frac{7}{8})$ and $(3\frac{1}{2}, 1\frac{7}{8})$
4. $(-2.5, \sqrt{3})$ and $(17.3, \sqrt{3})$
5. $(\pi, 4.8)$ and $(\pi, -9.6)$
6. $(-2, 7)$ and $(3, -5)$
7. $(3, -6)$ and $(9, 0)$
8. $(4, 17)$ and $(-3, 9)$
9. $(-3, -5)$ and $(5, -1)$
10. $(6, -3)$ and $(-4, 2)$

11. Find the perimeter of the triangle whose vertices are $A(-2, -3)$, $B(3, 9)$, and $C(-10, 12)$.

12. Prove that the triangle whose vertices are $P(1, 2)$, $Q(9, 2)$, and $R(5, 8)$ is isosceles.
13. $\triangle ABC$ has vertices $A(6, 0)$, $B(-4, 4)$, and $C(10, 4)$.
   (a) Find the perimeter of $\triangle ABC$.
   (b) Find the area of $\triangle ABC$.
14. Find the lengths of the diagonals of a quadrilateral $ABCD$ if $A = (4, -3)$, $B = (7, 10)$, $C = (-8, 2)$, and $D = (-1, -5)$.
15. The vertices of $\triangle PQR$ are $P(-1, -2)$, $Q(4, 0)$, and $R(2, 5)$. Prove that $\triangle PQR$ is a right isosceles triangle.
16. If the distance between $A(6, -2)$ and $B(0, y)$ is 10, find the possible y-coordinates of B.
17. Find the coordinates of the points on the x-axis whose distance from $(2, 8)$ is 10.
18. If $(a, -a)$ is a point in quadrant IV, prove that the triangle with vertices $(-5, 0)$, $(0, 5)$, and $(a, -a)$ is isosceles.
19. Given $D = (-2, 2)$, $E = (10, 2)$, $F = (4, y)$ with $\angle DFE$ a right angle, find the two distinct possible values of y.
20. Given $P = (-2, -7)$, $Q = (3, 3)$, $R = (6, 9)$, use the distance formula to show that $PQ + QR = PR$, and hence that P-Q-R.

■ Exercises 21–26 refer to the triangle whose vertices are $A = (2, 4)$, $B = (6, 8)$, and $C = (12, 2)$.

21. Draw line l_1 through A and parallel to the y-axis. Draw line l_2 through B, parallel to the x-axis, and intersecting l_1 at D. What are the coordinates of D?
22. Draw line l_3 through C, parallel to the x-axis, and intersecting l_1 at E. What are the coordinates of E?
23. What kind of quadrilateral is quadrilateral $BDEC$? Find $|BDEC|$.
24. What kind of triangles are $\triangle BDA$ and $\triangle CEA$? Find $|\triangle BDA|$ and $|\triangle CEA|$.
25. Copy and complete:

$$|BDEC| = |\triangle ABC| + |\triangle \boxed{?}| + |\triangle \boxed{?}|.$$

26. Find $|\triangle ABC|$.

27. CHALLENGE PROBLEM. Find $|\triangle PQR|$ if $P = (6, 0)$, $Q = (1, 5)$, and $R = (10, 8)$.
28. CHALLENGE PROBLEM. If $A = (0, 0)$, $D = (b, c)$, $B = (a, 0)$, and $C = (a + b, c)$, where a, b, c are positive numbers, prove that quadrilateral $ABCD$ is a parallelogram. (*Hint:* If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.)

11.5 THE MIDPOINT FORMULA

We know that the midpoint M of a segment $\overline{PQ}$ is the point between P and Q such that $PM = MQ$ or that $PM = \frac{1}{2}PQ$, or, similarly, $MQ = \frac{1}{2}PQ$. Suppose that the coordinates of P and Q are given and that we wish to find the coordinates of M, the midpoint of $\overline{PQ}$. For example, if $P = (2, 3)$ and $Q = (8, 5)$, we can find the coordinates (x, y) of M, the midpoint of $\overline{PQ}$, as follows. (See Figure 11-16.)

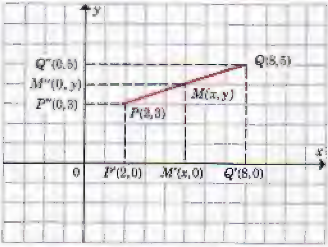


Figure 11-16

Let $P'(2, 0)$, $M'(x, 0)$, $Q'(8, 0)$ be the projections of P, M, Q, respectively, on the x-axis and $P''(0, 3)$, $M''(0, y)$, $Q''(0, 5)$ be the projections of P, M, Q, respectively, on the y-axis. Since M is the midpoint of $\overline{PQ}$, it follows from Theorem 10.8 that M' is the midpoint of $\overline{P'Q'}$ and M'' is the midpoint of $\overline{P''Q''}$. Therefore, by Definition 3.3, $2 < x < 8$ and $3 < y < 5$. By the definition of midpoint, we have

$$(1)\quad P'M' = M'Q' \qquad \text{and} \qquad (2)\quad P''M'' = M''Q''.$$

By Theorem 11.2,

$$P'M' = |x - 2| \qquad \text{and} \qquad M'Q' = |8 - x|.$$

Since $x - 2 > 0$ (Why?) and $8 - x > 0$, we get by substitution into (1),

$$x - 2 = 8 - x.$$

Therefore $2x = 10$ and $x = 5$.

Similarly, by Theorem 11.3,

$$P''M'' = |y - 3| \qquad \text{and} \qquad M''Q'' = |5 - y|.$$

Since $y - 3 > 0$ and $5 - y > 0$, we get by substitution into (2),

$$y - 3 = 5 - y.$$

Therefore $2y = 8$ and $y = 4$. Hence the coordinates of M, the midpoint of $\overline{PQ}$, are $(5, 4)$.

We can proceed, as in the preceding example, to find the coordinates of the midpoint of any segment if the coordinates of the endpoints of the segment are given. However, Theorem 11.5 provides us with a formula that will enable us to find the coordinates of the midpoint of a segment. The formula in Theorem 11.5 is often referred to as the midpoint formula.

THEOREM 11.5 If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points in an xy-plane, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

There are three cases to consider.

Case 1: P and Q are distinct points on a horizontal line.

Case 2: P and Q are distinct points on a vertical line.

Case 3: P and Q are distinct points on an oblique line.

We shall begin the proof for Case 3 and assign the remainder and the proofs of Cases 1 and 2 as exercises.

Proof of Case 3: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two distinct points on an oblique line in an xy-plane as shown in Figure 11-17 and let $M(x, y)$ be the midpoint of $\overline{PQ}$. Let $P'(x_1, 0)$, $M'(x, 0)$, and $Q'(x_2, 0)$ be the projections of P, M, and Q, respectively, on the x-axis. By Theorem 10.8, M' is the midpoint of $\overline{P'Q'}$. By Definition 3.3, x is between x_1 and x_2; thus

$$x_1 < x < x_2 \qquad \text{or} \qquad x_2 < x < x_1.$$

(In Figure 11-17 we have shown $x_1 < x < x_2$, but this order might be reversed if P and Q are chosen in a different way.)

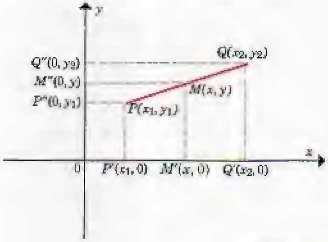


Figure 11-17

By the definition of midpoint,

$$P'M' = M'Q',$$

and by Theorem 11.2,

$$P'M' = |x - x_1|$$

and

$$M'Q' = |x_2 - x|.$$

Therefore

$$|x - x_1| = |x_2 - x|.$$

If $x_1 < x < x_2$, then $x - x_1 > 0$ and $x_2 - x > 0$ and we have $x - x_1 = x_2 - x$. Why? Therefore

$$2x = x_1 + x_2$$

and

$$x = \frac{x_1 + x_2}{2}.$$

If $x_2 < x < x_1$, then $x_1 - x > 0$ and $x - x_2 > 0$. In this case,

$$P'M' = |x - x_1| = x_1 - x$$

and

$$M'Q' = |x_2 - x| = x - x_2.$$

Thus

$$x - x_2 = x_1 - x,$$

and again we have

$$x = \frac{x_1 + x_2}{2}.$$

We have shown that if $M(x, y)$ is the midpoint of the segment whose endpoints are $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the abscissa of M is $\frac{x_1 + x_2}{2}$. In a similar way, it can be shown that the ordinate of M is $\frac{y_1 + y_2}{2}$. (You are asked to show this in the Exercises, thus completing the proof of Case 3.)

Example 1 Find the coordinates of the midpoint of the segment whose endpoints are the following:

1. $A(2, -3)$ and $B(2, 9)$.
2. $C(-12, 1)$ and $D(-3, 1)$.
3. $E(-2, 7)$ and $F(10, 12)$.

Solution:

1. Segment $\overline{AB}$ lies on a vertical line. Therefore $x = x_1 = x_2$ and

$$\frac{x_1 + x_2}{2} = \frac{x_1 + x_1}{2} = x_1.$$

Therefore the midpoint is

$$\begin{aligned} M &= \left(x_1, \frac{y_1 + y_2}{2}\right) \\ &= \left(2, \frac{-3 + 9}{2}\right) = (2, 3). \end{aligned}$$

2. Segment $\overline{CD}$ lies on a horizontal line. Therefore $y = y_1 = y_2$ and

$$\frac{y_1 + y_2}{2} = \frac{y_1 + y_1}{2} = y_1.$$

Therefore the midpoint is

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, y_1\right) \\ &= \left(\frac{-12 + (-3)}{2}, 1\right) = \left(-\frac{15}{2}, 1\right). \end{aligned}$$

3. Segment $\overline{EF}$ lies on an oblique line; hence the midpoint M of $\overline{EF}$ is the point

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-2 + 10}{2}, \frac{7 + 12}{2}\right) = \left(4, \frac{19}{2}\right). \end{aligned}$$

Example 2 The vertices of a triangle are $A(0, 5)$, $B(4, 3)$, and $C(-2, 1)$. Find the length of the median to $\overline{BC}$.

Solution: The median to $\overline{BC}$ is the segment whose endpoints are $A(0, 5)$ and M, the midpoint of $\overline{BC}$. The coordinates of M are $(1, 2)$ by the midpoint formula; hence

$$\begin{aligned} AM &= \sqrt{(0 - 1)^2 + (5 - 2)^2} \\ &= \sqrt{1 + 9} \\ &= \sqrt{10} \end{aligned}$$

by the Distance Formula.

EXERCISES 11.5

1. Complete the proof of Case 3 of Theorem 11.5 by showing that the ordinate of M is $\frac{y_1 + y_2}{2}$. (See Figure 11-17.)
2. Prove Case 1 of Theorem 11.5. (In this case, $y = y_1 = y_2$ in the statement of the theorem.)
3. Prove Case 2 of Theorem 11.5. (In this case, $x = x_1 = x_2$ in the statement of the theorem.)

In Exercises 4–10, find the midpoint of $\overline{AB}$ if A and B have the given coordinates.

4. $(-5, -2)$ and $(-5, 6)$
5. $(-3, 5)$ and $(8, 5)$
6. $(0, 0)$ and $(8, 10)$
7. $(0, 0)$ and $(-8, 10)$
8. $(1, 2)$ and $(6, 14)$
9. $(r, 7)$ and $(3r, -3)$
10. (a, b) and $(-5a, 7b)$

11. The vertices of a triangle are $P(2, -3)$, $Q(10, 1)$, and $R(4, 6)$. Find the midpoint of $\overline{PQ}$. Find the length of the median to $\overline{PQ}$.
12. The vertices of a triangle are $A(-3, -2)$, $B(1, 6)$, and $C(5, -2)$. Find the lengths of the medians to $\overline{AB}$ and $\overline{BC}$. How are the lengths of these two medians related to each other? What kind of triangle is $\triangle ABC$?

Exercises 13–15 refer to quadrilateral $ABCD$ whose vertices are the points $A = (0, 0)$, $B = (6, 0)$, $C = (8, 4)$, $D = (2, 4)$.

13. Find AC and BD.
14. Show that the midpoint of $\overline{AC}$ is the same point as the midpoint of $\overline{BD}$.
15. What kind of quadrilateral is $ABCD$?

Exercises 16–24 refer to $\triangle ABC$ whose vertices are the points $A = (2, 0)$, $B = (12, 0)$, $C = (7, 5\sqrt{3})$.

16. Prove that $\triangle ABC$ is equilateral.
17. Find the coordinates of D, the midpoint of $\overline{AB}$.
18. Find the coordinates of E, the midpoint of $\overline{BC}$.
19. Find the coordinates of F, the midpoint of $\overline{AC}$.
20. Show that the lengths of the three medians of $\triangle ABC$ are equal.
21. Find DE and show that $DE = \frac{1}{2}AC$.

22. Find EF and show that $EF = \frac{1}{2}AB$.
23. Find FD and show that $FD = \frac{1}{2}BC$.
24. Do the results of Exercises 21–23 prove that $\triangle DEF$ is equilateral?

Exercises 25–27 refer to $\triangle JSK$ whose vertices are the points $J = (0, 0)$, $S = (6, 0)$, $K = (0, 8)$.

25. What kind of triangle is $\triangle JSK$?
26. Find the coordinates of M, the midpoint of $\overline{SK}$.
27. Prove that $JM = \frac{1}{2}SK$.

In Exercises 28–31, the coordinates of two points A, M are given. Find the coordinates of the point B such that M is the midpoint of $\overline{AB}$.

28. $A = (1, 3)$, $M = (4, 7)$
29. $A = (4, 7)$, $M = (1, 3)$
30. $A = (-1, -8)$, $M = (6, 0)$
31. $A = (-6, -4)$, $M = (-3, -2)$

Exercises 32–36 refer to the segment $\overline{AB}$ whose endpoints are $A = (3, 2)$, $B = (11, 6)$ and the point $P(x, y)$ on $\overline{AB}$ such that $AP = \frac{3}{4}AB$.

32. Let A', P', B' be the projections of A, P, B, respectively, on the x-axis. Why does $\dfrac{A'P'}{A'B'} = \dfrac{AP}{AB} = \dfrac{3}{4}$?
33. What is the abscissa of P' in Exercise 32? What is the abscissa of P?
34. Let A'', P'', B'' be the projections of A, P, B, respectively, on the y-axis. Why does $\dfrac{A''P''}{A''B''} = \dfrac{AP}{AB} = \dfrac{3}{4}$?
35. What is the ordinate of P'' in Exercise 34? What is the ordinate of P?
36. What are the coordinates of P?
37. If $A = (-4, -2)$ and $B = (6, 3)$, find the coordinates of the point P on $\overline{AB}$ such that $AP = \frac{3}{5}AB$. (*Hint:* See Exercises 32–36.)
38. CHALLENGE PROBLEM. Given positive numbers a and b and a right triangle with vertices at $A = (0, b)$, $B = (a, 0)$, and $C = (0, 0)$, find the coordinates of M, the midpoint of $\overline{AB}$, and show that $CM = \frac{1}{2}AB$. Does this prove that for *any* right triangle, the median to the hypotenuse is one-half as long as the hypotenuse?
39. CHALLENGE PROBLEM. Given positive numbers a, b, c and a quadrilateral $ABCD$ with vertices at $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, and $D = (b, c)$,
    (a) prove that $ABCD$ is a parallelogram.
    (b) prove that the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other by showing that the midpoint of $\overline{AC}$ is the same point as the midpoint of $\overline{BD}$.

11.6 PARAMETRIC LINEAR EQUATIONS

If we are given a line in an xy-plane, it is often desirable to find the coordinates of points on that line. We know that two distinct points determine a line. Suppose that we know the xy-coordinates of two distinct points A and B on $\overleftrightarrow{AB}$. We should be able to find the xy-coordinates of any other point P on $\overleftrightarrow{AB}$ provided, of course, enough information is given to determine one and only one point P on $\overleftrightarrow{AB}$.

Example 1 Let $A = (1, 3)$, $B = (4, 7)$, and suppose that S, Q, R are points on $\overleftrightarrow{AB}$ such that S is on $\overrightarrow{AB}$, Q is on $\overrightarrow{AB}$, and R is on $opp\ \overrightarrow{AB}$. (See Figure 11-18.) Suppose, further, that

$$AS = \tfrac{2}{3}AB, \qquad AQ = 2AB, \qquad AR = AB,$$

and we wish to find the coordinates of S, Q, and R.

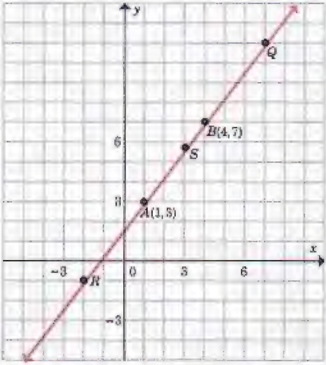


Figure 11-18

Let $S = (x, y)$. To find x and y we might proceed as follows. Let A', B', S' be the projections of A, B, S, respectively, on the x-axis, and let A'', B'', S'' be the projections of A, B, S, respectively, on the y-axis. Using Theorem 10.8, we get

$$\frac{2}{3} = \frac{AS}{AB} = \frac{A'S'}{A'B'} = \frac{x-1}{3} \qquad \text{and} \qquad x = 3,$$

$$\frac{2}{3} = \frac{AS}{AB} = \frac{A''S''}{A''B''} = \frac{y-3}{4} \qquad \text{and} \qquad y = 5\frac{2}{3}.$$

Similarly, we can find that $Q = (7, 11)$ and $R = (-2, -1)$. (Verify these results by setting up appropriate equations and solving them.) If P were any other point on $\overleftrightarrow{AB}$ such that

$$AP = kAB,$$

where k is any positive number or zero, we could find the coordinates of P by a computation similar to those above. Our objective, however, is to derive an expression from which the coordinates of any point on $\overleftrightarrow{AB}$ can be obtained by simple replacements.

In Chapter 3 we studied a coordinate system on a line. In this chapter we have already defined an xy-coordinate system in a plane, based on two line coordinate systems like those you studied in Chapter 3.

Let us consider now a third line coordinate system, a coordinate system on line $\overleftrightarrow{AB}$ with A as origin and B as unit point. We call it the k-coordinate system on $\overleftrightarrow{AB}$. It should be clear that we start with an xy-coordinate system based on a unit segment for distance. When we speak of "the distance" between two points in the xy-plane, we are talking about the distance based on the same unit segment that is used in setting up the xy-coordinate system. For every different choice of points A and B on a line l in the xy-plane, there is a different k coordinate system on l with A as origin and B as unit point. The k-distance between two points on l will usually be different from the xy-distance between those points.

The following table shows the x-coordinate, the y-coordinate, and the k-coordinate of the points A, B, S, Q, R that were shown in Example 1.

Point	x-Coordinate	y-Coordinate	k-Coordinate
A	1	3	0
B	4	7	1
S	3	$5\frac{2}{3}$	$\frac{2}{3}$
Q	7	11	2
R	-2	-1	-1
P	$\boxed{?}$	$\boxed{?}$	k

We shall derive the equations which show us how to compute the x- and y-coordinates of a point on $\overleftrightarrow{AB}$ in terms of its k-coordinate. First we shall show that the x-coordinates of the points of l form a coordinate system on l as do the y-coordinates.

Let l be any nonvertical line in an xy-plane. (See Figure 11-19.) Let $O(0, y_0)$, $I(1, y_1)$, $P(x, y)$ be points on l, and let O', I', P' be their respective projections on the x-axis. It follows from the Plane Separation Postulate and the properties of parallel lines that O', I', P' have the same betweenness relation as do their respective projections. For example, if O–I–P, then O'–I'–P'.

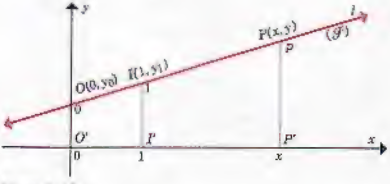


Figure 11-19

Let $\mathcal{S}$ denote the unique coordinate system on l with O as origin and I as unit point. Note that $\overline{OI}$ serves as a unit segment for $\mathcal{S}$ and that generally (except if l is parallel to the x-axis) distances "in $\mathcal{S}$" are different from distances "in the xy-coordinate system." Let p be the coordinate of P in the system $\mathcal{S}$. Since betweenness for points on l is the same as for their projections on the x-axis, it follows from the definition of a coordinate system on a line that p and x are both positive, or both negative, or both zero. It follows from Theorem 10.8 that

$$\frac{OP}{OI} = \frac{O'P'}{O'I'}.$$

Since these ratios are equal regardless of the distance function, we have

$$\frac{OP}{OI} = \frac{|p-0|}{|1-0|} = |p|, \quad \frac{O'P'}{O'I'} = \frac{|x-0|}{|1-0|} = |x|, \quad \text{and} \quad p = x.$$

Now $\mathcal{S}$ is the one-to-one correspondence that matches each point P on l with a number p. Since $p = x$, we see that the correspondence that matches each point of l with its x-coordinate is indeed a coordinate system. Similarly if l is a nonhorizontal line, then the correspondence that matches each point of l with its y-coordinate is a coordinate system on l. We state these results in our next theorem.

THEOREM 11.6 If l is any nonvertical (nonhorizontal) line in an xy-plane, then the one-to-one correspondence between the points of l and their x-coordinates (y-coordinates) is a coordinate system on l.

In the next theorem, as well as in many others throughout the rest of this book, there is the assertion, "k is real" within a set-builder symbol. This is short for "k is a real number."

THEOREM 11.7 If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two distinct points, then

$$\overleftrightarrow{AB} = \{(x, y) : x = x_1 + k(x_2 - x_1), y = y_1 + k(y_2 - y_1), k \text{ is real}\}.$$

If k is a real number and if P is the point (x, y) where $x = x_1 + k(x_2 - x_1)$ and $y = y_1 + k(y_2 - y_1)$, then

$$k = \frac{AP}{AB} \quad \text{and} \quad P \in \overrightarrow{AB} \quad \text{if} \quad k \geq 0;$$

$$-k = \frac{AP}{AB} \quad \text{and} \quad P \in opp\,\overrightarrow{AB} \quad \text{if} \quad k \leq 0.$$

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$ be any two distinct points. Suppose first that $\overleftrightarrow{AB}$ is neither vertical nor horizontal. Think of three coordinate systems on $\overleftrightarrow{AB}$ as suggested in Figure 11-20, the x-coordinate system and the y-coordinate system, determined by the xy-coordinate system (see Theorem 11.6), and the k-coordinate system in which A is the origin and B is the unit point.

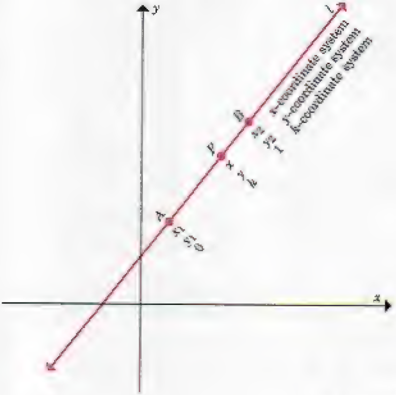


Figure 11-20

It follows from the Two-Coordinate-Systems Theorem (Theorem 3.6) and its corollary, each applied twice, that $\overleftrightarrow{AB}$ is the set of all points $P(x, y)$ such that

$$\frac{x - x_1}{x_2 - x_1} = \frac{k - 0}{1 - 0}, \qquad \frac{y - y_1}{y_2 - y_1} = \frac{k - 0}{1 - 0};$$

or

$$x - x_1 = k(x_2 - x_1), \qquad y - y_1 = k(y_2 - y_1);$$

or

$$x = x_1 + k(x_2 - x_1), \qquad y = y_1 + k(y_2 - y_1);$$

where $|k| = \dfrac{AP}{AB}$.

Since the definition of a coordinate system requires that betweenness for points agree with betweenness for coordinates, it follows that P is on $\overrightarrow{AB}$ if and only if $k \geq 0$, and P is on *opp* $\overrightarrow{AB}$ if and only if $k \leq 0$.

If $\overleftrightarrow{AB}$ is a vertical line, then the x-coordinate of every point P on $\overleftrightarrow{AB}$ is the same number, and there is no x-coordinate system on $\overleftrightarrow{AB}$. In this case $x_1 = x_2 = x$ for every point $P(x, y)$ on $\overleftrightarrow{AB}$ and the equation

$$x = x_1 + k(x_2 - x_1),$$

which simplifies to $x = x_1$, is still applicable. The proof for the nonvertical-nonhorizontal case is applicable to this case as far as the relation between y and k is concerned. Therefore the assertion of the theorem applies in the vertical case. Similarly, the theorem may be proved for the horizontal case.

In

$$\overleftrightarrow{AB} = \{(x, y) : x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\},$$

the "k is real" is put in because k is not mentioned before the braces and not before the colon within the braces, and we read the sentence as

"$\overleftrightarrow{AB}$ is the set of *all* points (x, y) such that

$$x = x_1 + k(x_2 - x_1),$$
$$y = y_1 + k(y_2 - y_1),$$

where k is real."

The equations

$$x = x_1 + k(x_2 - x_1) \qquad \text{and} \qquad y = y_1 + k(y_2 - y_1)$$

in the statement of Theorem 11.7 are called *parametric equations* for the line $\overleftrightarrow{AB}$ and k is called the *parameter*. A parameter is usually

thought of as a variable to which values may be assigned arbitrarily, and other variables are defined in terms of it. By assigning real number values to the parameter k in the parametric equations for $\overleftrightarrow{AB}$, we obtain ordered pairs of numbers corresponding to points on $\overleftrightarrow{AB}$ in the xy-coordinate system. Of course, we could assign values to x (in the nonvertical case) and find values of y so that the resulting ordered pairs correspond to points of $\overleftrightarrow{AB}$ or we could assign values to y (in the nonhorizontal case) and find values of x so that the resulting ordered pairs correspond to points of $\overleftrightarrow{AB}$. But it is easier, and it works in all cases, to assign values to k and compute values of x and of y so that the resulting ordered pairs correspond to points of $\overleftrightarrow{AB}$.

Note that parametric equations for a line are not unique. If l is a line, then there are many choices for the two points A and B of Theorem 11.7, and hence many pairs of parametric equations for l.

If we replace the coordinates (x_1, x_2) and (y_1, y_2) in the parametric equations of Theorem 11.7 with the coordinates (1, 3) and (4, 7) of the points A and B in Example 1, we obtain the parametric equations

$$x = 1 + 3k \qquad \text{and} \qquad y = 3 + 4k$$

for the line $\overleftrightarrow{AB}$ in that example. Recall that, in Example 1, S is a point on $\overleftrightarrow{AB}$ such that $AS = \frac{2}{3}AB$. Thus, by replacing k with the number $\frac{2}{3}$ in the parametric equations

$$x = 1 + 3k, \qquad y = 3 + 4k,$$

we obtain the coordinates $(3, 5\frac{2}{3})$ of the point S. Show that you get the xy-coordinates of the points Q and R on $\overleftrightarrow{AB}$ in Example 1 by assigning the values 2 and -1, respectively, to k in the parametric equations

$$x = 1 + 3k, \qquad y = 3 + 4k.$$

It follows from Theorem 11.7 that every line can be represented by set-builder notation and a pair of parametric equations. However, it is not true that every pair of parametric equations represents a line. For example, the set

$$S = \{(x, y) : x = 2 + k \cdot 0, y = 3 + k \cdot 0, k \text{ is real}\}$$

is a set whose only element is (2, 3). Note that in the statement of Theorem 11.7, the points A and B are distinct; hence not both of the coefficients of k in the parametric equations can be zero. That is, we can have $x_2 - x_1 = 0$ *or* $y_2 - y_1 = 0$, but we cannot have both $x_2 - x_1 = 0$ *and* $y_2 - y_1 = 0$.

Although not every pair of parametric equations represents a line, our next theorem provides us with a method for identifying those pairs of parametric equations that represent lines.

THEOREM 11.8 If a, b, c, d are real numbers, if b and d are not both zero, and if

$$S = \{(x, y) : x = a + bk, y = c + dk, k \text{ is real}\},$$

then S is a line.

Proof: Taking $k = 0$ and $k = 1$, we get two points in S, namely $A(a, c)$ and $B(a + b, c + d)$. By Theorem 11.7, parametric equations for $\overleftrightarrow{AB}$ are

$$x = a + k(a + b - a) = a + bk$$

and

$$y = c + k(c + d - c) = c + dk.$$

Therefore $\overleftrightarrow{AB} = \{(x, y) : x = a + bk, y = c + dk, k \text{ is real}\}$; hence $S = \overleftrightarrow{AB}$ and S is a line.

In addition to being able to write parametric equations for a line, we can also write parametric equations for a segment or a ray if we place the proper restrictions on the parameter k. We illustrate this technique in the following examples.

Example 2 Let $A = (4, 1)$ and $B = (2, -3)$. Using coordinates and parametric equations, express (1) $\overleftrightarrow{AB}$, (2) $\overrightarrow{AB}$, and (3) $\overline{AB}$.

Solution:

1. Substituting the coordinates of A and B into the equations of Theorem 11.7, we get $x = 4 - 2k$ and $y = 1 - 4k$ as parametric equations for $\overleftrightarrow{AB}$. Therefore
$$\overleftrightarrow{AB} = \{(x, y) : x = 4 - 2k, y = 1 - 4k, k \text{ is real}\}.$$
2. If P is a point on $\overleftrightarrow{AB}$, then $P \in \overrightarrow{AB}$ if and only if $k \geq 0$. Therefore
$$\overrightarrow{AB} = \{(x, y) : x = 4 - 2k, y = 1 - 4k, k \geq 0\}.$$
3. If P, A, B are three points on $\overleftrightarrow{AB}$ and if k_1, k_2, k_3 are the coordinates of P, A, B, respectively, then P is between A and B if and only if k_1 is between k_2 and k_3. Since the k-coordinates of A and B on $\overleftrightarrow{AB}$ are 0 and 1, respectively, then P is between A and B if and only if k is between 0 and 1. Therefore
$$\overline{AB} = \{(x, y) : x = 4 - 2k, y = 1 - 4k, 0 \leq k \leq 1\}.$$

Example 3 Given $A = (4, 1)$ and $B = (2, -3)$, find P on $\overrightarrow{AB}$ such that $AP = 4 \cdot AB$.

Solution: Taking $k = 4$ in the parametric equations of Example 2, we get

$$P = (x, y) = (-4, -15).$$

Example 4 Given $A = (4, 1)$ and $B = (2, -3)$, find P on *opp* $\overrightarrow{AB}$ such that $AP = 4 \cdot AB$.

Solution: Taking $k = -4$ in the parametric equations of Example 2, we get

$$P = (x, y) = (12, 17).$$

Example 5 Given $A = (4, 1)$ and $B = (2, -3)$, find C and D, the points of trisection of $\overline{AB}$.

Solution: Taking $k = \frac{1}{3}$ and $k = \frac{2}{3}$ in the parametric equations of Example 2, we get $C = (\frac{10}{3}, -\frac{1}{3})$ and $D = (\frac{8}{3}, -\frac{5}{3})$ as the points of trisection.

EXERCISES 11.6

In Exercises 1–5, the coordinates of two points A, B are given. Use parametric equations and set-builder notation to express $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, *opp* $\overrightarrow{AB}$, and $\overline{AB}$.

1. $A = (1, 4)$, $B = (3, 7)$
2. $A = (2, 2)$, $B = (5, 5)$
3. $A = (-1, 3)$, $B = (3, 0)$
4. $A = (2, 3)$, $B = (0, 0)$
5. $A = (-3, -2)$, $B = (0, 1)$

6–10. Find the coordinates of the midpoint of $\overline{AB}$ in Exercises 1–5. (*Hint:* Let $k = \frac{1}{2}$ in the parametric equations for $\overleftrightarrow{AB}$.)

11. Find the points of trisection of $\overline{AB}$ in Exercise 2.

In Exercises 12–17, a relation between AP and AB is given. If $A = (-2, -5)$ and $B = (4, 1)$, find the coordinates of P in $\overrightarrow{AB}$.

12. $AP = 2AB$
13. $AP = 25AB$
14. $4AP = 3AB$
15. $AP = \sqrt{2}AB$
16. $AP = 1.5AB$
17. $AP = \frac{1}{3}AB$

18–23. The instructions for Exercises 18–23 are the same as for Exercises 12–17, except that P is in $opp\ \overrightarrow{AB}$. Recall that $\dfrac{AP}{AB} = -k$ in this case. In Exercise 18, $AP = 2AB$ as in Exercise 12; in Exercise 19, $AP = 25AB$ as in Exercise 13; etc.

24. Find the coordinates of $P \in \overleftrightarrow{AB}$ if $A = (-1, 5)$, $B = (4, -3)$, and $AP = 3AB$. (There are two possible answers.)

25. Given $A = (0, 4)$, $B = (3, 0)$, and C is on $\overleftrightarrow{AB}$, find the y-coordinate of C if its x-coordinate is -2. (*Hint:* Obtain the parametric equations for $\overleftrightarrow{AB}$ and let $x = -2$ in one of these equations (Which one?). Solve this equation for k and use this value of k to find the y-coordinate of C.)

26. Given $A = (-1, 3)$, $B = (2, -3)$, and C is on $\overleftrightarrow{AB}$,
(a) find the y-coordinate of C if its x-coordinate is 5.
(b) find the abscissa of C if its ordinate is 8.

In Exercises 27–31, draw the graph of the set.

27. $\{(x, y) : 1 + 3k, y = 2 - k, k \text{ is real}\}$
28. $\{(x, y) : x = 3k, y = k, 0 \leq k \leq 3\}$
29. $\{(x, y) : x = -2 + k, y = -2k, k \geq 0\}$
30. $\{(x, y) : x = -k, y = k, k \leq 0\}$
31. $\{(x, y) : x = 5, y = 2 + k, -2 \leq k \leq 3\}$

Exercises 32–39 refer to the triangle whose vertices are $A = (3, 2)$, $B = (9, 4)$, and $C = (5, 8)$.

32. Find the coordinates of D, the midpoint of $\overline{AB}$.
33. Find the coordinates of E, the midpoint of $\overline{BC}$.
34. Find the coordinates of F, the midpoint of $\overline{AC}$.
35. Find the coordinates of the point R in $\overrightarrow{CD}$ such that $CR = \frac{2}{3}CD$. (Use the parametric equations for $\overleftrightarrow{CD}$, where $C = (x_1, y_1) = (5, 8)$, $D = (x_2, y_2) = (6, 3)$, and $k = \frac{2}{3}$.)
36. Find the coordinates of the point S in $\overrightarrow{BF}$ such that $BS = \frac{2}{3}BF$. (Use the parametric equations for $\overleftrightarrow{BF}$, where $B = (x_1, y_1)$, $F = (x_2, y_2)$, and $k = \frac{2}{3}$.)
37. Find the coordinates of the point T in $\overrightarrow{AE}$ such that $AT = \frac{2}{3}AE$.
38. Is $R = S = T$?
39. Show that the three medians of $\triangle ABC$ intersect in a point whose distance from each vertex is two-thirds of the length of the median from that vertex.

40. CHALLENGE PROBLEM. If a, b, c are positive numbers and if $A = (0, 0)$, $B = (a, 0)$, and $C = (b, c)$, show that the three medians of $\triangle ABC$ are concurrent at a point whose distance from each vertex is two-thirds of the length of the median from that vertex.

11.7 SLOPE

In this section we develop the idea of the *slope* of a line. Slope corresponds to the idea of the steepness of an inclined plane or the steepness of a stairway. If all the steps of a stairway are uniform, we may describe the steepness of the stairway in terms of the "rise" and "run" of one of its steps. The steepness of a uniform stairway is the number obtained by dividing the rise by the run of one of the steps. For example, we may say that the steepness of the stairway shown in Figure 11-21 is $\frac{2}{3}$.

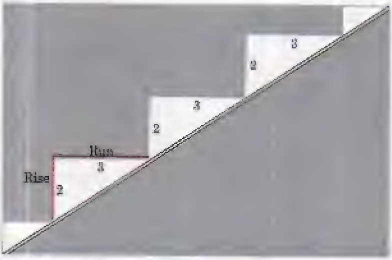


Figure 11-21

In our formal geometry, we define the steepness or slope of a line in a similar way. That is, we define the slope of a line in terms of the slope of any segment on the line and we define the slope of a segment in an xy-plane in terms of the coordinates of its endpoints. In informal geometry, we can think of the slope of a segment in somewhat the same way as we think of the slope of a step in a stairway; that is, in terms of its "rise divided by run." Figure 11-22 suggests that the slope of $\overline{AB}$ is $\frac{4}{6} = \frac{2}{3}$. Note that in terms of the coordinates (2, 3) and (8, 7) of the endpoints A and B of $\overline{AB}$, the rise is $|7 - 3| = 4$ and the run is $|8 - 2| = 6$.

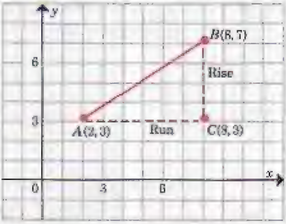


Figure 11-22

Thus, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$, where $x_1 \neq x_2$, we *could* define the slope m of $\overline{AB}$ as

$$m = \left|\frac{y_2 - y_1}{x_2 - x_1}\right|,$$

but we do not. The same formula without the absolute value symbols is not only easier to handle but it is more useful. The sign of the slope indicates whether the segment "slopes up or down." In Figure 11-22, segment $\overline{AB}$ has a positive number for its slope and we note that the segment slopes up as we view it from left to right. However, the segment $\overline{CD}$ shown in Figure 11-23 slopes down as we view it from left to right.

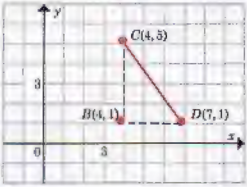


Figure 11-23

If we use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

to compute the slope of $\overline{CD}$, we obtain

$$\frac{1 - 5}{7 - 4} = -\frac{4}{3},$$

a negative number. This is one of the reasons why we do not include the absolute value symbols in our formula for the slope of a segment. That is, if the slope of a segment is a positive number, the segment slopes up when viewed from left to right, and if the slope is a negative number, the segment slopes down when viewed from left to right. We are ready now for our formal definition.

Definition 11.2 If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points and if $x_1 \neq x_2$, then the **slope** of $\overline{AB}$ is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points and $x_1 = x_2$, then $\overline{AB}$ is a vertical segment and slope is not defined for $\overline{AB}$ in this case.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points and $y_1 = y_2$, then $\overline{AB}$ is a horizontal segment and its slope is zero.

Note that in computing the slope of a segment $\overline{AB}$ it does not matter which endpoint of $\overline{AB}$ is designated (x_1, y_1) and which endpoint is designated (x_2, y_2). Thus, for the slope m of $\overline{AB}$ in Figure 11-22, we could write

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 3}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

or we could write

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 7}{2 - 8} = \frac{-4}{-6} = \frac{2}{3}.$$

Similarly, for the slope m of $\overline{CD}$ in Figure 11-23, we could write

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 5}{7 - 4} = -\frac{4}{3}$$

or

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{4 - 7} = -\frac{4}{3}.$$

As suggested earlier, the concept of the slope of a line is based on the concept of the slope of a segment. It seems reasonable that the slopes of all segments of a nonvertical line are equal, and we state this as our next theorem.

THEOREM 11.9 The slopes of all segments of a nonvertical line are equal.

Proof: Suppose that $\overleftrightarrow{AB}$ is any nonvertical line with $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then, by Theorem 11.7, $\overleftrightarrow{AB}$ can be expressed parametrically as

$$\overleftrightarrow{AB} = \{(x, y) : x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\}.$$

By Definition 11.2, the slope of $\overline{AB}$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

Let R and S be any two distinct points of $\overleftrightarrow{AB}$. There are two distinct numbers k_1 and k_2 such that

$$R = (x_1 + k_1(x_2 - x_1),\ y_1 + k_1(y_2 - y_1))$$

and

$$S = (x_1 + k_2(x_2 - x_1),\ y_1 + k_2(y_2 - y_1)).$$

By Definition 11.2, the slope of $\overline{RS}$ is

$$\frac{(y_1 + k_2(y_2 - y_1)) - (y_1 + k_1(y_2 - y_1))}{(x_1 + k_2(x_2 - x_1)) - (x_1 + k_1(x_2 - x_1))} = \frac{(y_2 - y_1)(k_2 - k_1)}{(x_2 - x_1)(k_2 - k_1)}$$
$$= \frac{y_2 - y_1}{x_2 - x_1} = m.$$

We have proved that any two segments of $\overleftrightarrow{AB}$ have the same slope, and the proof is complete.

> *Definition 11.3* The slope of a nonvertical line is equal to the slope of any of its segments. The slope of a nonvertical ray is equal to the slope of the line that contains the ray.

Example 1 Find the slope of the line which contains the points $A = (-2, 5)$ and $B = (4, 0)$.

Solution: $\overleftrightarrow{AB}$ is a nonvertical line and, by Definition 11.3, the slope of $\overleftrightarrow{AB}$ is equal to the slope of $\overline{AB}$. Therefore the slope of $\overleftrightarrow{AB}$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 5}{4 - (-2)} = -\frac{5}{6}.$$

We know that two distinct points *determine* a line in the sense that if any two distinct points are given, then there is exactly one line which contains them. It is also true that a nonvertical line is *determined* by any point on it and its slope. That is, there is exactly one line which contains a given point and has a given slope. We now state this formally.

THEOREM 11.10 Given a point A and a real number m, there is one and only one line which contains A and has slope m.

Proof:

Existence. Let a point $A = (x_1, y_1)$ and a slope m be given. Let B be the point $(x_1 + 1, y_1 + m)$. Then $\overleftrightarrow{AB}$ is a line which contains A and has slope m. (Show that the slope of $\overleftrightarrow{AB}$ is m.)

Uniqueness. Let $\overleftrightarrow{PQ}$ be any line which contains A and has slope m. Since $\overleftrightarrow{PQ}$ is a nonvertical line (Why?), it intersects the vertical line through $B(x_1 + 1, y_1 + m)$ in some point $R(x_1 + 1, y_2)$ as suggested in Figure 11-24. (The figure shows R and B as distinct points. We shall prove that they are actually the same point.) The slope of $\overleftrightarrow{PQ}$ is equal to

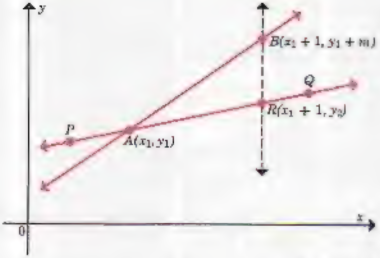


Figure 11-24

$$\frac{y_2 - y_1}{(x_1 + 1) - x_1} = \frac{y_2 - y_1}{1} = y_2 - y_1 = m.$$

Therefore $y_2 = y_1 + m$. Hence $R = B$ and $\overleftrightarrow{PQ} = \overleftrightarrow{AB}$. This proves that $\overleftrightarrow{AB}$ is the only line through A with slope m, and the proof is complete.

If we know the coordinates of two points on a line, we can use Theorem 11.7 to write parametric equations for the line. Theorem 11.10 implies that a line is determined by any point on it and its slope. Therefore we should be able to write parametric equations for a line passing through a given point and having a given slope. Our next theorem tells us how to do this.

THEOREM 11.11 The line l given by
1. $l = \{(x, y) : x = x_1 + k,\ y = y_1 + mk,\ k \text{ is real}\}$
or by
2. $l = \{(x, y) : x = x_1 + sk,\ y = y_1 + rk,\ k \text{ is real}\}$
is the line through (x_1, y_1) with slope $m = \frac{r}{s}$.

Proof: We shall establish (1) and (2) separately.

Proof of 1: Taking $k = 0$ and $k = 1$ in the parametric equations in (1), we get (x_1, y_1) and $(x_1 + 1, y_1 + m)$, two points on l. The slope of the segment joining these two points is

$$\frac{y_1 + m - y_1}{x_1 + 1 - x_1} = \frac{m}{1} = m.$$

Therefore l as given in (1) is the line through (x_1, y_1) with slope m.

Proof of 2: Taking $k = 0$ and $k = 1$ in the parametric equations in (2), we get (x_1, y_1) and $(x_1 + s, y_1 + r)$, two points on l. The slope of the segment joining these two points is

$$\frac{y_1 + r - y_1}{x_1 + s - x_1} = \frac{r}{s} = m.$$

Therefore l as given in (2) is the line through (x_1, y_1) with slope m, and the proof is complete.

Example 2 A line p passes through (2, 5) and has slope 3. Find the point on p whose abscissa is -2.

Solution: Using Theorem 11.11, we can express p as

$$p = \{(x, y) : x = 2 + k, y = 5 + 3k, k \text{ is real}\}.$$

We next set $x = -2$, obtaining $-2 = 2 + k$, or $k = -4$. Hence

$$y = 5 + 3(-4) = -7.$$

The point on p whose abscissa is -2 is $(-2, -7)$.

Example 3 Find the slope of the line

$$l = \{(x, y) : x = 2 - 3k, y = 3, k \text{ is real}\}.$$

Solution: Taking $k = 0$ and $k = 1$, we get (2, 3) and $(-1, 3)$, two points on the line. The slope of the segment joining these two points is

$$\frac{3 - 3}{-1 - 2} = \frac{0}{-3} = 0.$$

Therefore the slope of l is zero. Is l a vertical, horizontal, or oblique line?

We conclude this section with the following summary remarks regarding the slope of a line.

1. If a line has a positive number for its slope, then the line is oblique and slopes up when viewed from left to right. (See Figure 11-25a.)
2. If a line has a negative number for its slope, then the line is oblique and slopes down when viewed from left to right. (See Figure 11-25b.)
3. If a line has slope zero, then the line is horizontal. (See Figure 11-25c.)
4. Slope is not defined for a vertical line.

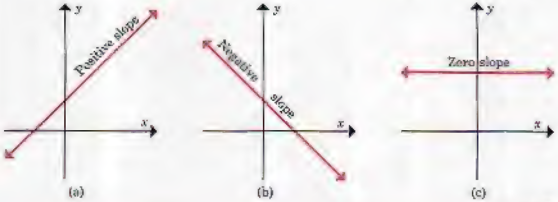


Figure 11-25

EXERCISES 11.7

In Exercises 1–10, find the slope of the segment joining the given points. Express each answer as a fraction in lowest terms.

1. (2, 5) and (5, 7)
2. (−1, −3) and (8, 3)
3. (0, −4) and (8, 0)
4. (−1, 4) and (2, −2)
5. (−3, −3) and (3, 3)
6. $(2\frac{1}{2}, -4)$ and $(-1, 4\frac{1}{2})$
7. $(-5.6, 3)$ and $(1.4, -\frac{1}{2})$
8. (1.2, −5) and (3.2, −5)
9. (2, 5) and (5, 2)
10. (7, 12) and (1, −3)

11-20. In Exercises 11–20, tell without plotting the points whether the given line (a) slopes up, (b) slopes down, or (c) is horizontal. In Exercise 11, the line is the one that passes through the two given points in Exercise 1. In Exercise 12, the line is the one that passes through the two given points in Exercise 2, and so on.

21. (a) On the same xy-plane, graph the line r through the two points of Exercise 1 and the line s through the two points of Exercise 2.
    (b) What appears to be true about these two lines, that is, how are r and s related?
    (c) How are the slopes of these two lines related?
22. (a) On the same xy-plane, graph the line p through the two points of Exercise 3 and the line q through the two points of Exercise 4.
    (b) What appears to be true about these two lines, that is, how are p and q related?
    (c) How are the slopes of these two lines related? (*Hint:* Find the product of the slope of p and the slope of q.)

In Exercises 23–27, find x or y (whichever is not given) so that the line through the two points will have the given slope.

23. $(4, 1)$ and $(x, 4)$, $m = 3$
24. $(-3, 1)$ and $(6, y)$, $m = -\frac{1}{3}$
25. $(x, 0)$ and $(-3, 5)$, $m = \frac{2}{3}$
26. $(4, -3)$ and $(0, y)$, $m = -\frac{5}{2}$
27. $(5, 12)$ and $(-2, y)$, $m = 0$

28. (a) On the same xy-plane, graph the line l through the two points of Exercise 25 and the line n through the two points of Exercise 26.
(b) What appears to be true about these two lines, that is, how are l and n related?
(c) How are the slopes of l and n related?

29. (a) Plot the quadrilateral $ABCD$ with vertices $A = (-2, -3)$, $B = (3, -2)$, $C = (4, 2)$, and $D = (-1, 1)$.
(b) Which pairs of sides have the same slope?
(c) Is $ABCD$ a parallelogram?

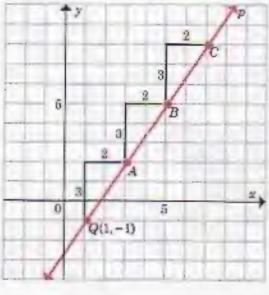
30. Write parametric equations for the line l through $(-2, 5)$ with slope 2.

31. Write parametric equations for the line p through $(1, -1)$ with slope $\frac{3}{2}$.

32. Use the parametric equations for line l obtained in Exercise 30 to find the coordinates of at least one more point on l and plot the graph of l.

33. Use the parametric equations for the line p obtained in Exercise 31 to find the coordinates of at least one more point on p and plot the graph of p.

34. One way to plot the graph of line p in Exercise 31 is suggested by the figure. Note that the point $Q(1, -1)$ is on p and the "slope fraction" $\frac{3}{2}$ tells us how to get from one point to another on p. The numerator 3 represents the difference in the ordinates of two points on p, and the denominator 2 represents the difference in the abscissas of the same two points. Thus, if we begin at the point $(1, -1)$, which is known to be on p, we arrive at another point on p by increasing the ordinate and abscissa of $(1, -1)$ by 3 and 2, respectively. If the slope is negative, it can be written as a fraction with a negative numerator and a positive denominator; for example, $\frac{-3}{2}$. In this example we could get from one point to another, in informal geometry, by moving 2 to the right and 3 down; in formal geometry, by adding 2 to the abscissa and subtracting 3 from the ordinate. Write the coordinates of the points A, B, C.



35. Use the method described in Exercise 34 to plot the graph of the line through $(-2, 1)$ with slope $\frac{2}{3}$.
36. Use the method described in Exercise 34 to plot the graph of the line through $(0, 6)$ with slope $-\frac{3}{4}$.
37. Use the method described in Exercise 34 to plot the graph of the line through $(0, 0)$ with slope 2. (*Hint:* $2 = \frac{2}{1}$.)

11.8 OTHER EQUATIONS OF LINES

In Section 11.6 we showed how a line can be expressed in terms of parametric equations. In this section we shall show how parametric equations for a line can be used to obtain other equations of the line. You are already familiar with most of these equations from your work in algebra. Perhaps the simplest equations for lines are those for horizontal and vertical lines.

THEOREM 11.12 If l is the horizontal line through (x_1, y_1), then

$$l = \{(x, y) : y = y_1\}.$$

Proof: Let (x_2, y_2) be any other point on the horizontal line l through the point (x_1, y_1). By Theorem 11.7,

$$l = \{(x, y) : x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\}.$$

Line l is horizontal so $y_1 = y_2$. Therefore

$$y = y_1 + k(y_2 - y_1) = y_1 + k \cdot 0 = y_1.$$

For any real number k,

$$x = x_1 + k(x_2 - x_1)$$

is a real number. Conversely, for every real number x, there is a real number k such that $x = x_1 + k(x_2 - x_1)$. The one and only k that works here is

$$k = \frac{x - x_1}{x_2 - x_1}.$$

It follows that l is the set of *all* points (x, y) such that $y = y_1$ and x is a real number, that is, that

$$l = \{(x, y) : y = y_1\}.$$

This completes the proof.

In the statement of Theorem 11.12 it should be clear that we could say that

$$l = \{(x, y) : x = x_1 + k(x_2 - x_1), y = y_1, k \text{ is real}\},$$

or

$$l = \{(x, y) : x \text{ is real and } y = y_1\},$$

or

$$l = \{(x, y : y = y_1, x \text{ is real}\}.$$

Since the equation $x = x_1 + k(x_2 - x_1)$ establishes a one-to-one correspondence between the set of all real numbers thought of as k-values and the set of all real numbers thought of as x-values, these set-builder symbols all denote the same set.

THEOREM 11.13 If l is the vertical line through (x_1, y_1), then

$$l = \{(x, y) : x = x_1\}.$$

Proof: Assigned as an exercise.

Example 1 If $P = (-4, 3)$, write an equation of (1) the vertical line through P and (2) the horizontal line through P.

Solution:

1. An equation of the vertical line through P is $x = -4$.
2. An equation of the horizontal line through P is $y = 3$.

Our next theorem tells us how to write an equation of a line if we know the coordinates of any two distinct points on the line.

THEOREM 11.14 (***The Two-Point Form***) If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are distinct points, and if $\overleftrightarrow{AB}$ is an oblique line, then

$$\overleftrightarrow{AB} = \left\{(x, y) : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}\right\}.$$

Proof: We are given that $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are distinct points on oblique line $\overleftrightarrow{AB}$. There are two things to prove.

1. If $P = (x, y)$ is a point of $\overleftrightarrow{AB}$, then $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$.

2. If $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$, then $P = (x, y)$ is a point of $\overleftrightarrow{AB}$.

Proof of 1: Suppose that $P = (x, y)$ and that P is a point on $\overleftrightarrow{AB}$. If $P = A$, then

$$x = x_1, \qquad y = y_1, \qquad \frac{x - x_1}{x_2 - x_1} = 0,$$

$$\frac{y - y_1}{y_2 - y_1} = 0, \qquad \text{and} \qquad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

If $P \neq A$, then by Theorem 11.9 the slope of $\overline{AP}$ equals the slope of $\overline{AB}$ since $\overline{AP}$ and $\overline{AB}$ are segments (not necessarily distinct) of the same line $\overleftrightarrow{AB}$. But the slope of $\overline{AB}$ is $\dfrac{y_2 - y_1}{x_2 - x_1}$ and the slope of $\overline{AP}$ is $\dfrac{y - y_1}{x - x_1}$. Therefore

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}.$$

Multiplying both sides of this last equation by $x - x_1$, then dividing both sides by $y_2 - y_1$, then simplifying, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

Therefore, in all cases, whether $A = P$ or $A \neq P$, if $P \in \overleftrightarrow{AB}$, then

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

Proof of 2: Suppose that $P = (x, y)$ and that

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

Either $x - x_1 = 0$ or $x - x_1 \neq 0$. If $x - x_1 = 0$, then $y - y_1 = 0$, $(x, y) = (x_1, y_1)$, $P = A$, and $P \in \overleftrightarrow{AB}$. If $x - x_1 \neq 0$, multiply both sides of

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \qquad \text{by} \qquad \frac{y_2 - y_1}{x - x_1}$$

to get

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}.$$

Since $\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope of $\overline{AB}$ and $\dfrac{y - y_1}{x - x_1}$ is the slope of $\overline{AP}$, this proves that if $x - x_1 \neq 0$, then the slopes of $\overline{AB}$ and $\overline{AP}$ are the same and, as we shall prove, $P \in \overleftrightarrow{AB}$.

Suppose, contrary to what we assert, that the slopes of $\overline{AB}$ and $\overline{AP}$ are equal, but that A, B, P are noncollinear as in Figure 11-26.

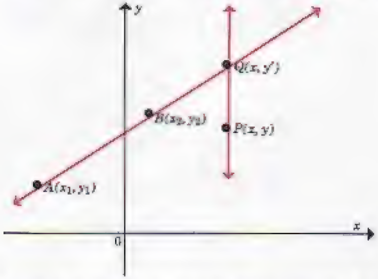


Figure 11-26

Let $Q(x, y')$ be the point in which the vertical line through P intersects $\overleftrightarrow{AB}$. It follows from Theorem 11.9 that the slopes of $\overline{AQ}$ and $\overline{AB}$ are equal, so the slope of $\overline{AQ}$ is equal to the slope of $\overline{AP}$, and

$$\frac{y' - y_1}{x - x_1} = \frac{y - y_1}{x - x_1},$$
$$y' - y_1 = y - y_1,$$
$$y' = y,$$
$$Q = P,$$

and the points A, B, P are collinear. Since our supposition that A, B, P are noncollinear leads to a contradiction, this proves that our supposition is false. Therefore A, B, P are collinear and P lies on $\overleftrightarrow{AB}$. This completes the proof in all cases, whether $x - x_1 = 0$ or $x - x_1 \neq 0$, that if

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

and $P = (x, y)$, then $P \in \overleftrightarrow{AB}$. This completes the proof of Theorem 11.14.

The equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

in the statement of Theorem 11.14 is often written in the form

(1) $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Show how Equation (1) is obtained from the one in the statement of Theorem 11.14. Since $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are distinct points on the oblique line $\overleftrightarrow{AB}$ of Theorem 11.14, then

$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

is the slope of $\overleftrightarrow{AB}$. Substituting m for $\frac{y_2 - y_1}{x_2 - x_1}$ in Equation (1), we obtain

$$(2) \qquad y - y_1 = m(x - x_1)$$

for $\overleftrightarrow{AB}$. It should be noted that Equation (2) still holds if $\overleftrightarrow{AB}$ is a horizontal line since, in this case, $m = 0$ and Equation (2) reduces to $y = y_1$. Thus, if we know the slope of a line and the coordinates of any point on the line, we can write an equation of the line. When an equation of a line is written in the form of Equation (2), it is often referred to as the *point slope form*. We have proved the following theorem.

THEOREM 11.15 (***The Point Slope Form***) If l is the line through $A = (x_1, y_1)$ with slope m, then

$$l = \{(x, y) : y - y_1 = m(x - x_1)\}$$

Example 1 If $A = (-2, 3)$ and $B = (4, 6)$, write an equation of $\overleftrightarrow{AB}$.

Solution: Substituting the coordinates of A and B in the Two-Point Form of an equation, we obtain

$$\frac{x - (-2)}{4 - (-2)} = \frac{y - 3}{6 - 3}$$

or

$$\frac{x + 2}{6} = \frac{y - 3}{3}$$

as an equation of $\overleftrightarrow{AB}$.

If we multiply both sides of the equation in Example 1 by 6, and add $-2y + 6$ to both sides, we obtain

$$x - 2y + 8 = 0,$$

which is of the form

$$Ax + By + C = 0,$$

with $\qquad A = 1, B = -2, \text{ and } C = 8.$

This latter form is often referred to as the *general form* of a *linear equation*, that is, of an equation whose graph is a line. Although we shall not do so here, it can be proved using the theorems of this chapter that if A, B, C are real numbers with A and B not both zero, then $Ax + By + C = 0$ is an equation of a line. The converse statement is also true, that is, every line has an equation of the form $Ax + By + C = 0$ in which A, B, C are real and A and B are not both zero. What is the graph of $Ax + By + C = 0$ if $A = 0, B = 0, C \neq 0$? if $A = 0, B = 0, C = 0$?

Example 2 Write an equation of the line through $(3, -4)$ with slope $-\frac{2}{3}$ and put the equation in general form.

Solution: Substituting $(3, -4)$ for (x_1, y_1) and $-\frac{2}{3}$ for m in the Point Slope Form of an equation, we obtain

$$y - (-4) = -\tfrac{2}{3}(x - 3)$$

or

$$y + 4 = -\tfrac{2}{3}(x - 3).$$

Multiplying both sides of this last equation by 3 and adding $2x - 6$ to both sides, we obtain $2x + 3y + 6 = 0$ as an equation of the line in general linear form.

Every nonvertical line intersects the y-axis (Why?) in a point whose coordinates are $(0, b)$, where b is a real number. The number b is often called *the y-intercept* of the line. Similarly, every nonhorizontal line intersects the x-axis in a point whose coordinates are $(a, 0)$, where a is a real number. The number a is called *the x-intercept* of the line. It should be clear that the x- and y-intercepts of a line can be obtained from an equation of the line as follows:

1. If the line is a nonhorizontal line, the x-intercept is obtained by substituting 0 for y in an equation of the line and solving the resulting equation for x.
2. If the line is a nonvertical line, the y-intercept is obtained by substituting 0 for x in an equation of the line and solving the resulting equation for y.

Example 3 Find the x- and y-intercepts of the line whose equation is

$$3x - 4y + 8 = 0.$$

Solution: Substituting 0 for y in the equation $3x - 4y + 8 = 0$, we obtain $3x + 8 = 0$, or $x = -2\frac{2}{3}$. Hence the x-intercept is $-2\frac{2}{3}$. Substituting 0 for x in the equation, we obtain $-4y + 8 = 0$, or $y = 2$. Hence the y-intercept is 2.

If we substitute $(0, b)$ for (x_1, y_1) in the Point Slope Form of an equation of a line, we obtain

$$y - b = m(x - 0)$$

or

$$y = mx + b.$$

It should be clear that when an equation of a nonvertical line is put in the form $y = mx + b$, the slope and y-intercept of the line can be read directly from the equation. This form, which is often called the slope y-intercept form, is especially convenient when one wants to draw the graph of a line whose equation is given.

Example 4 An equation of the line l is $3x + 2y - 8 = 0$.

1. Put the equation in slope y-intercept form.
2. Draw the graph of l on an xy-plane.

Solution:

1. To put the equation $3x + 2y - 8 = 0$ in slope y-intercept form we add $-3x + 8$ to both sides and then divide both sides by 2. The resulting equation is $y = -\frac{3}{2}x + 4$.
2. The graph of l is shown in Figure 11-27. Note that the slope and y-intercept of l can be read directly from the equation $y = -\frac{3}{2}x + 4$. Thus, if we start at the point where $y = 4$ on the y-axis [that is, the point $(0, 4)$] and "trace out" a slope of $-\frac{3}{2}$, we arrive at a second point on l. These two points determine l and hence they determine the graph of l.

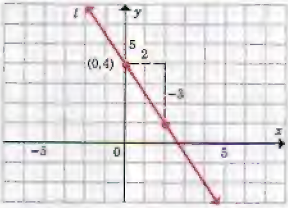


Figure 11-27

As you might expect, two nonvertical lines are parallel to each other if and only if they have the same slope. If you worked Exercises 22 and 28 of Section 11.7, you should have discovered a relationship between the slopes of two lines that are oblique and perpendicular. We state these properties of parallel nonvertical lines and of perpendicular oblique lines as the last two theorems of this section.

THEOREM 11.16 Two nonvertical lines are parallel if and only if their slopes are equal.

Proof: Let two nonvertical lines r and s be given. We have two things to prove.

1. If $r \parallel s$, then their slopes are equal.
2. If the slopes of r and s are equal, then $r \parallel s$.

Proof of 1: Suppose that r and s are nonvertical lines and that $r \parallel s$. If $r = s$, then r and s are the same line and hence they have the same slope. Suppose $r \neq s$, as shown in Figure 11-28. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points on r. Let the vertical lines through P and Q intersect s in $P'(x_1, y_1 + h)$ and $Q'(x_2, y_2 + k)$, respectively.

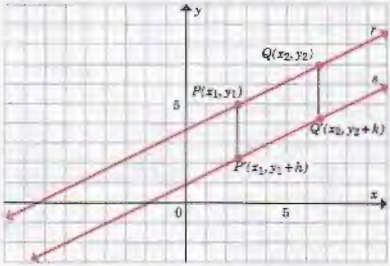


Figure 11-28

Then $PP'Q'Q$ is a parallelogram (Why?) and $PP' = QQ'$. But $PP' = |h|$ and $QQ' = |k|$. Why? Therefore $|h| = |k|$. Since h and k are either both positive or both negative, we have $h = k$. The slope of $\overline{PQ}$ (and hence of r) is $\dfrac{y_2 - y_1}{x_2 - x_1}$. The slope of $\overline{P'Q'}$ (and hence of s) is

$$\frac{(y_2 + k) - (y_1 + h)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

since $h = k$. Therefore the slopes of r and s are equal.

Proof of 2: Suppose that r and s are nonvertical lines and that their slopes are equal. If $r = s$, then by definition $r \parallel s$. Suppose that $r \neq s$. Let m be the slope of r. Then s also has slope m. Now, either r and s are

parallel or they are not. Suppose they are not parallel. Then they have exactly one point $P(x_1, y_1)$ in common. Thus we have two distinct lines passing through the same point and having the same slope. This contradicts Theorem 11.10. Therefore our supposition that r and s are not parallel is incorrect; hence it follows that $r \parallel s$, and the proof is complete.

THEOREM 11.17 Two oblique lines are perpendicular if and only if the product of their slopes is -1.

Proof. Let l_1 and l_2 be two given oblique lines with slopes m_1 and m_2, respectively. We have two statements to prove.

1. If $l_1 \perp l_2$, then $m_1 \cdot m_2 = -1$.
2. If $m_1 \cdot m_2 = -1$, then $l_1 \perp l_2$.

Before proceeding with the proof we wish to comment on the adjective "oblique" in the statement of the theorem. Would the statement that results if "oblique" is erased be a theorem? No, it would not. For if one of two lines is not oblique, then those two lines are perpendicular if and only if one of them is horizontal and the other is vertical. Since a vertical line has no slope, there would be no product of slopes in this case.

We now proceed to the proof of statements 1 and 2.

Let p_1 and p_2 be the lines through $(0, 0)$ and parallel to l_1 and l_2, respectively, as shown in Figure 11-29. Then the slope of p_1 is m_1 and the slope of p_2 is m_2. Why? Since neither p_1 nor p_2 is a vertical line, they both intersect the line

$$l = \{(x, y) : x = 1\}.$$

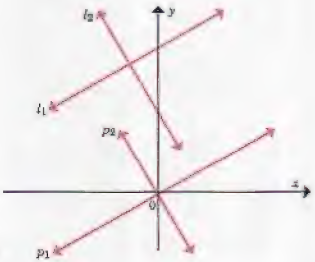


Figure 11-29

Let $A = (1, y_1)$ and $B = (1, y_2)$ be the points of intersection of l with lines p_1 and p_2, respectively, as shown in Figure 11-30.

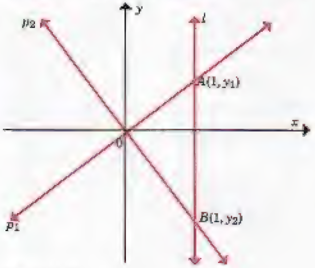


Figure 11-30

Then the slope of p_1 is

$$m_1 = \frac{y_1 - 0}{1 - 0} = y_1,$$

and the slope of p_2 is

$$m_2 = \frac{y_2 - 0}{1 - 0} = y_2.$$

Therefore $A = (1, m_1)$ and $B = (1, m_2)$. Now, $l_1 \perp l_2$ if and only if $p_1 \perp p_2$.

It follows from the Pythagorean Theorem and its converse (applied to $\triangle OAB$ in Figure 11-30) that

$$p_1 \perp p_2 \quad \text{if and only if} \quad (OA)^2 + (OB)^2 = (AB)^2.$$

From the distance formula, we get

$$(OA)^2 = 1 + m_1^2,$$
$$(OB)^2 = 1 + m_2^2,$$

and

$$(AB)^2 = (m_2 - m_1)^2.$$

Thus

$$\begin{aligned} p_1 \perp p_2 \quad & \text{if and only if} \quad 1 + m_1^2 + 1 + m_2^2 = (m_2 - m_1)^2, \\ & \text{if and only if} \quad 2 + m_1^2 + m_2^2 = m_2^2 - 2m_1m_2 + m_1^2, \\ & \text{if and only if} \quad 2 = -2m_1m_2, \\ & \text{if and only if} \quad m_1m_2 = -1. \end{aligned}$$

Therefore $l_1 \perp l_2$ if and only if $m_1m_2 = -1$, and the proof is complete.

Example 5 If

$$l_1 = \{(x, y) : 5x - 2y + 4 = 0\},$$
$$l_2 = \{(x, y) : 2x + 5y - 15 = 0\},$$

and

$$l_3 = \{(x, y) : 5x - 2y - 8 = 0\},$$

show that $l_1 \parallel l_3$ and that $l_1 \perp l_2$.

Solution: Putting the equations of l_1, l_2, l_3 in slope y-intercept form, we get

$$l_1 = \{(x, y) : y = \tfrac{5}{2}x + 2\},$$
$$l_2 = \{(x, y) : y = -\tfrac{2}{5}x + 3\},$$

and

$$l_3 = \{(x, y) : y = \tfrac{5}{2}x - 4\}.$$

Since $\frac{5}{2}$ = slope of l_1 = slope of l_3, it follows that $l_1 \parallel l_3$. Also, since the product of the slopes of l_1 and l_2 equals

$$\tfrac{5}{2} \cdot (-\tfrac{2}{5}) = -1,$$

it follows that $l_1 \perp l_2$. It is also true that $l_2 \perp l_3$. Why?

EXERCISES 11.8

1. Prove Theorem 11.13.

In Exercises 2–6, use the Two-Point Form to write an equation of the line containing the given points, and put each equation in general form (that is, the form $Ax + By + C = 0$).

2. (1, 5) and (3, 4)
3. (0, 3) and (−5, 0)
4. (0, −3) and (5, 0)
5. (0, 0) and (−1, 6)
6. (−2, 2) and (2, −2)

In Exercises 7–12, use the Point Slope Form to write an equation of the line which contains the given point and has the given slope, and put each equation in general form.

7. (3, 5) and $m = 1$
8. (−2, 1) and $m = -1$
9. (0, 0) and $m = \frac{3}{5}$
10. (5, 0) and $m = -\frac{5}{3}$
11. (−3, −7) and $m = -1$
12. (−5, −3) and $m = \frac{3}{5}$

In Exercises 13–16, determine which word, parallel or perpendicular, would make a true statement.

13. The lines of Exercises 7 and 8 are $\boxed{?}$.
14. The lines of Exercises 8 and 11 are $\boxed{?}$ and distinct.
15. The lines of Exercises 9 and 12 are $\boxed{?}$. Are these two lines distinct?
16. The lines of Exercises 9 and 10 are $\boxed{?}$.

17. Write an equation (in general form) of the line which contains the point $(3, 8)$ and is parallel to the line whose equation is $2x - 3y = 10$. (*Hint:* What is the slope of the line whose equation is $2x - 3y = 10$?)
18. Write an equation (in general form) of the line which contains the point $(3, 8)$ and is perpendicular to the line whose equation is $2x - 3y = 10$.
19. Write an equation (in general form) of the line which contains the origin and is perpendicular to the line whose equation is $y = x$.

In Exercises 20–25, an equation of a line is given. In each exercise, (a) put the equation in slope y-intercept form, (b) find the slope of the line, and (c) find the x- and y-intercepts of the line.

20. $2x - 3y - 12 = 0$
21. $3x + 2y = 16$
22. $4x - 6y = -8$
23. $y + x = 0$
24. $4x - 2y = 11$
25. $5x + 4y + 13 = 0$

26. Which pairs, if any, of the lines of Exercises 20–25 are parallel? Which pairs, if any are perpendicular?
27. Find, without graphing, the coordinates of the point of intersection of the lines

$$p = \{(x, y) : x + 3y = 6\}$$

and

$$q = \{(x, y) : 5x + 4y = -3\}.$$

(*Hint:* Note that p is not parallel to q (show this) and hence the lines intersect in exactly one point.) Let (x_1, y_1) be the point of intersection of lines p and q. Then, since (x_1, y_1) is on p, $x_1 + 3y_1 = 6$ or

(1) $$y_1 = -\tfrac{1}{3}x_1 + 2.$$

Also since (x_1, y_1) is on q, $5x_1 + 4y_1 = -3$ or

(2) $$y_1 = -\tfrac{5}{4}x_1 - \tfrac{3}{4}.$$

Apply the substitution property of equality to Equations (1) and (2) and find x_1 and y_1.

28. Show without graphing that lines

$$p = \{(x, y) : 2x - 3y = 12\}$$

and

$$q = \{(x, y) : 4y + x = -5\}$$

are not parallel, and find the coordinates of their point of intersection. (See Exercise 27.)

Exercises 29–39 refer to the triangle whose vertices are $A = (1, 1)$, $B = (9, 3)$, and $C = (7, 9)$. In each case where an equation of a line is asked for, write the equation in slope y-intercept form. (It may help to draw the figure in an xy-plane and label the points as you need them.)

29. Find the coordinates of the midpoint D of $\overline{AB}$.
30. Find the coordinates of the midpoint E of $\overline{BC}$.
31. Find the coordinates of the midpoint F of $\overline{AC}$.
32. Write an equation of the line through A and E.
33. Write an equation of the line through B and F.
34. Write an equation of the line through C and D.
35. Show that the lines $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{CD}$ intersect in the same point.
36. Write an equation of the line through E and F.
37. Write an equation of the line through A and B.
38. Show that $\overleftrightarrow{EF} \parallel \overleftrightarrow{AB}$ and hence that $\overline{EF} \parallel \overline{AB}$.
39. Show that $EF = \frac{1}{2}AB$.

40. If $a \neq 0$ and $b \neq 0$, an equation of the form $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is called the *intercept form* of an equation of a line. Show that the line whose equation is $\dfrac{x}{a} + \dfrac{y}{b} = 1$ contains the points $(a, 0)$ and $(0, b)$.
41. Put the equation $3x + 4y = 12$ in intercept form (see Exercise 40) and read the x- and y-intercepts directly from the equation.
42. Given that $p = \{(x, y) : x = 3\}$.

    (a) Is $(3, 7) \in p$?
    (b) Is $(3, 17) \in p$?
    (c) Is $(4, 4) \in p$?
    (d) Is $(3, -9) \in p$?

43. Given that $q = \{(x, y) : y = -4\}$:

    (a) Is $(-4, 5) \in q$?
    (b) Is $(-4, \sqrt{3}) \in q$?
    (c) Is $(-4, -\pi) \in q$?
    (d) Is $(4, -4) \in q$?

44. CHALLENGE PROBLEM. Use theorems of this chapter to prove the following statement: If A, B, C are real numbers with A and B not both zero, then $Ax + By + C = 0$ is an equation of a line.

45. CHALLENGE PROBLEM. Use theorems of this chapter to prove the following statement: Every line has an equation of the form $Ax + By + C = 0$ in which A, B, C are real and A and B are not both zero. (Note that this statement is the converse of the statement in Exercise 44.)

11.9 PROOFS USING COORDINATES

We have defined an xy-coordinate system in a plane and have used coordinates as tools in much of our work in this chapter. Given a plane, there are many xy-coordinate systems in that plane. In constructing a proof of a geometric theorem, it is wise to select a convenient xy-coordinate system that fits the problem and, at the same time, reduces the number of symbols needed in the proof. Such a selection yields no loss of generality, yet reduces the amount and difficulty of work involved. We illustrate with our next theorem which appeared as a corollary in Chapter 10.

THEOREM 11.18 A segment which joins the midpoints of two sides of a triangle is parallel to the third side and has half the length of the third side.

We shall give two proofs of Theorem 11.18. In the first proof, we select an arbitrary xy-coordinate system in the plane of the triangle without any regard to the position of the vertices and sides of the given triangle. In our second proof, we "pick" an xy-coordinate system in the plane of the triangle in such a way as to reduce the number of symbols needed in the proof.

Proof I: Let $\triangle ABC$ in any xy-plane be given. (See Figure 11-31.)

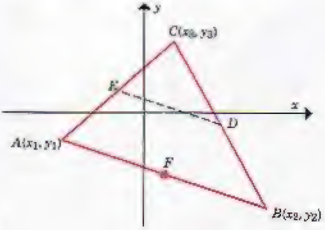


Figure 11-31

Suppose that

$$A = (x_1, y_1), \qquad B = (x_2, y_2), \qquad C = (x_3, y_3).$$

Let D, E, F be the midpoints of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Then

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right) \quad \text{and} \quad E = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right). \qquad \text{Why?}$$

Using the Distance Formula, we get

$$\begin{aligned}(DE)^2 &= \left(\frac{x_2 + x_3}{2} - \frac{x_1 + x_3}{2}\right)^2 + \left(\frac{y_2 + y_3}{2} - \frac{y_1 + y_3}{2}\right)^2,\\ &= \left(\tfrac{1}{2}(x_2 - x_1)\right)^2 + \left(\tfrac{1}{2}(y_2 - y_1)\right)^2,\\ &= \tfrac{1}{4}[(x_2 - x_1)^2 + (y_2 - y_1)^2].\end{aligned}$$

But, by the same formula,

$$(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Therefore

$$(DE)^2 = \tfrac{1}{4}(AB)^2,$$

or

$$DE = \tfrac{1}{2}(AB).$$

Next, we prove that $\overline{DE} \parallel \overline{AB}$. Suppose that $\overline{AB}$ is a vertical segment. Then,

$$x_1 = x_2 \qquad \text{and} \qquad \frac{x_2 + x_3}{2} = \frac{x_1 + x_3}{2}.$$

Therefore

$$D = \left(\frac{x_1 + x_3}{2}, \frac{y_2 + y_3}{2}\right),$$

$$E = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right),$$

and $\overline{DE}$ is a vertical segment. Hence $\overline{DE} \parallel \overline{AB}$. If $\overline{AB}$ is not a vertical segment, then its slope is $\dfrac{y_2 - y_1}{x_2 - x_1}$. The slope of $\overline{DE}$ is

$$\frac{\dfrac{y_2 + y_3}{2} - \dfrac{y_1 + y_3}{2}}{\dfrac{x_2 + x_3}{2} - \dfrac{x_1 + x_3}{2}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Therefore, since $\overline{DE}$ and $\overline{AB}$ have the same slope, $\overline{DE} \parallel \overline{AB}$. This completes the proof so far as the segment $\overline{DE}$ is concerned. In a similar way, we can prove that $EF = \frac{1}{2}BC$ and $\overline{EF} \parallel \overline{BC}$ and that $DF = \frac{1}{2}AC$ and $\overline{DF} \parallel \overline{AC}$.

Proof II: Let $\triangle ABC$ be given. In the plane of this triangle there is an xy-coordinate system with the origin at A, with $\overleftrightarrow{AB}$ as the x-axis, with the x-coordinate of B positive, and with the y-coordinate of C positive. (See Figure 11-32.)

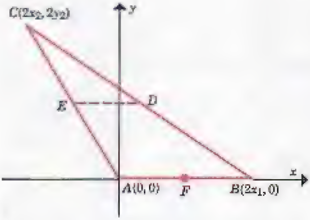


Figure 11-32

Let x_1, x_2, y_2 be real numbers such that $B = (2x_1, 0)$, $C = (2x_2, 2y_2)$. Let D, E, F be the midpoints of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Then

$$D = (x_1 + x_2, y_2), \qquad E = (x_2, y_2),$$

the slope of $\overline{DE} = 0$, the slope of $\overline{AB} = 0$,

$$DE = |(x_1 + x_2) - x_2| = |x_1| = x_1,$$

and

$$AB = |2x_1| = 2x_1.$$

Therefore $\overline{DE} \parallel \overline{AB}$ and $DE = \frac{1}{2}AB$. This completes the proof for $\overline{DE}$ and this is all that we need to prove, since $\overline{DE}$ might be any one of the three segments which joins the midpoints of two sides of the given triangle.

To prove the statement of the theorem for the segment which joins the midpoints of two sides, we first label the triangle so that $\overline{AC}$ and $\overline{CB}$ are those two sides and then proceed as above. Thus each of the three parts of the proof uses a different coordinate system, but what we write in each case is the same. For example, Figure 11-33 shows another picture of the triangle shown in Figure 11-32. However, it shows a different xy-coordinate system and a different labeling of the vertices. Vertex C in Figure 11-32 becomes vertex A in Figure 11-33, A becomes B, and B becomes C, and we have made the x-axis look horizontal.

You should note that the proof that $DE = \frac{1}{2}AB$ and $\overline{DE} \parallel \overline{AB}$ would proceed exactly as before, but $\overline{DE}$ in Figure 11-33 is not the same segment $\overline{DE}$ as in Figure 11-32. The same applies for the segment $\overline{AB}$.

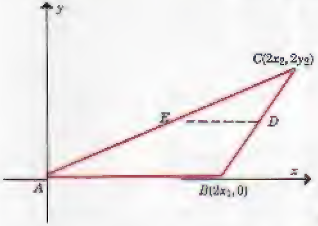


Figure 11-33

It is clear that Proof II is simpler than Proof I and is to be preferred. In general, if a proof using coordinates involves a polygon, it is usually easier to construct a proof if we select an xy-coordinate system in the plane of the polygon satisfying one or both of the following conditions:

1. Let the origin be a vertex of the polygon and let the positive part of the x-axis contain one of the sides of the polygon.
2. If the polygon contains a right angle as one of its angles, let the origin be the vertex of the right angle and let the positive parts of the x- and y-axes contain the sides of the right angle.

THEOREM 11.19 The medians of a triangle are concurrent in a point (centroid) which is two-thirds of the distance from each vertex to the midpoint of the opposite side.

Proof: Let $\triangle ABC$ be given. Select an xy-coordinate system in the plane of this triangle with the origin at A, with $\overleftrightarrow{AB}$ as the x-axis, with the abscissa of B positive, and with the ordinate of C positive. (See Figure 11-34.) Let a, b, c be numbers such that $B = (6a, 0)$, $C = (6b, 6c)$. Let D, E, F be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively.

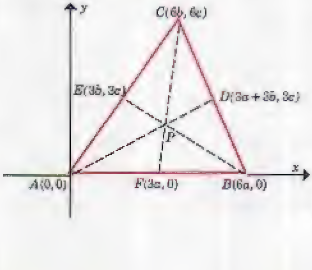


Figure 11-34

Then $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are the medians of $\triangle ABC$. We must prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent at some point P and that $AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BE$, and $CP = \frac{2}{3}CF$.

The midpoint of $\overline{AB}$ is $F(3a, 0)$. We can express $\overline{CF}$ parametrically as follows:

$$\overline{CF} = \{(x, y) : x = 6b + (3a - 6b)k, y = 6c + (0 - 6c)k, 0 \le k \le 1\}.$$

The point P on $\overline{CF}$ such that $CP = \frac{2}{3}CF$ can be obtained by setting $k = \frac{2}{3}$ in the parametric equations for $\overline{CF}$. Thus

$$\begin{aligned} x &= 6b + (3a - 6b) \cdot \tfrac{2}{3} \\ &= 6b + 2a - 4b \\ &= 2b + 2a \end{aligned}$$

and

$$\begin{aligned} y &= 6c + (0 - 6c) \cdot \tfrac{2}{3} \\ &= 6c - 4c \\ &= 2c. \end{aligned}$$

Therefore

$$P = (2a + 2b, 2c).$$

Similarly, the midpoint of $\overline{BC}$ is

$$D = (3a + 3b, 3c)$$

and

$$\begin{aligned} \overline{AD} = \{(x, y) : x &= 0 + (3a + 3b - 0)k, \\ y &= 0 + (3c - 0)k, 0 \le k \le 1\}. \end{aligned}$$

The point P' on $\overline{AD}$ such that $AP' = \frac{2}{3}AD$ is obtained by setting $k = \frac{2}{3}$. Thus

$$\begin{aligned} x &= (3a + 3b) \cdot \tfrac{2}{3} \\ &= 2a + 2b \end{aligned}$$

and

$$\begin{aligned} y &= (3c) \cdot \tfrac{2}{3} \\ &= 2c. \end{aligned}$$

Therefore

$$P' = (2a + 2b, 2c).$$

The midpoint of $\overline{CA}$ is $E = (3b, 3c)$ and

$$\overline{BE} = \{(x, y) : x = 6a + (3b - 6a)k, y = 0 + (3c - 0)k, 0 \le k \le 1\}.$$

The point P'' on $\overline{BE}$ such that $BP'' = \frac{2}{3}BE$ is obtained by setting $k = \frac{2}{3}$.

Thus

$$\begin{aligned} x &= 6a + (3b - 6a) \cdot \tfrac{2}{3} \\ &= 6a + 2b - 4a \\ &= 2a + 2b \end{aligned}$$

and

$$\begin{aligned} y &= (3c) \cdot \tfrac{2}{3} \\ &= 2c. \end{aligned}$$

Therefore

$$P'' = (2a + 2b, 2c).$$

We have shown that

$$P = P' = P'' = (2a + 2b, 2c),$$

that $CP = \frac{2}{3}CF$, $AP = \frac{2}{3}AD$, and that $BP = \frac{2}{3}BE$. Therefore the medians of a triangle are concurrent at a point which is two-thirds of the distance from each vertex to the midpoint of the opposite side, and the proof is complete.

In the second sentence of this proof we could have taken a, b, c as real numbers such that $B = (a, 0)$ and $C = (b, c)$. The resulting expressions for the coordinates of D, E, F, and P would have involved many fractions. We avoided these fractions by taking a, b, c so that $B = (6a, 0)$ and $C = (6b, 6c)$.

You may feel that Theorem 11.19 would be easier to prove without using coordinates. It is possible to construct such a proof using the theorems, postulates, and definitions that we have established before this chapter. You might be interested in trying to do so.

As indicated in the statement of the theorem the point of intersection of the medians of a triangle is its centroid. In informal geometry we think of it as the balance point; it is the point where a cardboard triangular region of uniform thickness balances. In calculus the idea of moments of mass (extending the idea of weight times distance in teeter-totter exercises) is introduced and extended to develop a theory of centroids for plane figures. The centroid of a triangle is an example of a centroid as the concept is developed formally in calculus.

Our last example of this section is a theorem that you will find helpful in working some of the exercises at the end of the section.

THEOREM 11.20 Let quadrilateral $ABCD$ with $A = (0, 0)$, $B = (a, 0)$, $D = (b, c)$ be given. $ABCD$ is a parallelogram if and only if $C = (a + b, c)$.

Figure 11-35 shows one possible orientation of the given quadrilateral $ABCD$ in an xy-plane. However, our proof depends only on A being at the origin and B being on the x-axis as given in the theorem. We must prove two things.

1. If $C = (a + b, c)$, then $ABCD$ is a parallelogram.
2. If $ABCD$ is a parallelogram, then $C = (a + b, c)$.

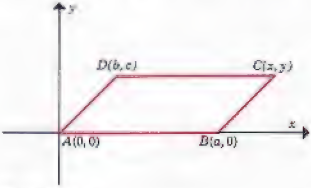


Figure 11-35

Proof of 1: If $C = (a + b, c)$, the slope of $\overline{CD}$ is

$$\frac{c - c}{a + b - b} = \frac{0}{a} = 0.$$

Also, the slope of $\overline{AB} = \frac{0}{a} = 0$. Therefore $\overline{CD} \parallel \overline{AB}$. Since $\overline{AB}$ and $\overline{CD}$ are horizontal segments (Why?), we have

$$AB = |a|$$

and

$$CD = |a + b - b| = |a|.$$

Therefore $AB = CD$ and $ABCD$ is a parallelogram.

Proof of 2: If $ABCD$ is a parallelogram, we must prove that $C = (a + b, c)$. Since $ABCD$ is a parallelogram, $\overline{AB} \parallel \overline{CD}$. The slope of $\overline{AB}$ is 0; therefore the slope of $\overline{CD}$ is 0. Let $C = (x, y)$; then the slope of $\overline{CD}$ is

$$\frac{y - c}{x - b} = 0.$$

Therefore $y - c = 0$ and $y = c$. We have $AB = CD$ (Why?) and

$$AB = |a| \qquad \text{and} \qquad CD = |x - b|.$$

Therefore

$$|a| = |x - b|.$$

Now, if $a > 0$, then $x > b$, and if $a < 0$, then $x < b$. (If $a > 0$ and $x < b$, or $a < 0$ and $x > b$, then C and B would be on opposite sides of $\overleftrightarrow{AD}$ and $ABCD$ would not be a parallelogram.) It follows that $a = x - b$ or that $x = a + b$. Therefore

$$C = (a + b, c)$$

and the proof is complete.

EXERCISES 11.9

Unless stated otherwise, use coordinates to prove the theorems in this set of exercises. Many of these theorems have appeared as theorems or exercises earlier in the text. We include them here since they can be proved easily using coordinates.

1. Prove:

   THEOREM 11.21 If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

(*Hint:* Let $A = (0, 0)$, $B = (a, 0)$, $C = (x, y)$, and $D = (b, c)$ be the vertices of the quadrilateral and suppose the diagonals of the quadrilateral bisect each other. Show that $x = a + b$, $y = c$, and then apply Theorem 11.20.)

2. Prove:

   THEOREM 11.22 The diagonals of a parallelogram bisect each other.

(This is the converse of Theorem 11.21. By Theorem 11.20, the vertices of a parallelogram may be taken as $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, and $D = (b, c)$. Show that the midpoint of $\overline{AC}$ is the same point as the midpoint of $\overline{BD}$.)

3. Prove:

   THEOREM 11.23 If the vertices of a parallelogram are $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, $D = (b, c)$, then the parallelogram is a rectangle if and only if $b = 0$.

(You must prove (1) if $ABCD$ is a rectangle, then $b = 0$ and (2) if $b = 0$, then $ABCD$ is a rectangle.)

4. Prove:

   THEOREM 11.24 The diagonals of a rectangle are congruent.

(Let $ABCD$ be the given rectangle. Use Theorem 11.23 and prove $AC = BD$.)

5. Justify the steps in the proof of the following theorem.

THEOREM 11.25 If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Proof: Let $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, $D = (b, c)$ be the vertices of the given parallelogram. What theorem justifies our writing $C = (a + b, c)$?

We have

$$AC = BD.$$

Therefore

$$(AC)^2 = (BD)^2,$$
$$(AC)^2 = (a + b)^2 + c^2,$$

and $$(BD)^2 = (b - a)^2 + c^2.$$

Therefore

$$(a + b)^2 + c^2 = (b - a)^2 + c^2.$$

Simplifying,

$$a^2 + 2ab + b^2 + c^2 = b^2 - 2ab + a^2 + c^2,$$
$$2ab = -2ab,$$

and $$4ab = 0.$$

Since $4a \neq 0$, we can divide both sides of the last equation by $4a$, obtaining $b = 0$. Therefore $ABCD$ is a rectangle.

6. Justify the steps in the proof of the following theorem.

THEOREM 11.26 A rectangle is a square if and only if its diagonals are perpendicular.

(See Figure 11-36.) By Theorem 11.23 we may write $A = (0, 0)$, $B = (a, 0)$, $C = (a, c)$, $D = (0, c)$ for the vertices of the given rectangle. There are two things to prove.

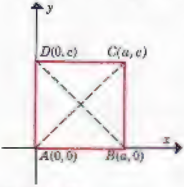


Figure 11-36

1. If $\overline{AC} \perp \overline{BD}$, then $ABCD$ is a square.
2. If $ABCD$ is a square, then $\overline{AC} \perp \overline{BD}$.

Proof of 1: We are given $\overline{AC} \perp \overline{BD}$. The slope of $\overline{AC}$ is $\frac{c}{a}$ and the slope of $\overline{BD}$ is $\frac{c}{-a}$. Therefore

$$\frac{c}{a} \cdot \frac{c}{-a} = -1.$$

Therefore

$$\frac{c^2}{-a^2} = -1, \qquad c^2 = a^2, \qquad \text{and} \qquad |c| = |a|.$$

But

$$|c| = BC \qquad \text{and} \qquad |a| = AB.$$

Therefore $BC = AB$ and $ABCD$ is a square.

Proof of 2: We are given $ABCD$ is a square. Therefore

$$AB = BC, \qquad |a| = |c|, \qquad \text{and} \qquad a^2 = c^2.$$

Let m_1 be the slope of $\overline{AC}$ and m_2 be the slope of $\overline{BD}$. Then

$$m_1 = \frac{c}{a}, \qquad m_2 = -\frac{c}{a}, \qquad \text{and } m_1 m_2 = -\frac{c^2}{a^2}.$$

But

$$\frac{c^2}{a^2} = 1, \qquad \text{so} \qquad m_1 m_2 = -1.$$

Therefore $\overline{AC} \perp \overline{BD}$.

7. Justify the steps in the proof of the following theorem.

THEOREM 11.27 If the vertices of a parallelogram are $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, and $D = (b, c)$, then the parallelogram is a rhombus if and only if $a^2 = b^2 + c^2$.

There are two things to prove.

1. If $a^2 = b^2 + c^2$, then $ABCD$ is a rhombus.
2. If $ABCD$ is a rhombus, then $a^2 = b^2 + c^2$.

Proof of 1: We are given that $a^2 = b^2 + c^2$, so

$$|a| = \sqrt{b^2 + c^2}.$$

By the Distance Formula,

$$AD = \sqrt{b^2 + c^2}.$$

Therefore $AD = |a|$. Also, $AB = |a|$. Therefore $AB = AD$ and $ABCD$ is a rhombus.

Proof of 2: $ABCD$ is a rhombus, so

$$AD = AB$$

and

$$\sqrt{b^2 + c^2} = |a|.$$

Therefore

$$a^2 = b^2 + c^2,$$

and the proof is complete.

8. Prove:

THEOREM 11.28 If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

(*Hint:* Let $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, and $D = (b, c)$ be the vertices of the given parallelogram. Let m_1 and m_2 be the slopes of $\overline{AC}$ and $\overline{BD}$, respectively. Using Theorem 11.27 of Exercise 7 and $m_1 \cdot m_2 = -1$, show that $a^2 = b^2 + c^2$.)

9. Recall that a trapezoid is a quadrilateral with at least one pair of parallel sides which are called the bases of the trapezoid. The other two sides are called the legs of the trapezoid. The segment joining the midpoints of the legs is called the median of the trapezoid. Prove the following theorem.

THEOREM 11.29 The median of a trapezoid is parallel to each of the bases and its length is one-half the sum of the lengths of the two bases.

(*Hint:* Let $A = (0, 0)$, $B = (2a, 0)$, $C = (2d, 2c)$, and $D = (2b, 2c)$ be the vertices of the given trapezoid. Then $\overline{AB}$ and $\overline{CD}$ are the parallel bases. Let E and F be the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. Show that $\overline{EF} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{CD}$, and that

$$EF = \tfrac{1}{2}(AB + CD).)$$

10. A trapezoid is isosceles if its legs are congruent. (See Exercise 9.) Prove the following theorem.

THEOREM 11.30 A trapezoid is isosceles if its diagonals are congruent.

11. Prove:

THEOREM 11.31 If a line bisects one side of a triangle and is parallel to a second side, then the line bisects the third side of the triangle.

12. Prove:

THEOREM 11.32 The midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.

(*Hint:* Let $A = (0, 0)$, $B = (2a, 0)$, and $C = (0, 2b)$, where a, b are positive numbers, be the vertices of the given right triangle.)

13. Complete the proof of the following theorem.

THEOREM 11.33 The lines which contain the altitudes of a triangle are concurrent. (Their common point is called the **orthocenter** of the triangle.) (See Figure 11-37).

Proof: Let $A = (a, 0)$, $B = (b, 0)$, $C = (0, c)$, and suppose that $a < b$, $0 < c$ as shown in Figure 11-37. Let A', B', C' be the feet of the per-

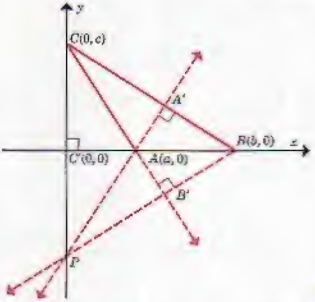


Figure 11-37

pendiculars from A, B, C to $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$, respectively. You are to prove that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are concurrent at some point $P(x_1, y_1)$. (Note that $C'(0, 0)$ is the origin.)

The slope of $\overleftrightarrow{BC}$ is $\frac{c-0}{0-b} = -\frac{c}{b}$. Since $\overleftrightarrow{AA'} \perp \overleftrightarrow{BC}$, the slope of $\overleftrightarrow{AA'}$ is $\frac{b}{c}$ (Why?). Using the point slope form of an equation, we may express $\overleftrightarrow{AA'}$ as

$$\overleftrightarrow{AA'} = \{(x, y) : y = \frac{b}{c}(x - a)\},$$

since $(a, 0)$ is a point on $\overleftrightarrow{AA'}$. We may express $\overleftrightarrow{CC'}$ as

$$\overleftrightarrow{CC'} = \{(x, y) : x = 0\}.$$

Now write an equation for $\overleftrightarrow{BB'}$. Let $P(x_1, y_1)$ be the point of intersection of lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{CC'}$, and let $P'(x_2, y_2)$ be the point of intersection of lines $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$. Solve the equations for $\overleftrightarrow{AA'}$, $\overleftrightarrow{CC'}$ "simultaneously" to find $P(x_1, y_1)$ and solve the equations for $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ simultaneously to find $P'(x_2, y_2)$. Show that $P = P'$ and hence the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are concurrent at P.

14. Prove Theorem 11.18 by use of definitions, postulates, and theorems studied before this chapter: that is, without the use of coordinates. (Perhaps you did this in an exercise of Chapter 10.)
15. Use Theorem 11.18 and theorems, definitions, and postulates studied before this chapter to prove Theorem 11.29 without the use of coordinates.
16. CHALLENGE PROBLEM. Prove Theorem 11.19 without the use of coordinates.

CHAPTER SUMMARY

In this chapter we used the idea of a coordinate system on a line to define an xy-coordinate system in a plane. We showed that there is a one-to-one correspondence between the set of all points in a plane and the set of all ordered pairs of real numbers. The key theorems in this chapter are:

THEOREM 11.4 If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are any two points in an xy-plane, then

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

THEOREM 11.5 If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points in an xy-plane, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

THEOREM 11.7 If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two distinct points, then

$$\overleftrightarrow{AB} = \{(x, y) : x = x_1 + k(x_2 - x_1), y = y_1 + k(y_2 - y_1), k \text{ is real}\}.$$

If k is a real number and if $P = (x, y)$ where $x = x_1 + k(x_2 - x_1)$, $y = y_1 + k(y_2 - y_1)$, then

$$\begin{array}{llll} AP = k(AB) & \text{and} \quad P \in \overrightarrow{AB} & \text{if} & k \geq 0; \\ AP = -k(AB) & \text{and} \quad P \in opp\,\overrightarrow{AB} & \text{if} & k \leq 0. \end{array}$$

The formula in Theorem 11.4 is called the DISTANCE FORMULA. The formula in Theorem 11.5 is called the MIDPOINT FORMULA. The equations $x = x_1 + k(x_2 - x_1)$ and $y = y_1 + k(y_2 - y_1)$ in Theorem 11.7 are called PARAMETRIC EQUATIONS for the line $\overleftrightarrow{AB}$, and k is called the PARAMETER. We proved that the converse of Theorem 11.7 also holds; that is, if a, b, c, d are real numbers, if b and d are not both zero, and if

$$S = \{(x, y) : x = a + bk, y = c + dk, k \text{ is real}\},$$

then S is a line.

We defined the SLOPE of a nonvertical line to be the slope of any one of its segments. We defined the slope of a nonvertical segment with endpoints $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ to be $\dfrac{y_2 - y_1}{x_2 - x_1}$. We showed that two nonvertical lines are parallel if and only if they have equal slopes and that two oblique lines are perpendicular if and only if the product of their slopes is -1. Slope is not defined for a vertical line, but the slope of a horizontal line is zero.

We proved that if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two distinct points on a nonvertical line, then

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

is an equation of the line. We called this form of equation the TWO-POINT FORM for an equation of a line. We proved that a line with slope m and passing through $P(x_1, y_1)$ has an equation of the form

$$y - y_1 = m(x - x_1)$$

and called this form the POINT SLOPE FORM for an equation of a line. We proved that a line with slope m and y-intercept b has an equation of the form

$$y = mx + b$$

and called this form the SLOPE y-INTERCEPT FORM for an equation of a line. We showed that an equation of a vertical line through $P(x_1, y_1)$ is $x = x_1$ and that an equation of a horizontal line through $P(x_1, y_1)$ is $y = y_1$. We called the form $Ax + By + C = 0$ the GENERAL FORM for an equation of a line.

Finally, we showed how coordinates could be used to construct proofs for some geometric theorems and observed that, in some cases, proofs using cordinates are easier than those using previously established definitions, postulates, and theorems.

REVIEW EXERCISES

Graph each of the sets indicated in Exercises 1–10.

1. $\{(x, y) : x = 5, 0 < y < 5\}$
2. $\{(x, y) : y = -3, -2 \leq x \leq 7\}$
3. $\{(x, y) : x = 1 + 2k, y = 2 + 3k, k \text{ is real}\}$
4. $\{(x, y) : x = 1 + 2k, y = 2 + 3k, k \geq 0\}$
5. $\{(x, y) : x = -2 + k, y = 1 - 2k, k \leq 0\}$
6. $\{(x, y) : x = k, y = 3 - 2k, 2 \leq k \leq 6\}$
7. $\{(x, y) : 1 < x < 4 \text{ or } 2 < y < 5\}$
8. $\{(x, y) : 1 < x < 4 \text{ and } 2 < y < 5\}$
9. $\{(x, y) : y > 2\}$
10. $\{(x, y) : y = \frac{2}{3}x + 5\}$

In Exercises 11–15, the endpoints A and B of a segment $\overline{AB}$ are given. Find (a) the slope of $\overline{AB}$ (in lowest terms), (b) the midpoint of $\overline{AB}$, and (c) the distance AB.

11. $A = (2, 5)$ and $B = (-2, 3)$
12. $A = (-1, -3)$ and $B = (2, -9)$
13. $A = (2, -5)$ and $B = (7, 7)$
14. $A = (-2, 1)$ and $B = (2, 3)$
15. $A = (-3, 1)$ and $B = (7, 1)$

16. Which segments in Exercises 11–15 are
    (a) parallel?
    (b) perpendicular?
    (c) congruent?
17. Write, in general form, an equation of the line which contains the points $P = (-1, -3)$ and $Q = (2, -9)$.
18. Write, in general form, an equation of the line which has a slope of $\frac{1}{2}$ and contains the point $R = (-2, 1)$.
19. Prove that the line of Exercise 17 is perpendicular to the line of Exercise 18.
20. Write, in slope y-intercept form, an equation of the line with slope $\frac{2}{3}$ and y-intercept 6.

In Exercises 21–25, an equation of a line is given. In each exercise, (a) put the equation in slope y-intercept form, (b) write the slope of the line, and (c) write the x- and y-intercepts of the line.

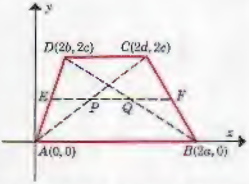
21. $3x + 2y = 12$
22. $2x - y = 7$
23. $15x - 21y = 7$
24. $x + y = 0$
25. $x - y = 0$

26. Show that $\triangle SKM$ is a right isosceles triangle if $S = (3, 4)$, $K = (-1, 5)$, and $M = (-2, 1)$.
27. Given $A = (1, 0)$, $B = (4, 3)$, express $\overleftrightarrow{AB}$ using set-builder notation and parametric equations.
28. Given A and B as in Exercise 27, express $\overline{AB}$ using set-builder notation and parametric equations.
29. Given A and B as in Exercise 27, find the trisection points of $\overline{AB}$.
30. Write an equation of the vertical line through (2, 5).
31. Write an equation of the horizontal line through (2, 5).
32. Write an equation of the line through $(3, -7)$ and parallel to the line with equation $y = 3x + 5$.
33. Write an equation of the line through $(3, -7)$ and perpendicular to the line with equation $y = 3x + 5$.
34. Given the line with equation $5x - 6y = 60$, write the equation of this line in slope y-intercept form.
35. Given the line of Exercise 34, write the equation of this line in intercept form.
36. Given $p = \{(x, y) : x = 3\}$ and $q = \{(x, y) : 2x = 6\}$, explain why $p \parallel q$.

Exercises 37–47 refer to the rectangle $PQRS$ whose vertices are $P = (-1, -4)$, $Q = (5, -4)$, $R = (5, 3)$, $S = (-1, 3)$.

37. Find the midpoint of $\overline{PR}$.
38. Find the midpoint of $\overline{SQ}$.
39. Show that $PQ = SR$.
40. Show that $PR = SQ$.
41. Write parametric equations for $\overleftrightarrow{PQ}$.
42. Write parametric equations for $\overleftrightarrow{PR}$.
43. Find A on $\overrightarrow{PR}$ such that $PA = 4PR$.
44. Find B on $\overrightarrow{PR}$ such that $PB = \frac{2}{3}PR$.
45. Find C on *opp* $\overrightarrow{PR}$ such that $PC = \frac{2}{3}PR$.
46. Write parametric equations for the line through S and perpendicular to $\overline{PR}$.
47. Write parametric equations for the line through R and parallel to $\overline{SQ}$.

48. Let the trapezoid $ABCD$ have vertices $A = (0, 0)$, $B = (2a, 0)$, $C = (2d, 2c)$, and $D = (2b, 2c)$, as shown in the figure. Let a, b, c, d be positive numbers such that $b < d < a$. Let E and F be the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. Let $\overline{EF}$ intersect $\overline{AC}$ at P and $\overline{BD}$ at Q. Show that P is the midpoint of $\overline{AC}$, that Q is the midpoint of $\overline{BD}$, and that $PQ = \frac{1}{2}(AB - CD)$.



49. Prove, using coordinates, that the set of all points in a plane equidistant from two given points in the plane is the perpendicular bisector of the segment joining the given points. (*Hint:* Let the two given points in an xy-plane be $A = (-a, 0)$ and $B = (a, 0)$. Then the y-axis is the perpendicular bisector of $\overline{AB}$ in the given xy-plane.) There are two things to prove.
    (a) If $P(x, y)$ is on the y-axis, then $AP = PB$.
    (b) If $AP = PB$ and P is in the xy-plane, then P is on the y-axis.
50. CHALLENGE PROBLEM. Prove that the area S of a triangle whose vertices are $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$ is given by the formula
$$S = \tfrac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_1y_3 - x_2y_1|.$$

Chapter 12

Bradley Smith/Photo Researchers

Coordinates in Space

12.1 A COORDINATE SYSTEM IN SPACE

In Chapter 3 we introduced the fundamental idea of a coordinate system on a line, or a line coordinate system, or a one-dimensional coordinate system, as it is sometimes called. In Chapter 11, we defined a coordinate system in a plane and called it an xy-coordinate system. An xy-coordinate system is a one-to-one correspondence between all the points in a plane and all the ordered pairs of real numbers. Each point has two coordinates. An xy-coordinate system is a two-dimensional coordinate system. In this chapter we introduce the idea of a coordinate system in space and called it an xyz-coordinate system. In this system each point of space is matched with an ordered triple of numbers. It is a three-dimensional coordinate system.

Let a unit segment and the distance function based on it be given. All distances will be relative to this unit segment unless otherwise indicated.

Let $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ be any two perpendicular lines and let $\overleftrightarrow{OZ}$ be the unique line that is perpendicular to each at their point of intersection. Let I, J, K be points on $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$, $\overleftrightarrow{OZ}$, respectively, such that

$$OI = OJ = OK = 1.$$

On $\overleftrightarrow{OX}$ there is a unique line coordinate system with O as origin and I as unit point. On $\overleftrightarrow{OY}$ there is a unique line coordinate system with O as origin and J as unit point. On $\overleftrightarrow{OZ}$ there is a unique line coordinate system with O as origin and K as unit point. We call these coordinate systems the x-coordinate system on $\overleftrightarrow{OX}$, the y-coordinate system on $\overleftrightarrow{OY}$, and the z-coordinate system on $\overleftrightarrow{OZ}$. We refer to $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$, $\overleftrightarrow{OZ}$ as the x-axis, the y-axis, and the z-axis, respectively. We refer to them collectively as the **coordinate axes.**

The plane containing the x- and y-axes is called the **xy-plane.** The plane containing the x- and z-axes is called the **yz-plane.** The plane containing the y- and z-axes is called the **yz-plane.** We refer to these three planes collectively as the **coordinate planes.**

From the theorems regarding parallelism and perpendicularity in Chapter 8 it should be clear that all lines parallel to the z-axis are perpendicular to the xy-plane, that all lines parallel to the y-axis are perpendicular to the xz-plane, and that all lines parallel to the x-axis are perpendicular to the yz-plane. It should also be clear that all planes parallel to the xy-plane are perpendicular to the z-axis, that all planes parallel to the xz-plane are perpendicular to the y-axis, and that all planes parallel to the yz-plane are perpendicular to the x-axis.

Figure 12-1 suggests the x-, y-, and z-coordinate systems. The parts of the axes with negative coordinates are shown by dashed lines. They are not "hidden" from view by the coordinate axes in the figure. However, they are hidden from view by the coordinate planes. Thus the negative part of the x-axis is behind the yz-plane; the negative part of the y-axis is hidden by the xz-plane; the negative part of the z-axis is hidden by the xy-plane. In drawing pictures you should use your own judgment about whether a dashed segment is better than a solid one.

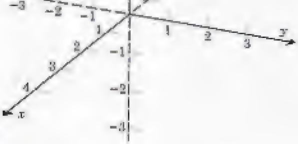


Figure 12-1

In Figure 12-1, the positive parts of the x-, y-, and z-axes are arranged as the thumb, forefinger, and middle finger, respectively, of a right hand when it is held as suggested in Figure 12-2. An xyz-coordinate system with axes oriented in this manner is called a right-handed coordinate system. If the unit points on the axes are selected so that the positive parts of the x-, y-, z-axes conform to the orientation of the thumb, forefinger, and middle finger of the left hand when the thumb and forefinger are extended and the middle finger is folded, the xyz-coordinate system is called a left-handed coordinate system. The figures in this book are for a right-handed system.

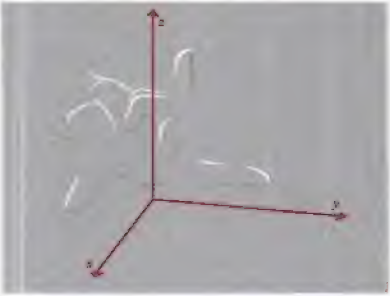


Figure 12-2

There is an xy-coordinate system in the xy-plane determined by O, I, J as in Chapter 11. This system is a one-to-one correspondence between the set of all points in the xy-plane and the set of all ordered pairs of real numbers. Similarly, there is an xz-coordinate system in the xz-plane and a yz-coordinate system in the yz-plane.

Let P be any point in space. Figure 12-3 shows P as not lying on any of the coordinate planes. However, the following discussion which leads to the definition of an xyz-coordinate system applies to any point. For special positions of P some of the labeled points that are distinct in Figure 12-3 may not be distinct. For example, P and P_{xy} may be the same point.

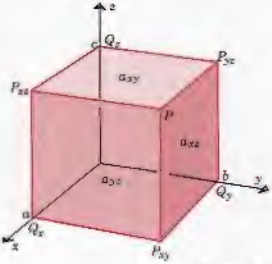


Figure 12-3

Let P_{xy}, P_{xz}, P_{yz} be the projections of P on the xy-plane, the xz-plane, and the yz-plane, respectively. Let α_{xy}, α_{xz}, α_{yz} be the planes through P and parallel to the xy-plane, the xz-plane, and the yz-plane, respectively. Then α_{xy} contains P, P_{xz}, and P_{yz}; α_{xz} contains P, P_{xy}, and P_{yz}; and α_{yz} contains P, P_{xz}, and P_{xy}.

Let Q_x, Q_y, Q_z be the points in which α_{yz}, α_{xz}, α_{xy} intersect the x-axis, the y-axis, and the z-axis, respectively. We are now ready to set up an xyz-coordinate system.

The x-coordinate of P is the x-coordinate of Q_x; the y-coordinate of P is the y-coordinate of Q_y; the z-coordinate of P is the z-coordinate of Q_z. We write

$$P = (a, b, c) \qquad \text{or} \qquad P(a, b, c)$$

to indicate that the x-, y-, z-coordinates of P are a, b, c, respectively.

If (a, b, c) is any ordered triple of real numbers, then there is one and only one point P such that

$$P = P(a, b, c).$$

It is the intersection of three planes, one parallel to the xz-plane and cutting the x-axis at the point whose x-coordinate is a, etc. The correspondence between the set of all ordered triples of real numbers and the set of all points is a one-to-one correspondence. For if (a, b, c) and (d, e, f) are different triples of numbers, then one or more of the following inequalities must hold:

$$a \neq d, \qquad b \neq e, \qquad c \neq f.$$

Suppose, for example, that $a \neq d$. Then $P(a, b, c)$ and $R(d, e, f)$ lie in distinct planes parallel to the yz-plane and therefore $P \neq R$.

Definition 12.1 Given an x-axis, a y-axis, and a z-axis, the one-to-one correspondence between all the points in space and all the ordered triples of real numbers in which each point P corresponds to the ordered triple (a, b, c) where a, b, c are the x-, y-, z-coordinates, respectively, of P is the **xyz-coordinate system.**

Since the correspondence between points and triples is one-to-one, a system of names for the triples is a suitable system of names for the points. Thus $(5, 6, -3)$ is an ordered triple of real numbers. It is also a point. It is the point whose x-, y-, and z-coordinates are 5, 6, and -3, respectively.

Example 1 If
$$A = \{(x, y, z) : x = 2, y = 3, z = 4\},$$
then
$$A = \{(2, 3, 4)\},$$
that is, A is the set whose only element is the point $(2, 3, 4)$.

Example 2 If
$$B = \{(x, y, z) : x = -2, y = 3\},$$
then B is the set of all points (x, y, z) such that the x-coordinate is -2 and the y-coordinate is 3, that is, B is the line through $(-2, 3, 0)$ and parallel to the z-axis.

Example 3 If
$$C = \{(x, y, z) : y = 7\},$$
then C is the set of all points whose y-coordinate is 7, that is, C is the set of all points that are 7 units to the right of the xz-plane; hence C is the plane that is parallel to the xz-plane and 7 units to the right of it.

Example 4
$$D = \{(x, y, z) : x = 2 \text{ and } y = 3\}.$$
The use of "and" in this set-builder notation is the same as the use of a comma between statements of conditions in a set-builder notation. In other words,
$$D = \{(x, y, z) : x = 2, y = 3\}$$
is the set of all points $(2, 3, z)$, that is, the line parallel to the z-axis and passing through $(2, 3, 0)$.

Example 5
$$E = \{(x, y, z) : x = 2 \text{ or } y = 3\}.$$
Clearly, E is not the set D of Example 4. If you do not understand clearly the distinction between the use of "and" and "or" in a set-builder notation, you should review Chapter 1. E is the set of all points with x-coordinate 2 or y-coordinate 3, hence the set of all points lying either in the plane $x = 2$ or in the plane $y = 3$. In other words, E is the union of two planes, each parallel to the z-axis, one of them parallel to the yz-plane and 2 units in front of it, the other parallel to the xz-plane and 3 units to the right of it, assuming that the axes are as in Figure 12-1.

Example 6

$$F = \{(x, y, z) : x = 1 + 2k, y = 2 + 3k, k \text{ is real}\}.$$

Now

$$\{(x, y) : x = 1 + 2k, y = 2 + 3k, k \text{ is real}\}$$

is a line l in the xy-plane with slope $\frac{3}{2}$ that passes through $(1, 2)$. The set-builder notation for F places no restriction on the z-coordinate. Therefore, if (x, y) is any point on l and z is any real number, then (x, y, z) is a point of F. Conversely, if (x, y, z) is any point on F, then (x, y) is a point of l. Think of l' as the set

$$\{(x, y, z) : x = 1 + 2k, y = 2 + 3k, z = 0, k \text{ is real}\}.$$

Then l' is the same set as l. If we think of a point of l, we think of it as a point (x, y) in an xy-plane. If we think of the same point as a point of l', we think of it as a point $(x, y, 0)$ in an xyz-space. Although the names of the points are different, the sets l and l' are the same. The graph of the set l' is shown in Figure 12-4.

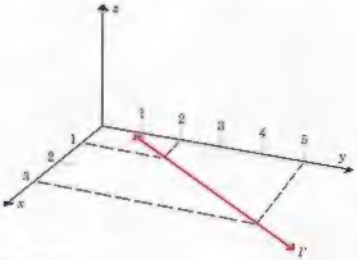


Figure 12-4

Now l' is a part of F. Indeed, F is the set l' and all points directly above it and all points directly below it. It may be helpful to think of it as the union of all lines parallel to the z-axis that pass through a point of l'. But any way you look at F, you really have not seen it unless you recognize it as a plane, the plane through l' and parallel to the z-axis.

Example 7

$$G = \{(x, y, z) : z < 3\}.$$

G is the set of all points (x, y, z) whose z-coordinate is less than 3. Now $z = 3$ is an equation of the horizontal plane, say α, that is 3 units above the xy-plane. Then G is the halfspace that is the underside of α.

Example 8

$$H = \{(x, y, z) : 1 \leq z \leq 3\}.$$

H is a slab or zone of space bounded by two horizontal planes. H is the union of the horizontal planes with equations $z = 1$ and $z = 3$ and all of the space that lies between them.

Example 9

$$I = \{(x, y, z) : x^2 + y^2 = 25, z = 0\}.$$

I is a circle in the xy-plane. Its center is the point $(0, 0, 0)$. Its radius is 5.

Example 10

$$J = \{(x, y, z) : x^2 + y^2 \leq 25, 0 \leq z \leq 6\}.$$

J is a solid right circular cylinder. Its axis is a part of the z-axis. Its radius is 5. Its lower base lies in the xy-plane. Its height is 6.

Example 11

$$K = \{(x, y, z) : x = 2 \text{ and } x = 3\}.$$

There is no number x such that $x = 2$ and $x = 3$. Therefore there is no point (x, y, z) such that $x = 2$ and $x = 3$. Therefore $K = \varnothing$, the null set.

Example 12

$$L = \{(x, y, z) : x^2 - 3x + 2 = 0\}.$$

Now $x^2 - 3x + 2 = 0$ is true if and only if $(x - 1)(x - 2) = 0$; hence if and only if $x = 1$ or $x = 2$. Therefore L is the union of two planes, each parallel to the yz-plane, one of them 1 unit in front of it, the other 2 units in front of it.

EXERCISES 12.1

In Exercises 1–35, a set S of points is given in set-builder notation. In each case, describe the set S in words, assuming that the axes appear as in Figure 12-1. (*Hint*: For Exercises 20, 21, and 22, compare with Example 12.)

1. $S = \{(x, y, z) : x = 0\}$
2. $S = \{(x, y, z) : y = 0\}$
3. $S = \{(x, y, z) : z = 0\}$
4. $S = \{(x, y, z) : x = 0, y = 0\}$

5. $S = \{(x, y, z) : x = 0, z = 0\}$
6. $S = \{(x, y, z) : y = 0, z = 0\}$
7. $S = \{(x, y, z) : x = 0, y = 0, z = 0\}$
8. $S = \{(x, y, z) : x = 1, y = 2, z = 2\}$
9. $S = \{(x, y, z) : x = 1, y = 2, 1 \leq z \leq 5\}$
10. $S = \{(x, y, z) : x = 1, 1 \leq y \leq 3, z = -1\}$
11. $S = \{(x, y, z) : x \geq 1, y = 3, z = 1\}$
12. $S = \{(x, y, z) : y \leq 3\}$
13. $S = \{(x, y, z) : x = 3, y \leq 3\}$
14. $S = \{(x, y, z) : 1 \leq x \leq 2, 3 \leq y \leq 5, -1 \leq z \leq 3\}$
15. $S = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$
16. $S = \{(x, y, z) : x > 379\}$
17. $S = \{(x, y, z) : x = 1, y = 2\}$
18. $S = \{(x, y, z) : x = 1 \text{ or } y = 2\}$
19. $S = \{(x, y, z) : x = 5 \text{ or } x = 7\}$
20. $S = \{(x, y, z) : z^2 + 3z + 2 = 0\}$
21. $S = \{(x, y, z) : z^2 + 2z + 1 = 0\}$
22. $S = \{(x, y, z) : x^2 = 16\}$
23. $S = \{(x, y, z) : x^2 + y^2 = 25\}$
24. $S = \{(x, y, z) : x^2 + y^2 = 5, z = 5\}$
25. $S = \{(x, y, z) : y = x + 3, z = 4\}$
26. $S = \{(x, y, z) : y = x\}$
27. $S = \{(x, y, z) : z = y\}$
28. $S = \{(x, y, z) : 3x + 4y = 12, x \geq 0, y \geq 0, 1 \leq z \leq 2\}$
29. $S = \left\{(x, y, z) : \dfrac{x-1}{2} = \dfrac{y-5}{3}, z = 0\right\}$
30. $S = \left\{(x, y, z) : \dfrac{x+3}{2} = \dfrac{z-1}{1}, y = 0\right\}$
31. $S = \{(x, y, z) : x = 1 + 2k, y = 2 - k, z = 0, 1 \leq k \leq 2\}$
32. $S = \{(x, y, z) : x = 1 + 2k, y = 2 - k, 1 \leq k \leq 2\}$
33. $S = \{(x, y, z) : x^2 + y^2 \leq 25, z = 3\}$
34. $S = \{(x, y, z) : y^2 + z^2 \leq 25, x = 3\}$
35. $S = \{(x, y, z) : x \neq 0, y \neq 0, z \neq 0\}$

In Exercises 36–50, use set-builder notation to express the set S.

36. S is the xy-plane.
37. S is the xz-plane.
38. S is the yz-plane.
39. S is the x-axis.
40. S is the y-axis.

41. S is the plane through (2, 6, 7) and parallel to the yz-plane.
42. S is the plane through (2, 6, 7) and parallel to the xz-plane.
43. S is the plane through (2, 6, 7) and parallel to the xy-plane.
44. S is the line through (2, 6, 7) and perpendicular to the yz-plane.
45. S is the line through (2, 6, 7) and perpendicular to the xz-plane.
46. S is the line through (2, 6, 7) and perpendicular to the xy-plane.
47. S is the line in the xy-plane which contains (2, 3, 0) and (3, 7, 0).
48. S is the ray $\overrightarrow{AB}$, with $A = (5, 3, 0)$ and $B = (4, 6, 0)$.
49. S is the segment $\overline{PQ}$, with $P = (2, 1, 0)$ and $Q = (0, -7, 0)$.
50. S is the segment $\overline{RS}$, with $R = (2, 1, 1)$ and $S = (0, -1, 1)$.

12.2 A DISTANCE FORMULA

In this section we develop a formula for the distance between two points expressed in terms of their coordinates. We begin by considering an example.

Example 1 Let $A = (3, 2, 1)$ and $B = (5, 4, 2)$. (See Figure 12-5.) Let $A_2 = (3, 2, 0)$, $B_1 = (5, 4, 1)$, $B_2 = (5, 4, 0)$, $B_3 = (0, 4, 1)$, and $B_4 = (0, 4, 2)$. Then $AA_2B_2B_1$ is a rectangle and $AB_1 = A_2B_2$. Also $BB_1B_3B_4$ is a rectangle and $B_1B = B_3B_4$.

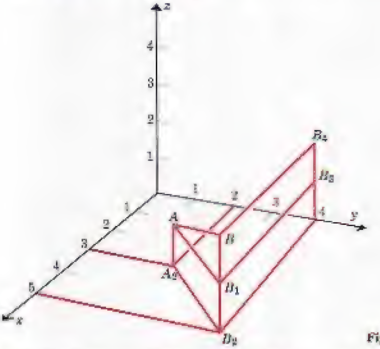


Figure 12-5

To find A_2B_2 we use the Distance Formula and the x- and y-coordinates of A_2 and B_2. In the xy-coordinate system, $A_2 = (3, 2)$ and $B_2 = (5, 4)$. Therefore

$$A_2B_2 = \sqrt{(5-3)^2 + (4-2)^2} = \sqrt{8} = 2\sqrt{2}.$$

To find B_3B_4 we use the Distance Formula and the y- and z-coordinates of B_3 and B_4. In the yz-coordinate system $B_3 = (4, 1)$ and $B_4 = (4, 2)$. Therefore

$$B_3B_4 = \sqrt{(4 - 4)^2 + (1 - 2)^2} = \sqrt{1} = 1.$$

Now $\overleftrightarrow{BB_1}$ is perpendicular to the xy-plane. It is also perpendicular to every plane parallel to the xy-plane. In particular, $\overleftrightarrow{BB_1}$ is perpendicular to the plane with equation $z = 1$. Then $\overleftrightarrow{BB_1}$ is perpendicular to every line in that plane through B_1. Therefore $\overleftrightarrow{BB_1} \perp \overleftrightarrow{AB_1}$ and $\triangle AB_1B$ is a right triangle. It follows from the Pythagorean Theorem that

$$\begin{aligned}
(AB)^2 &= (AB_1)^2 + (B_1B)^2 \\
(AB)^2 &= (A_2B_2)^2 + (B_3B_4)^2 \\
(AB)^2 &= (\sqrt{8})^2 + 1^2 \\
(AB)^2 &= 8 + 1 = 9 \\
AB &= 3
\end{aligned}$$

Following the procedure used in this example we shall prove the next theorem.

THEOREM 12.1 (***Distance Formula Theorem***) The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof: Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be given. Let

$$\begin{aligned}
P_1 &= (x_1, y_1, 0) \\
Q_1 &= (x_2, y_2, z_1) \\
Q_2 &= (x_2, y_2, 0) \\
Q_3 &= (0, y_2, z_1) \\
Q_4 &= (0, y_2, z_2)
\end{aligned}$$

Figure 12-6 shows these seven points as distinct points. Depending on the values of the coordinates, the points may turn out not to be distinct. For example, if $z_1 = 0$, then $P = P_1$.

We proceed to find PQ in terms of the coordinates of P and Q. If P and Q_1 are distinct points, then they lie on a plane parallel to the xy-plane and $\overleftrightarrow{PQ_1}$ is parallel to the xy-plane. Therefore

(1) $P_1Q_2Q_1P$ is a rectangle,

or (2) $P = P_1$ and $Q_1 = Q_2$,

or (3) $P = Q_1$ and $P_1 = Q_2$.

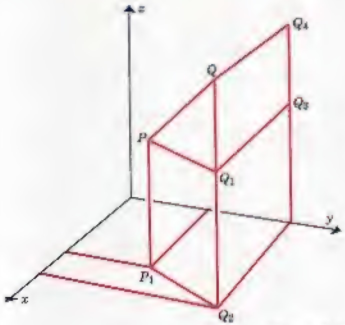


Figure 12-8

In all three cases it is true that

$$(PQ_1)^2 = (P_1Q_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Similarly, (1) $QQ_1Q_3Q_4$ is a rectangle, or (2) $Q = Q_4$ and $Q_1 = Q_3$, or (3) $Q = Q_1$ and $Q_4 = Q_3$. In all three cases it is true that $(Q_1Q)^2 = (Q_3Q_4)^2 = (z_2 - z_1)^2$.

If $Q = Q_1$, then $z_2 = z_1$, $(z_2 - z_1)^2 = 0$, and

$$(PQ)^2 = (PQ_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

If $P = Q_1$, then $x_2 = x_1$, $y_2 = y_1$, $(x_2 - x_1)^2 = 0$, $(y_2 - y_1)^2 = 0$, and

$$(PQ)^2 = (Q_1Q)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

If $Q \neq Q_1$ and $P \neq Q_1$, then $\overline{QQ_1} \perp \overline{PQ_1}$, and it follows from the Pythagorean Theorem that

$$(PQ)^2 = (PQ_1)^2 + (QQ_1)^2,$$
$$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

We have now exhausted all cases. For if P and Q are any points and if Q_1 is related to P and Q as indicated in the first sentence of this proof, then $P = Q_1$; or $Q = Q_1$; or P, Q, Q_1 are three distinct points. We have proved in all of these cases that

$$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Therefore

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXERCISES 12.2

In Exercises 1–5, find the distance between P and Q.

1. $P = (0, 0, 0)$, $Q = (3, 4, 5)$
2. $P = (2, -1, 7)$, $Q = (2, 1, -7)$
3. $P = (7, 8, 1)$, $Q = (0, 9, 0)$
4. $P = (0, 0, 5)$, $Q = (3, 4, 0)$
5. $P = (8, -1, 5)$, $Q = (1, -1, 6)$

6. Use the Distance Formula to show that the following points are collinear: $A = (-1, 5, 2)$, $B = (5, 7, 10)$, $C = (2, 6, 6)$.
7. Use the Distance Formula to show that the following points are not collinear: $A = (0, 0, 0)$, $B = (5, 3, -4)$, $C = (15, 10, -12)$.
8. If $A = (1, 3, -2)$, $B = (2, 7, 1)$, $C = (2, 6, 0)$, is $\triangle ABC$ a right triangle?
9. If $A = (5, 3, 2)$, $B = (7, 8, 1)$, $C = (5, 2, -3)$, is $\triangle ABC$ a right triangle?
10. Find the perimeter of $\triangle DEF$ if $D = (0, 0, 0)$, $E = (0, 6, 8)$, $F = (10, 6, 8)$.

In Exercises 11–14, find the distance between $P(2, -1, 7)$ and the given plane α.

11. $\alpha = \{(x, y, z) : x = -5\}$
12. $\alpha = \{(x, y, z) : y = 0\}$
13. $\alpha = \{(x, y, z) : z = 0\}$
14. $\alpha = \{(x, y, z) : y = -1\}$

In Exercises 15–18, find the distance between the line l given by

$$l = \{(x, y, z) : x = 3, y = 4\}$$

and the given plane α which is parallel to it. (The distance between a line and a plane parallel to it is not defined in the book. Write a suitable definition for this distance.)

15. $\alpha = \{(x, y, z) : x = 0\}$
16. $\alpha = \{(x, y, z) : x = -5\}$
17. $\alpha = \{(x, y, z) : y = 6\}$
18. $\alpha = \{(x, y, z) : y = 4\}$

In Exercises 19–23, sketch a figure to represent the given set. (The graphs will be three-dimensional.)

19. $A = \{(x, y, z) : 1 \leq x \leq 2, 2 \leq y \leq 4, 0 \leq z \leq 10\}$
20. $B = \{(x, y, z) : 1 \leq x \leq 2, 6 \leq y \leq 8, 4 \leq z \leq 6\}$
21. $C = \{(x, y, z) : 1 \leq x \leq 2, 10 \leq y \leq 12, 0 \leq z \leq 10\}$
22. $D = \{(x, y, z) : 1 \leq x \leq 2, 14 \leq y \leq 16, 4 \leq z \leq 5 \text{ or } 6 \leq z \leq 7\}$
23. $E = \{(x, y, z) : 1 \leq x \leq 2, 18 \leq y \leq 20, 0 \leq z \leq 10\}$

24. CHALLENGE PROBLEM. On one large sheet graph the union of the sets in Exercises 19–23.

12.3 PARAMETRIC EQUATIONS FOR A LINE IN SPACE

Before studying Section 12.3 you should review the work on parametric equations in Chapter 11. By use of a proof similar to the one for Theorem 11.7 the following theorem can be established. A proof of this theorem is not included here.

THEOREM 12.2 If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are any two distinct points, then

$$\overleftrightarrow{PQ} = \left\{ (x, y, z) : \begin{array}{l} x = x_1 + k(x_2 - x_1), \\ y = y_1 + k(y_2 - y_1), \text{ and } k \text{ is real} \\ z = z_1 + k(z_2 - z_1), \end{array} \right\}$$

If $R = (a, b, c)$, where

$$\begin{aligned} a &= x_1 + k(x_2 - x_1), \\ b &= y_1 + k(y_2 - y_1), \\ c &= z_1 + k(z_2 - z_1), \end{aligned}$$

is a point of $\overleftrightarrow{PQ}$, then

$$\begin{array}{lll} R \text{ is the point } P & \text{if} & k = 0, \\ R \in \overrightarrow{PQ} \quad \text{and} \quad PR = k \cdot PQ & \text{if} & k \geq 0, \\ R \in opp\ \overrightarrow{PQ} \quad \text{and} \quad PR = -k \cdot PQ & \text{if} & k \leq 0. \end{array}$$

Example 1 Given $P = (0, 0, 0)$ and $Q = (3, 1, -2)$, find the point R on $\overrightarrow{PQ}$ such that $PR = 3 \cdot PQ$ and the point S on *opp* $\overrightarrow{PQ}$ such that $PS = 3 \cdot PQ$.

Solution:

$$\overleftrightarrow{PQ} = \{(x, y, z) : x = 3k,\ y = k,\ z = -2k,\ k \text{ is real}\}$$

To get R, set $k = 3$. Then $R = (9, 3, -6)$.
To get S, set $k = -3$. Then $S = (-9, -3, 6)$.

Example 2 Given $A = (5, 1, -2)$ and $B = (7, 3, -2)$, find the midpoint M of $\overline{AB}$.

Solution:

$$\overleftrightarrow{AB} = \{(x, y, z) : x = 5 + 2k,\ y = 1 + 2k,\ z = -2,\ k \text{ is real}\}.$$

The k-coordinate of A is 0; the k-coordinate of B is 1; of M is $\frac{1}{2}$.

$$M = (5 + 2 \cdot \tfrac{1}{2}, 1 + 2 \cdot \tfrac{1}{2}, -2) = (6, 2, -2).$$

Example 3 Given $E = (5, 7, 9)$ and $F = (-2, 3, -4)$, write parametric equations for $\overrightarrow{EF}$.

Solution:

$$\begin{aligned} x &= 5 - 7k \\ y &= 7 - 4k \\ z &= 9 - 13k \\ k &\geq 0 \end{aligned}$$

Example 4 Given $H = (5, -3, 14)$ and $I = (-2, 5, -1)$, write parametric equations for $\overline{HI}$

(1) with a parameter k so that $k = 0$ at H and $k = 1$ at I.
(2) with a parameter h so that $h = 0$ at I and $h = 1$ at H.
(3) with a parameter t so that $t = 0$ at H and $t = 10$ at I.

Solution:

(1) $x = 5 - 7k$	(2) $x = -2 + 7h$	(3) $x = 5 - 0.7t$
$y = -3 + 8k$	$y = 5 - 8h$	$y = -3 + 0.8t$
$z = 14 - 15k$	$z = -1 + 15h$	$z = 14 - 1.5t$
$0 \leq k \leq 1$	$0 \leq h \leq 1$	$0 \leq t \leq 10$

Example 5 Given $J = (5, 7, 3)$ and $K = (7, -1, 5)$, find an equation for the perpendicular bisecting plane of $\overline{JK}$. A simplified equation is desired.

Solution: Let α be the perpendicular bisecting plane of $\overline{JK}$. Then $P(x, y, z)$ is a point of α if and only if

$$JP = PK,$$

$$(JP)^2 = (PK)^2,$$

$$(x-5)^2 + (y-7)^2 + (z-3)^2 = (x-7)^2 + (y+1)^2 + (z-5)^2,$$

$$x^2 - 10x + 25 + y^2 - 14y + 49 + z^2 - 6z + 9$$
$$= x^2 - 14x + 49 + y^2 + 2y + 1 + z^2 - 10z + 25,$$

$$-10x + 14x - 14y - 2y - 6z + 10z$$
$$+ 25 + 49 + 9 - 49 - 1 - 25 = 0,$$

$$4x - 16y + 4z + 8 = 0,$$

$$x - 4y + z + 2 = 0.$$

Then $x - 4y + z + 2 = 0$ is "an equation of α." This means that

$$\alpha = \{(x, y, z) : x - 4y + z + 2 = 0\}.$$

EXERCISES 12.3

In Exercises 1–4, find the coordinates of the point P satisfying the given condition.

1. $A = (0, 0, 0)$, $B = (3, 3, 3)$, $P \in \overrightarrow{AB}$, $AP = \frac{4}{3} \cdot AB$.
2. $A = (0, 0, 0)$, $B = (3, 3, 3)$, $P \in opp\ \overrightarrow{AB}$, $AP = \frac{4}{3} \cdot AB$.
3. $A = (5, -2, 6)$, $B = (1, -3, 2)$, $P \in \overrightarrow{BA}$, $AP = 2 \cdot AB$.
4. $A = (-3, 2, 5)$, $B = (5, 0, -1)$, $P \in \overline{AB}$, $AP = PB$.

In Exercises 5–10, $A = (0, 0, 0)$, $B = (6, -3, 12)$, $C = (-9, 18, 9)$.

5. Find D, the midpoint of $\overline{BC}$.
6. Find E, the midpoint of $\overline{AC}$.
7. Find F, the midpoint of $\overline{AB}$.
8. Find G_1, the point that is two-thirds of the way from A to D.
9. Find G_2, the point that is two-thirds of the way from B to E.
10. Find G_3, the point that is two-thirds of the way from C to F.

11. Given $S = \{(x, y, z) : 3x + 4y + 5z = 50\}$, which of the following points are elements of S: $A(5, 5, 3)$, $B(0, 0, 5)$, $C(-1, -2, 12)$?
12. CHALLENGE PROBLEM. Find the point on the x-axis that is equidistant from $A(2, 3, 1)$ and $B(7, 2, 1)$.
13. CHALLENGE PROBLEM. If S is the set of all points each of which is at a distance of 7 units from the origin, find an equation (in simplified form) for S. What is a set of points like S usually called?
14. CHALLENGE PROBLEM. If T is the set of all points, each of which is at a distance of 3 units from the point $(3, -1, 2)$, find an equation (in simplified form) for T.
15. CHALLENGE PROBLEM. Given

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 125\},$$
$$T = \{(x, y, z) : x = 5\}, \qquad \text{and} \qquad Q = (5, 0, 0),$$

prove that all points P in the intersection of S and T are at the same distance from Q.

16. CHALLENGE PROBLEM. Let planes α and β and points, O, A, A_1, B, B_1, be given as follows:

$$\alpha = \{(x, y, z) : x + y + z = 6\},$$
$$\beta = \{(x, y, z) : x + y + z = 9\},$$
$$O = (0, 0, 0), A = (2, 2, 2), A_1 = (6, 0, 0),$$
$$B = (3, 3, 3), B_1 = (9, 0, 0).$$

(a) Show that A and A_1 are points of α.
(b) Show that B and B_1 are points of β.

(c) Show that $\overline{OA} \perp \overline{AA_1}$.
(d) Show that $\overline{OB} \perp \overline{BB_1}$.
(e) Find a point A_2 in α such that A, A_1, A_2 are noncollinear.
(f) Find a point B_2 in β such that B, B_1, B_2 are noncollinear.
(g) Show that $\overline{OA} \perp \overline{AA_2}$.
(h) Show that $\overline{OB} \perp \overline{BB_2}$.
(i) Show that $\overline{OA} \perp \alpha$.
(j) Show that $\overline{OB} \perp \beta$.
(k) Show that O, A, B are collinear.
(l) Find the distance between α and β.

17. CHALLENGE PROBLEM. Let planes α and β and line l be given as follows:

$$\alpha = \{(x, y, z) : x + y + 2z = 8\},$$
$$\beta = \{(x, y, z) : x + y + 2z = 2\},$$
$$l = \left\{(x, y, z) : x = y = \frac{z - 1}{2}\right\}.$$

(a) Find the point A in which l intersects α.
(b) Find the point B in which l intersects β.
(c) Prove that $\overline{AB} \perp \alpha$.
(d) Prove that $\overline{AB} \perp \beta$.
(e) Find the distance between α and β.

12.4 EQUATIONS OF PLANES

In Chapter 11 we used coordinates to write equations of lines. For example,

$$3x + 4y + 7 = 0$$

is an equation of a line in an xy-plane. We say that

$$3x + 4y + 7 = 0$$

is a linear equation in x and y. The equation

$$ax + by + c = 0$$

is called the general linear equation in x and y. It is natural to extend this terminology and to call

$$ax + by + cz + d = 0$$

the general linear equation in x, y, z. In this section we show that, if an xyz-coordinate system has been set up, every plane has an equation of the form

$$ax + by + cz + d = 0$$

with a, b, c, d real numbers and with a, b, c not all zero, and conversely, that every equation of this form is an equation of a plane. What is the situation if a, b, and c are all zero? More specifically, what is the graph of $0x + 0y + 0z + 1 = 0$? Of $0x + 0y + 0z + 0 = 0$?

We begin with several examples, assuming that an xyz-coordinate system has been set up in space.

Example 1 Let $O = (0, 0, 0)$, $P = (3, 4, -6)$, and let α be the plane that is perpendicular to $\overline{OP}$ at P. We shall express α in terms of the coordinates of the points on it using set-builder notation. Now

$$\overrightarrow{OP} = \{(x, y, z) : x = 3k,\ y = 4k,\ z = -6k,\ k \geq 0\}.$$

Let Q be the point on $\overrightarrow{OP}$ that is twice as far from O as P is from O. To find the coordinates of Q, set $k = 2$, then $Q = (6, 8, -12)$, P is the midpoint of $\overline{OQ}$, and α is the perpendicular bisecting plane of $\overline{OQ}$.

The perpendicular bisecting plane of a segment is the set of all points equidistant from the endpoints of the segment. Using the Distance Formula, we find that the distance between $O(0, 0, 0)$ and (x, y, z) is

$$\sqrt{x^2 + y^2 + z^2}$$

and that the distance between $P(6, 8, -12)$ and (x, y, z) is

$$\sqrt{(x - 6)^2 + (y - 8)^2 + (z + 12)^2}.$$

Therefore

$$\alpha = \{(x, y, z) : \sqrt{x^2 + y^2 + z^2} = \sqrt{(x - 6)^2 + (y - 8)^2 + (z + 12)^2}\},$$

$$\alpha = \{(x, y, z) : x^2 + y^2 + z^2 = x^2 - 12x + 36 + y^2 - 16y + 64 + z^2 + 24z + 144\},$$

$$\alpha = \{(x, y, z) : 12x + 16y - 24z - 244 = 0\},$$

$$\alpha = \{(x, y, z) : 3x + 4y - 6z - 61 = 0\}.$$

Example 2 Let $O = (0, 0, 0)$, $P = (3, 4, -6)$, and let β be the plane that is perpendicular to $\overline{OP}$ at O. Note that β is parallel to the plane α of Example 1 and that α does not contain O, but that β does. Now

$$\overleftrightarrow{OP} = \{(x, y, z) : x = 3k,\ y = 4k,\ z = -6k,\ k \text{ is real}\}.$$

Let R be the point on *opp* $\overrightarrow{OP}$ such that $OR = OP$. To find the coordinates of R, set $k = -1$, then $R = (-3, -4, 6)$, and O is the midpoint of $\overline{RP}$ and β is the perpendicular bisecting plane of $\overline{RP}$.

β is the set of all points (x, y, z) each of which is at the same distance from $(3, 4, -6)$ as it is from $(-3, -4, 6)$. Therefore

$$\begin{aligned}
\beta = \{(x, y, z) :\ & \sqrt{(x-3)^2 + (y-4)^2 + (z+6)^2} \\
& = \sqrt{(x+3)^2 + (y+4)^2 + (z-6)^2}\},\\
\beta = \{(x, y, z) :\ & x^2 - 6x + 9 + y^2 - 8y + 16 + z^2 + 12z + 36 \\
& = x^2 + 6x + 9 + y^2 + 8y + 16 + z^2 - 12z + 36\},\\
\beta = \{(x, y, z) :\ & -12x - 16y + 24z = 0\},\\
\beta = \{(x, y, z) :\ & 3x + 4y - 6z = 0\}.
\end{aligned}$$

Example 3 Let γ be the following set of points in xyz-space:

$$\gamma = \{(x, y, z) : x - 2y + 3z - 4 = 0\}.$$

We shall show that γ is a plane.

Note in Example 1 that an equation of α is

$$3x + 4y - 6z - 61 = 0,$$

that the coefficients of x, y, z in this equation are $3, 4, -6$, that the foot of the perpendicular from O to α is $(3, 4, -6)$, that

$$3^2 + 4^2 + (-6)^2 = 61,$$

and that 61 is the negative of the constant term in the equation of α. This suggests that we transform the given equation of γ,

$$x - 2y + 3z - 4 = 0,$$

into an equivalent equation in which the sum of the squares of the coefficients is the negative of the constant term, that is, an equation

$$ax + by + cz + d = 0$$

where $a^2 + b^2 + c^2 = -d$.

We multiply through by k and then determine a positive value of k so that the resulting equation of γ has the desired property. We get

$$kx - 2ky + 3kz - 4k = 0$$

and we want

$$k^2 + (-2k)^2 + (3k)^2 = 4k \qquad \text{with } k > 0.$$

Therefore

$$k^2 + 4k^2 + 9k^2 = 4k, \qquad 14k^2 = 4k, \qquad k = \tfrac{2}{7}.$$

Therefore

$$\gamma = \{(x, y, z) : \tfrac{2}{7}x - \tfrac{4}{7}y + \tfrac{6}{7}z - \tfrac{8}{7} = 0\}.$$

Note that

$$(\tfrac{2}{7})^2 + (-\tfrac{4}{7})^2 + (\tfrac{6}{7})^2 = \tfrac{4}{49} + \tfrac{16}{49} + \tfrac{36}{49} = \tfrac{56}{49} = \tfrac{8}{7},$$

and that $\frac{8}{7}$ is the negative of the constant term of our "adjusted" equation of γ.

Taking a clue from Example 1, we suspect that γ is the plane that is perpendicular at $(\frac{2}{7}, -\frac{4}{7}, \frac{6}{7})$ to the line through $(0, 0, 0)$ and $(\frac{2}{7}, -\frac{4}{7}, \frac{6}{7})$, and hence that it is the perpendicular bisecting plane of the segment with endpoints $(0, 0, 0)$ and $(\frac{4}{7}, -\frac{8}{7}, \frac{12}{7})$. To show that it is, we let γ' be that perpendicular bisecting plane and we show that $\gamma' = \gamma$.

Now γ' is the set of all points (x, y, z) each of which is the same distance from $(0, 0, 0)$ as it is from $(\frac{4}{7}, -\frac{8}{7}, \frac{12}{7})$. Therefore

$$\begin{aligned}
\gamma' &= \{(x, y, z) : \sqrt{x^2 + y^2 + z^2} \\
&\qquad = \sqrt{(x - \tfrac{4}{7})^2 + (y + \tfrac{8}{7})^2 + (z - \tfrac{12}{7})^2}\},\\
\gamma' &= \{(x, y, z) : x^2 + y^2 + z^2 \\
&\qquad = x^2 - \tfrac{8}{7}x + \tfrac{16}{49} + y^2 + \tfrac{16}{7}y + \tfrac{64}{49} + z^2 - \tfrac{24}{7}z + \tfrac{144}{49}\},\\
\gamma' &= \{(x, y, z) : \tfrac{2}{7}x - \tfrac{4}{7}y + \tfrac{6}{7}z - \tfrac{8}{7} = 0\}.
\end{aligned}$$

Therefore $\gamma' = \gamma$, and γ is a plane. Since

$$\gamma = \{(x, y, z) : x - 2y + 3z - 4 = 0\},$$

this proves that

$$x - 2y + 3z - 4 = 0$$

is an equation of a plane.

Example 4 Let δ be the following set:

$$\delta = \{(x, y, z) : x - 2y + 3z = 0\}.$$

Since $(0, 0, 0)$ is an element of δ, it follows that δ is a set of points that contains the origin. We shall show that δ is a plane.

Taking a clue from Example 2, we suspect that δ is the plane that is perpendicular at the origin to the line through the origin and $(1, -2, 3)$. We suspect further that δ is the perpendicular bisecting plane of the segment with endpoints $(1, -2, 3)$ and $(-1, 2, -3)$. Let δ' be that perpendicular bisecting plane. We shall show that $\delta' = \delta$.

We have

$$
\begin{aligned}
\delta' = \{(x, y, z) :\ & \sqrt{(x-1)^2 + (y+2)^2 + (z-3)^2} \\
& \quad = \sqrt{(x+1)^2 + (y-2)^2 + (z+3)^2}\},\\
\delta' = \{(x, y, z) :\ & x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 - 6z + 9 \\
& \quad = x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 + 6z + 9\},\\
\delta' = \{(x, y, z) :\ & -4x + 8y - 12z = 0\},\\
\delta' = \{(x, y, z) :\ & x - 2y + 3z = 0\}.
\end{aligned}
$$

Therefore $\delta' = \delta$, and δ is a plane.

We are now ready for several theorems on linear equations in x, y, and z.

> *THEOREM 12.3* Given an xyz-coordinate system, every plane has a linear equation.

Proof: Let a plane α be given. We consider two cases.

Case 1. $O(0, 0, 0)$ does not lie on α.
Case 2. $O(0, 0, 0)$ lies on α.

Proof of Case 1: Let $P(a, b, c)$ be the foot of the perpendicular from O to α. Then a, b, c are not all zero. Let $Q = (2a, 2b, 2c)$. Then P is the midpoint of $\overline{OQ}$, and α is the perpendicular bisecting plane of $\overline{OQ}$. Then

$$
\begin{aligned}
\alpha = \{(x, y, z) :\ & \sqrt{x^2 + y^2 + z^2} \\
& \quad = \sqrt{(x-2a)^2 + (y-2b)^2 + (z-2c)^2}\},\\
\alpha = \{(x, y, z) :\ & ax + by + cz = a^2 + b^2 + c^2\}.
\end{aligned}
$$

Therefore in Case 1, α has a linear equation.

Proof of Case 2: Let $P(a, b, c)$ be a point distinct from O on the line perpendicular to α at O. Then a, b, c are not all zero. Let $Q = (-a, -b, -c)$. Then O is the midpoint of $\overline{PQ}$ and α is the perpendicular bisecting plane of $\overline{PQ}$. Then

$$
\begin{aligned}
\alpha = \{(x, y, z) :\ & \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \\
& \quad = \sqrt{(x+a)^2 + (y+b)^2 + (z+c)^2}\},\\
\alpha = \{(x, y, z) :\ & x^2 - 2ax + a^2 + y^2 - 2by + b^2 + z^2 - 2zc + c^2 \\
& \quad = x^2 + 2ax + a^2 + y^2 + 2by + b^2 + z^2 + 2zc + c^2\},\\
\alpha = \{(x, y, z) :\ & ax + by + cz = 0\}.
\end{aligned}
$$

Therefore in Case 2, α has a linear equation.

THEOREM 12.4 Given an xyz-coordinate system, the graph of every linear equation

$$ax + by + cz + d = 0,$$

in which a, b, c, d are real numbers and a, b, c are not all zero, is a plane.

Proof: Let an xyz-coordinate system and an equation

$$ax + by + cz + d = 0,$$

with a, b, c not all zero, be given. We consider two cases.

Case 1. $d \neq 0$.
Case 2. $d = 0$.

Proof of Case 1: Suppose $d \neq 0$ and let

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}.$$

Let k be a number, not zero, to be specified later. Then

$$\alpha = \{(x, y, z) : a'x + b'y + c'z + d' = 0\},$$

where $a' = ka$, $b' = kb$, $c' = kc$, and $d' = kd$. Taking a clue from Example 3, we shall determine k, with $k \neq 0$, so that

$$d' = -(a')^2 - (b')^2 - (c')^2.$$

Substituting we get

$$kd = -(ka)^2 - (kb)^2 - (kc)^2.$$

Solving for k we get

$$\begin{aligned} kd &= -k^2a^2 - k^2b^2 - k^2c^2, \\ d &= -ka^2 - kb^2 - kc^2, \\ d &= k(-a^2 - b^2 - c^2), \\ k &= \frac{-d}{a^2 + b^2 + c^2}, \end{aligned}$$

and this is our specified value of k. Note that since a, b, c are not all zero, then at least one of the nonnegative numbers, a^2, b^2, c^2, must be positive, and $a^2 + b^2 + c^2$ is positive. Hence the expression for k in terms of a, b, c, d is mathematically acceptable since the indicated division is division by a number that is not zero.

Checking we find that

$$\begin{aligned} d' &= kd = k^2(-a^2 - b^2 - c^2) = -(ka)^2 - (kb)^2 - (kc)^2, \\ d' &= -(a')^2 - (b')^2 - (c')^2. \end{aligned}$$

Let $P = (a', b', c')$ and $Q = (2a', 2b', 2c')$. Let α' be the perpendicular bisecting plane of $\overline{OQ}$. Then

$$\begin{aligned}
\alpha' &= \{(x, y, z) : \sqrt{x^2 + y^2 + z^2} \\
&\qquad\qquad = \sqrt{(x - 2a')^2 + (y - 2b')^2 + (z - 2c')^2}\},\\
&= \{x, y, z) : x^2 + y^2 + z^2 = x^2 - 4a'x + 4(a')^2 \\
&\qquad\qquad + y^2 - 4b'y + 4(b')^2 + z^2 - 4c'z + 4(c')^2\},\\
&= \{(x, y, z) : a'x + b'y + c'z - (a')^2 - (b')^2 - (c')^2 = 0\},\\
&= \{(x, y, z) : a'x + b'y + c'z + d' = 0\},\\
&= \{(x, y, z) : kax + kby + kcz + kd = 0\},\\
&= \{(x, y, z) : ax + by + cz + d = 0\},
\end{aligned}$$

and $\alpha' = \alpha$. Since α' is a plane, it follows that α is a plane, as we wished to prove.

Proof of Case 2: Suppose that $d = 0$ and let

$$\alpha = \{(x, y, z) : ax + by + cz = 0\}.$$

Then

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}.$$

Let $P = (a, b, c)$, $Q = (-a, -b, -c)$, and let α' be the perpendicular bisecting plane of $\overline{PQ}$. We shall show that $\alpha' = \alpha$, and this will prove that

$$ax + by + cz + d = 0$$

is an equation of a plane.

Now

$$\begin{aligned}
\alpha' &= \{(x, y, z) : \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} \\
&\qquad\qquad = \sqrt{(x + a)^2 + (y + b)^2 + (z + c)^2}\},\\
\alpha' &= \{(x, y, z) : x^2 - 2ax + a^2 + y^2 - 2by + b^2 + z^2 - 2zc + c^2 \\
&\qquad = x^2 + 2ax + a^2 + y^2 + 2by \\
&\qquad\qquad + b^2 + z^2 + 2zc + c^2\},\\
\alpha' &= \{(x, y, z) : -4ax - 4by - 4cz = 0\},\\
\alpha' &= \{(x, y, z) : ax + by + cz = 0\}.
\end{aligned}$$

Therefore $\alpha' = \alpha$.

THEOREM 12.5 If a, b, c, d are real numbers with a, b, c not all zero, then the plane

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}$$

is perpendicular to the line through $O(0, 0, 0)$ and $P(a, b, c)$.

Proof: If $d = 0$, then as in the proof of Case 2 of Theorem 12.4 it follows that α is perpendicular to the line $\overleftrightarrow{PQ}$, where

$$Q = (-a, -b, -c) \qquad \text{and} \qquad P = (a, b, c).$$

But O is the midpoint of $\overline{PQ}$. Therefore $\overleftrightarrow{OP} = \overleftrightarrow{PQ}$ and $\alpha \perp \overleftrightarrow{OP}$.

If $d \neq 0$, then as in the proof of Case 1 of Theorem 12.4, it follows that α is perpendicular to the line $\overleftrightarrow{OQ}$, where

$$Q = (2a', 2b', 2c').$$

Now

$$\overleftrightarrow{OQ} = \{(x, y, z) : x = 2a't,\ y = 2b't,\ z = 2c't,\ t \text{ is real}\}.$$

(We have used t as the parameter since k was used for another purpose in the proof of Theorem 12.4.) Setting $t = \dfrac{1}{2k}$, we obtain a particular point on $\overleftrightarrow{OQ}$, namely $(\dfrac{a'}{k}, \dfrac{b'}{k}, \dfrac{c'}{k}) = P(a, b, c)$. Since α is perpendicular to $\overleftrightarrow{OQ}$, and since $\overleftrightarrow{OQ} = \overleftrightarrow{OP}$, it follows that α is perpendicular to $\overleftrightarrow{OP}$.

Example 5 Find an equation of the plane perpendicular to $\overleftrightarrow{OP}$ at P if $O = (0, 0, 0)$ and $P = (2, -7, 0)$.

Solution: For every real number d, $2x - 7y + d = 0$ is an equation of a plane perpendicular to $\overleftrightarrow{OP}$. (Which theorem or theorems of this section are we using here?) To get a satisfactory equation we fix d so that the plane contains P. Substituting the coordinates of P, we get

$$2 \cdot 2 - 7(-7) + d = 0 \qquad \text{and} \qquad d = -53.$$

Therefore

$$2x - 7y - 53 = 0$$

is an equation of the plane perpendicular to $\overleftrightarrow{OP}$ at P.

Example 6 Find an equation of the plane perpendicular to $\overline{OP}$ at O if $O = (0, 0, 0)$ and $P = (2, -7, 0)$.

Solution: For every real number d, $2x - 7y + d = 0$ is an equation of a plane perpendicular to $\overleftrightarrow{OP}$. The origin $O(0, 0, 0)$ is a point of this plane if $2 \cdot 0 - 7 \cdot 0 + d = 0$, that is, if $d = 0$. Therefore

$$2x - 7y = 0$$

is an equation of the plane perpendicular to $\overline{OP}$ at O.

Example 7 Find an equation of the plane perpendicular to $\overleftrightarrow{OP}$ at P if $O = (0, 0, 0)$ and $P = (7, 4, -5)$.

Solution:

$$\begin{aligned}
7x + 4y - 5z + d &= 0 \\
7 \cdot 7 + 4 \cdot 4 - 5(-5) + d &= 0 \\
90 + d &= 0 \\
d &= -90 \\
7x + 4y - 5z - 90 &= 0
\end{aligned}$$

Example 8 Given

$$\alpha = \{(x, y, z) : 5x - 2y + 4z - 8 = 0\},$$

find a point P such that α is perpendicular to the line through $O(0, 0, 0)$ and P.

Solution: $P = (5, -2, 4)$.

Example 9 Given

$$\alpha = \{(x, y, z) : 2x - 7z = 0\},$$

find a point P such that α is perpendicular to the line through $O(0, 0, 0)$ and P.

Solution: $P = (2, 0, -7)$.

Example 10 Find an equation of the plane α that contains the points $O(0, 0, 0)$, $A(1, 1, 1)$, and $B(1, 7, -3)$.

Solution: We want real numbers a, b, c, d with a, b, c not all zero such that

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}.$$

Since O, A, B are points of α, it must be true that

$$\begin{aligned}
&(1) \qquad a \cdot 0 + b \cdot 0 + c \cdot 0 + d = 0, \\
&(2) \qquad a \cdot 1 + b \cdot 1 + c \cdot 1 + d = 0, \\
&(3) \qquad a \cdot 1 + b \cdot 7 + c \cdot (-3) + d = 0.
\end{aligned}$$

From (1) we deduce that $d = 0$. From (2) and (3) we deduce by subtraction that

$$(4) \qquad -6b + 4c = 0.$$

Now neither b nor c can be zero. Why? To complete the solution we take any number except 0 for b and solve for the corresponding a and c. Thus, if $b = -2$, then $c = -3$, $a = 5$, and

$$\alpha = \{(x, y, z) : 5x - 2y - 3z = 0\}.$$

To check, substitute coordinates as follows:

$$\begin{aligned}
O&: \quad 5\cdot 0 - 2\cdot 0 - 3\cdot 0 = 0 - 0 - 0 = 0.\\
A&: \quad 5\cdot 1 - 2\cdot 1 - 3\cdot 1 = 5 - 2 - 3 = 0.\\
B&: \quad 5\cdot 1 - 2\cdot 7 - 3\cdot(-3) = 5 - 14 + 9 = 0.
\end{aligned}$$

Example 11 Find an equation of the plane that contains the three points $P(1, 5, 7)$, $Q(-1, 2, -4)$, $R(2, 1, -5)$.

Solution:

(1) $ax + by + cz + d = 0$
(2) $a\cdot 1 + b\cdot 5 + c\cdot 7 + d = 0, a + 5b + 7c + d = 0$
(3) $a\cdot(-1) + b\cdot 2 + c\cdot(-4) + d = 0, -a + 2b - 4c + d = 0$
(4) $a\cdot 2 + b\cdot 1 + c\cdot(-5) + d = 0, 2a + b - 5c + d = 0$

From (2) and (3) we get (5), and from (3) and (4) we get (6).

(5) $2a + 3b + 11c = 0$
(6) $-3a + b + c = 0$

From (6) we get (7), and from (7) and (5) we get (8).

(7) $-9a + 3b + 3c = 0$
(8) $11a + 8c = 0$

Take $a = 8$; then $c = -11$. From (6) we get $b = 35$ and from (2) we get $d = -106$. A satisfactory answer is

$$8x + 35y - 11z - 106 = 0.$$

Check:

$$\begin{aligned}
P&: \quad 8\cdot 1 + 35\cdot 5 - 11\cdot 7 - 106\\
&\qquad = 8 + 175 - 77 - 106 = 0.\\
Q&: \quad 8\cdot(-1) + 35\cdot 2 - 11\cdot(-4) - 106\\
&\qquad = -8 + 70 + 44 - 106 = 0.\\
R&: \quad 8\cdot 2 + 35\cdot 1 - 11\cdot(-5) - 106\\
&\qquad = 16 + 35 + 55 - 106 = 0.
\end{aligned}$$

THEOREM 12.6 Consider the plane

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}.$$

If $a = 0$, α is parallel to the x-axis. If $b = 0$, α is parallel to the y-axis. If $c = 0$, α is parallel to the z-axis.

Proof: Suppose first that $a = 0$ and $d = 0$. Then

$$\alpha = \{(x, y, z) : by + cz = 0\}$$

and for every real number x, α contains the point $(x, 0, 0)$. Therefore α contains the x-axis and is parallel to it.

Suppose next that $a = 0$ and $d \neq 0$. Then

$$\alpha = \{(x, y, z) : by + cz + d = 0\}$$

and since

$$b \cdot 0 + c \cdot 0 + d \neq 0,$$

it follows that no point $(x, 0, 0)$ is a point of α and hence that the x-axis is parallel to α.

Similarly, it may be shown that if $b = 0$, then α is parallel to the y-axis, and that if $c = 0$, then α is parallel to the z-axis.

Example 12 Let α, β, γ, δ be given as follows:

$$\begin{aligned}
\alpha &= \{(x, y, z) : 3x + 4y - 12 = 0\},\\
\beta &= \{(x, y, z) : 3x - 2z = 0\},\\
\gamma &= \{(x, y, z) : 5y = 8\},\\
\delta &= \{(x, y, z) : 2x - 5 = 0\}.
\end{aligned}$$

(See Figure 12-7.) α is parallel to the z-axis. α contains the line l in the xy-plane with x- and y-intercepts 4 and 3, respectively, and every line parallel to the z-axis that passes through a point of l.

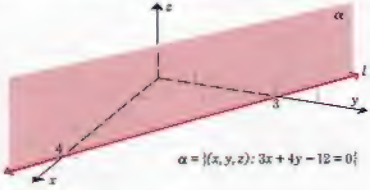
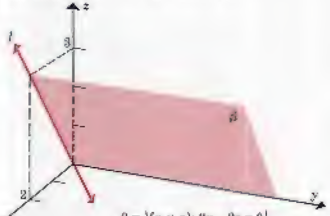


Figure 12-7

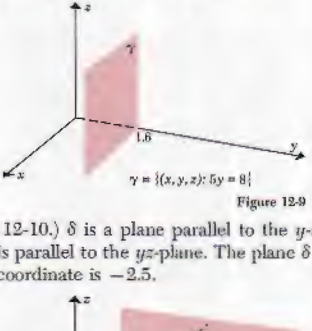
(See Figure 12-8.) β is parallel to the y-axis; in fact, it contains the y-axis. The plane β contains the line l in the xz-plane that passes through the points $(0, 0, 0)$ and $(2, 0, 3)$. It contains every line parallel to the y-axis that passes through a point of l.



$\beta = \{(x, y, z)\colon 3x - 2z = 0\}$

Figure 12-8

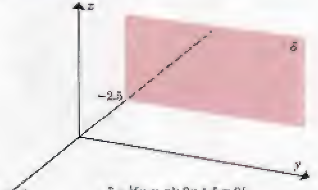
(See Figure 12-9.) γ is a plane parallel to the x-axis and the z-axis; hence it is parallel to the xz-plane. γ contains every point whose y-coordinate is 1.6 and every point of γ has y-coordinate 1.6.



$\gamma = \{(x, y, z)\colon 5y = 8\}$

Figure 12-9

(See Figure 12-10.) δ is a plane parallel to the y-axis and to the z-axis; hence it is parallel to the yz-plane. The plane δ is the set of all points whose x-coordinate is -2.5.



$\delta = \{(x, y, z)\colon 2x + 5 = 0\}$

Figure 12-10

EXERCISES 12.4

Let a plane $\alpha = \{(x, y, z) : 4x - 3y + 180 = 0\}$ be given. In Exercises 1–10, determine if the given point is a point of α.

1. $(0, 60, 0)$
2. $(45, 0, 0)$
3. $(0, 0, 180)$
4. $(0, 0, 0)$
5. $(0, 0, 37)$
6. $(0, 60, 137)$
7. $(-45, 0, 237)$
8. $(-30, 20, 10)$
9. $(7, 69, \frac{1}{3})$
10. $(-100, -73\frac{1}{3}, 66\frac{2}{3})$

In Exercises 11–20, an equation of a plane is given. Find three numbers a, b, c, not all zero, such that the given plane is perpendicular to the line through the origin and $P(a, b, c)$.

11. $-3x - 7y + 14 = 0$
12. $2x - y + z - 73 = 0$
13. $2y = 3z$
14. $y = 3x + 5$
15. $x = 3y - z$
16. $x + y - z + 1 = 0$
17. $x - y + 10z = 0$
18. $x - y + 10z + 10 = 0$
19. $y + 10x - 4z = 0$
20. $x - y = z - 1$

In Exercises 21–25, simplify the given equation to an equivalent equation in the form of the general linear equation $ax + by + cz + d = 0$. If one of the coefficients is zero, that term may be omitted in the simplified form. Thus we accept $3x + 4y = 0$ in place of $3x + 4y + 0z + 0 = 0$.

21. $\sqrt{x^2 + y^2 + z^2} = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$
22. $\sqrt{(x-2)^2 + (y-3)^2 + z^2} = \sqrt{(x+2)^2 + (y+3)^2 + z^2}$
23. $\sqrt{(x-1)^2 + (y-1)^2 + (z+4)^2} = \sqrt{(x-12)^2 + (y-5)^2 + z^2}$
24. $\sqrt{(x+4)^2 + (y+4)^2 + (z+4)^2} = \sqrt{x^2 + y^2 + z^2}$
25. $\sqrt{(x-1)^2 + y^2 + (z+1)^2} = \sqrt{(x-1)^2 + (y-1)^2 + (z+1)^2}$

In Exercises 26–30, find a linear equation in simplified form for the plane that contains the three given points.

26. $(0, 0, 0)$, $(8, 0, 0)$, $(-3, 0, 1)$
27. $(1, 3, 1)$, $(0, 1, 1)$, $(2, 1, 1)$
28. $(5, 2, 3)$, $(-3, 6, 7)$, $(0, 4, 6)$
29. $(8, 0, -4)$, $(0, 4, 3)$, $(4, 4, 1)$
30. $(3, 0, 1)$, $(-9, 0, -3)$, $(51, 10, 17)$

In Exercises 31–40, an equation of a plane is given. Find the coordinates of its intercepts on the coordinate axes. If it does not intersect an axis, write "none." The answer to Exercise 31 is given as a sample.

31. $2x - 3z + 30 = 0$

x-intercept: $(-15, 0, 0)$
y-intercept: None
z-intercept: $(0, 0, 10)$

32. $3x + 5y - 2z - 20 = 0$
33. $x - y + z = 0$
34. $5x - 4y + 100 = 0$
35. $2x - y + z = 0$
36. $x = 6$
37. $y = -3$
38. $z = -4$
39. $x + y + z = 100$
40. $x + y + z + 100 = 0$

In Exercises 41–50, state whether the given set S is a plane, the union of two distinct planes, a line, the union of two distinct lines, a set consisting of a single point, or the null set.

41. $S = \{(x, y, z) : x = 2, y = 3, z = 4\}$
42. $S = \{(x, y, z) : x + y + z = 5\}$
43. $S = \{(x, y, z) : x + y = 5\}$
44. $S = \{(x, y) : x + y = 5\}$
45. $S = \{(x, y) : x = 5\}$
46. $S = \{(x, y, z) : x = 5 \text{ or } x = 10\}$
47. $S = \{(x, y, z) : x + y + z = 10 \text{ and } x + y + z = 20\}$
48. $S = \{(x, y, z) : x + y + z = 10 \text{ or } x + y + z = 20\}$
49. $S = \{(x, y, z) : x + y + 2z = 10 \text{ and } x - 2y + 2z = 10\}$
50. $S = \{(x, y, z) : x + y + 2z = 10, x - y + 2z = 10, z = 0\}$

12.5 SYMMETRIC EQUATIONS FOR A LINE

Since a linear equation in x and y represents a line in an xy-plane and a linear equation in x, y, z represents a plane in xyz-space, it is natural to ask how we might represent a line in terms of coordinates in xyz-space and to do it without using a parameter. A clue to the answer is provided by some of the exercises of Exercises 12.4. As usual, we assume an xyz-coordinate system is given.

Example 1 If $P = (2, 1, 7)$ and $Q = (5, -1, -3)$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{x-2}{5-2} = \frac{y-1}{-1-1} = \frac{z-7}{-3-7}\right\}.$$

The notation

$$\frac{x-2}{5-2} = \frac{y-1}{-1-1} = \frac{z-7}{-3-7}$$

is a short way of saying that

$$\frac{x-2}{5-2} = \frac{y-1}{-1-1} \qquad \text{and} \qquad \frac{y-1}{-1-1} = \frac{z-7}{-3-7}.$$

To see that the representation in terms of these equations is correct, note that

$$\frac{x-2}{5-2}=\frac{y-1}{-1-1} \qquad \text{and} \qquad \frac{y-1}{-1-1}=\frac{z-7}{-3-7}$$

are equations of distinct planes, one of them parallel to the z-axis and the other parallel to the x-axis. Note that both P and Q lie on each of these planes. (Check this by substituting coordinates.) Hence $\overleftrightarrow{PQ}$ is precisely the intersection of these two planes and the representation we gave in terms of "symmetric" equations is a satisfactory one. Actually, there is more symmetry in this situation than we have indicated so far. It is natural to think of $\overleftrightarrow{PQ}$ as the intersection of three planes related to the three expressions equated to each other in the set-builder notation. Thus every point of $\overleftrightarrow{PQ}$ lies on each of the following three planes, α, β, and γ, which are parallel, respectively, to the x-, y-, and z-axes:

$$\alpha=\left\{(x, y, z) : \frac{y-1}{-1-1}=\frac{z-7}{-3-7}\right\},$$

$$\beta=\left\{(x, y, z) : \frac{x-2}{5-2}=\frac{z-7}{-3-7}\right\},$$

$$\gamma=\left\{(x, y, z) : \frac{x-2}{5-2}=\frac{y-1}{-1-1}\right\}.$$

Generalizing the situation in Example 1, we obtain the following theorem.

THEOREM 12.7 If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two distinct points with $x_1 \neq x_2$, $y_1 \neq y_2$, $z_1 \neq z_2$, then

$$\overleftrightarrow{PQ}=\left\{(x, y, z) : \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\right\}.$$

Proof: Let

$$\alpha=\left\{(x, y, z) : \frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\right\},$$

$$\beta=\left\{(x, y, z) : \frac{x-x_1}{x_2-x_1}=\frac{z-z_1}{z_2-z_1}\right\},$$

$$\gamma=\left\{(x, y, z) : \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}\right\}.$$

Now $P(x_1, y_1, z_1)$ is a point of α since $\dfrac{y_1 - y_1}{y_2 - y_1} = \dfrac{z_1 - z_1}{z_2 - z_1} = 0$

and $Q(x_2, y_2, z_2)$ is a point of α since $\dfrac{y_2 - y_1}{y_2 - y_1} = \dfrac{z_2 - z_1}{z_2 - z_1} = 1.$

Similarly, we can show that P and Q each lie on both β and γ.

Also, α, β, γ are three distinct planes. For α contains the point $(x_1 - 1, y_1, z_1)$, whereas β and γ do not; β contains $(x_1, y_1 - 1, z_1)$, whereas α and γ do not; γ contains $(x_1, y_1, z_1 - 1)$, whereas α and β do not. The -1 here may appear to border on the magic. Actually it does not. For example, if x_0 is any nonzero number whatsoever, then α contains $(x_1 - x_0, y_1, z_1)$, whereas β and γ do not.

Therefore (x, y, z) is a point of $\overleftrightarrow{PQ}$ if and only if it is a point of all three of the planes α, β, γ, and hence the representation using symmetric equations is a valid one.

THEOREM 12.8 If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_1)$ are distinct points with $x_1 \neq x_2$ and $y_1 \neq y_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, z = z_1\right\}.$$

Proof:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \qquad \text{and} \qquad z = z_1$$

are equations of two distinct planes each containing both P and Q. Therefore $\overleftrightarrow{PQ}$ is the intersection of two planes, and the representation given in the statement of the theorem is a valid one.

Similar proofs may be given for Theorems 12.9 and 12.10.

THEOREM 12.9 If $P(x_1, y_1, z_1)$ and $Q(x_2, y_1, z_2)$ are distinct points with $x_1 \neq x_2$ and $z_1 \neq z_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{x - x_1}{x_2 - x_1} = \frac{z - z_1}{z_2 - z_1}, y = y_1\right\}.$$

THEOREM 12.10 If $P(x_1, y_1, z_1)$ and $Q(x_1, y_2, z_2)$ are two distinct points with $y_1 \neq y_2$ and $z_1 \neq z_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, x = x_1\right\}.$$

Proofs of Theorems 12.11, 12.12, 12.13 are assigned as exercises.

THEOREM 12.11 If $P(x_1, y_1, z_1)$ and $Q(x_1, y_1, z_2)$ are two distinct points, then

$$\overleftrightarrow{PQ} = \{(x, y, z) : x = x_1 \text{ and } y = y_1\}.$$

THEOREM 12.12 If $P(x_1, y_1, z_1)$ and $Q(x_1, y_2, z_1)$ are two distinct points, then

$$\overleftrightarrow{PQ} = \{(x, y, z) : x = x_1 \text{ and } z = z_1\}.$$

THEOREM 12.13 If $P(x_1, y_1, z_1)$ and $Q(x_2, y_1, z_1)$ are two distinct points, then

$$\overleftrightarrow{PQ} = \{(x, y, z) : y = y_1 \text{ and } z = z_1\}.$$

EXERCISES 12.5

In Exercises 1–5, a line is given in terms of symmetric equations. In each case, find the missing coordinates of P, Q, R so that they will be points of the given line.

1. $\left\{(x, y, z) : \dfrac{x-1}{1} = \dfrac{y-2}{3} = \dfrac{z-4}{-5}\right\}$,
   $P(1, \boxed{?}, \boxed{?})$, $Q(2, \boxed{?}, \boxed{?})$, $R(3, \boxed{?}, \boxed{?})$

2. $\left\{(x, y, z) : \dfrac{x-3}{4} = \dfrac{y+1}{3} = \dfrac{z-10}{-2}\right\}$,
   $P(3, \boxed{?}, \boxed{?})$, $Q(7, \boxed{?}, \boxed{?})$, $R(11, \boxed{?}, \boxed{?})$

3. $\left\{(x, y, z) : \dfrac{x+3}{-5} = \dfrac{y-1}{3} = \dfrac{z+4}{2}\right\}$,
   $P(\boxed{?}, 1, \boxed{?})$, $Q(\boxed{?}, 4, \boxed{?})$, $R(\boxed{?}, -2, \boxed{?})$

4. $\left\{(x, y, z) : \dfrac{x+1}{6} = \dfrac{y+2}{-3} = \dfrac{z-1}{4}\right\}$,
   $P(\boxed{?}, \boxed{?}, -7)$, $Q(\boxed{?}, \boxed{?}, -3)$, $R(\boxed{?}, \boxed{?}, 1)$

5. $\left\{(x, y, z) : \dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}\right\}$,
   $P(0, \boxed{?}, \boxed{?})$, $Q(1, \boxed{?}, \boxed{?})$, $R(2, \boxed{?}, \boxed{?})$

In Exercises 6–10, a line is given. To which of the coordinate axes, if any, is it parallel?

6. $\left\{(x, y, z) : \dfrac{x-1}{3} = \dfrac{y+2}{2} = \dfrac{z-1}{5}\right\}$
7. $\left\{(x, y, z) : \dfrac{x-1}{3} = \dfrac{y+2}{2}, z = 1\right\}$
8. $\left\{(x, y, z) : \dfrac{y+2}{2} = \dfrac{z-1}{5}, x = 1\right\}$
9. $\left\{(x, y, z) : \dfrac{x-1}{3} = \dfrac{z-1}{5}, y = -2\right\}$
10. $\{x, y, z) : x = 3, y = 4\}$

In Exercises 11–15, a line is given in terms of parametric equations. Represent the line using symmetric equations.

11. $\{(x, y, z) : x = 1 + 3k, y = 2 - 2k, z = 3 - 4k, k \text{ is real}\}$
12. $\{(x, y, z) : x = -1 - k, y = 2 + 4k, z = 5 - k, k \text{ is real}\}$
13. $\{(x, y, z) : x = k, y = 2k, z = 3k, k \text{ is real}\}$
14. $\{(x, y, z) : x = 3 - 4k, y = 2k, z = 1 - k, k \text{ is real}\}$
15. $\{x, y, z) : x = 2 + k(5 - 2), y = 3 + k(5 - 3), z = 2 + k(1 - 2), k \text{ is real}\}$

In Exercises 16–20, a plane and a line are given. Find their point of intersection.

16. $\{(x, y, z) : 3x + 4y + 5z = 60\}, \left\{(x, y, z) : \dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{5}\right\}$
17. $\{(x, y, z) : 2x + y - 10 = 0\}, \{(x, y, z) : x = 5 \text{ and } z = 3\}$
18. $\{(x, y, z) : x - y + z = 0\}, \left\{(x, y, z) : \dfrac{x-1}{1} = \dfrac{y-1}{1} = \dfrac{z-1}{1}\right\}$
19. $\{(x, y, z) : 2x - y + z + 5 = 0\},$
$\left\{(x, y, z) : \dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-4}{5}\right\}$
20. $\{(x, y, z) : x + y + z + 1 = 0\}, \{(x, y, z) : x = y = z\}$

21. Prove Theorem 12.9.
22. Prove Theorem 12.10.
23. Prove Theorem 12.11.
24. Prove Theorem 12.12.
25. Prove Theorem 12.13.

CHAPTER SUMMARY

We began this chapter by defining an xyz-COORDINATE SYSTEM. Starting with a pair of perpendicular lines called the x- and the y-axes, we know that there is one and only one line through their intersection that is perpendicular to both of them. We call this line the z-AXIS. The one and only point that lies on all three of the coordinate axes is called the ORIGIN. There are three coordinate axes and three coordinate planes. Each axis lies in two of the coordinate planes and is perpendicular to the other one. All lines perpendicular to a coordinate plane are parallel to one of the coordinate axes. All lines parallel to a coordinate axis are perpendicular to one and the same coordinate plane. If a plane is parallel to a coordinate plane, then it is perpendicular to a coordinate axis and to all lines parallel to that axis.

In our work with coordinates in space we stated and proved the DISTANCE FORMULA THEOREM and stated a theorem regarding parametric equations for a line. These theorems are natural extensions of some theorems for the two-dimensional case in Chapter 11. The Distance Formula and parametric equations for a line are useful tools in solving problems involving solid geometry. Linear equations may be used to represent planes, and combinations of them in what we call symmetric form may be used to represent lines.

REVIEW EXERCISES

In Exercises 1–8, eight points are given as follows: $A(0, 0, 0)$, $B(3, 0, 0)$, $C(3, 4, 0)$, $D(0, 4, 0)$, $E(0, 0, 2)$, $F(3, 0, 2)$, $G(3, 4, 2)$, $H(0, 4, 2)$.

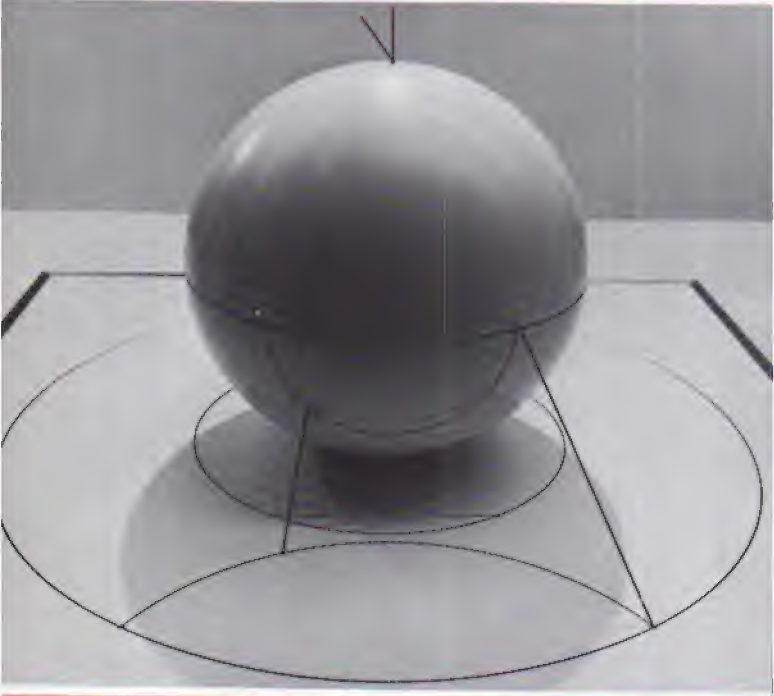
1. Draw a graph of the rectangular solid with vertices A, B, C, D, E, F, G, H.
2. Prove that $\angle ABF \cong \angle DBF \cong \angle CBF \cong \angle EFG$.
3. Find the measure of dihedral angle H–EA–F.
4. Prove that $\overline{AG} \cong \overline{EC} \cong \overline{BH} \cong \overline{FD}$.
5. Find the distance from point A to plane CDH.
6. Find the distance from point A to plane FDH.
7. Find the distance between plane ABC and plane EFG.
8. Let I, J, K be the midpoints of $\overline{EH}$, $\overline{HG}$, $\overline{DC}$, respectively. Find the distance between the planes IJK and EGC.

In Exercises 9–13, find the distance between the two given points.

9. (x_1, y_1, z_1) and (x_2, y_2, z_2).
10. (a, b, c) and (d, e, f).
11. (p, q, r) and (p, q, t).
12. $(7, 8, 0)$ and $(10, 12, 0)$.
13. $(3, 4, 5)$ and $(8, 9, 10)$.

In Exercises 14–22, there are two given points: $A(1, 2, 3)$ and $B(4, 5, 6)$.

14. Express $\overleftrightarrow{AB}$ using parametric equations and set-builder notation.
15. Express $\overrightarrow{AB}$ using parametric equations and set-builder notation.
16. Express $\overrightarrow{BA}$ using parametric equations and set-builder notation.
17. Express *opp* $\overrightarrow{AB}$ using parametric equations and set-builder notation.
18. Express *opp* $\overrightarrow{BA}$ using parametric equations and set-builder notation.
19. Find C on $\overrightarrow{AB}$ if $AC = 10 \cdot AB$.
20. Find C on *opp* $\overrightarrow{AB}$ if $AC = 10 \cdot AB$.
21. Find the trisection points of $\overline{AB}$.
22. Find the two points which divide $\overline{AB}$ internally and externally in the ratio $\frac{2}{1}$.
23. Given $A(5, 1, 1)$, $B(3, 1, 0)$, $C(4, 3, -2)$, $D(6, 3, -1)$, prove that quadrilateral $ABCD$ is a parallelogram.
24. Given $A(2, 4, 1)$, $B(1, 2, -2)$, $C(5, 0, -2)$, prove that $\triangle ABC$ is a right triangle.
25. Find three noncollinear points on the plane given by the equation
$$3x + 4y - z + 15 = 0.$$
26. Find three distinct points on the line given by the symmetric equation
$$\frac{x-2}{3} = \frac{y-4}{2} = \frac{z+3}{5}.$$
27. Write an equation of the plane that is perpendicular at $(2, 4, -3)$ to the line of Exercise 26.
28. Find the x-, y-, and z-intercepts of the plane α where
$$\alpha = \{(x, y, z) : 3x - 7y - 6z + 5 = 0\}.$$
29. Express the line l where
$$l = \{(x, y, z) : x = 1 + 2k, y = 1 - 3k, z = k, k \text{ is real}\},$$
using symmetric equations but no parameter.
30. Consider the line l and the plane α perpendicular to l at $O(0, 0, 0)$, given as follows:
$$l = \left\{(x, y, z) : \frac{x}{1} = \frac{y}{2} = \frac{z}{-4}\right\},$$
$$\alpha = \{(x, y, z) : x + 2y - 4z = 0\}.$$
Find two distinct points P and Q on l such that α is the perpendicular bisecting plane of $\overline{PQ}$.



Chapter 13

Erick Hartmann/Magnum Photos

Circles and Spheres

13.1 INTRODUCTION

As the title suggests, this chapter is concerned with properties of circles and spheres, some of which you may already have studied in your earlier work in mathematics.

The first part of the chapter deals with the intersection properties of a circle and a line in the plane of the circle and the intersection properties of a sphere and a plane.

The second part of the chapter is about the degree measure of arcs of a circle and properties of certain angles in relation to arcs, secants, tangents, and chords. We also consider properties of lengths of secant-segments, tangent-segments, and chords.

13.2 CIRCLES AND SPHERES: BASIC DEFINITIONS

Up to now, in our formal geometry, we have not discussed "curved figures," that is, figures not made up of segments, rays, or lines. Perhaps the simplest of these figures are the circle, the sphere, and portions of a circle, called arcs. We begin with some formal definitions.

Definition 13.1 (See Figure 13-1.) Let r be a positive number and let O be a point in a given plane. The set of all points P in the given plane such that $OP = r$ is called a **circle**. The given point O is called the **center** of the circle, the given number r is called the **radius** of the circle, and the number $2r$ is called the **diameter** of the circle.

A circle is a curve and encloses a portion of a plane. As an illustration, consider a circular disk. The edge of the disk is what we have in mind when we think of a circle. The edge of the disk together with its interior points is what we have in mind when we think of a *circular region*. We will have more to say about a circular region (that is, a circle and its interior) in Chapter 14.

Question: Is the center of a circle a point of the circle? Explain.

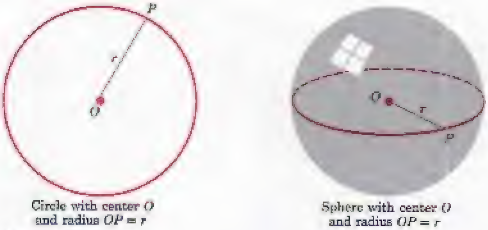


Figure 13-1

Definition 13.2 (See Figure 13-1.) Let r be a positive number and let O be a point in space. The set of all points P in space such that $OP = r$ is called a **sphere**. The given point O is called the **center** of the sphere, the given number r is called the **radius** of the sphere, and the number $2r$ is called the **diameter** of the sphere.

A sphere is a surface and encloses a portion of space. As an illustration, consider a ball. The surface of the ball is what we have in mind when we think of a sphere. The surface of the ball together with its interior points is what we have in mind when we think of a *spherical region*. We will have more to say about a spherical region (that is, a sphere and its interior) in Chapter 15.

Definition 13.3 Two or more coplanar circles, or two or more spheres, with the same center are said to be **concentric.**

In Figure 13-2, Q is the common center of three concentric circles with radii (plural of *radius*) r_1, r_2, and r_3, respectively.

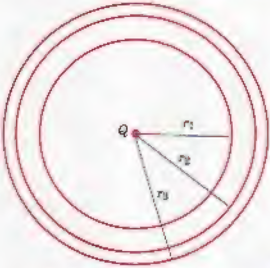


Figure 13-2

Let O be a point in a given plane α. Then we can choose an xy-coordinate system in α such that the origin is at O. Let C be a circle in α with center at O and radius r. Let $P(x, y)$ be any point on C as shown in Figure 13-3.

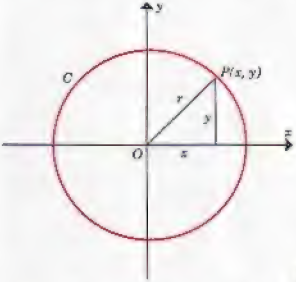


Figure 13-3

Then, by the Distance Formula, $OP = r = \sqrt{(x-0)^2 + (y-0)^2}$. Therefore $r = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = r^2$.

Conversely, if $P'(x, y)$ is any point such that $x^2 + y^2 = r^2$, then $\sqrt{(x-0)^2 + (y-0)^2} = r$, $OP' = r$, and P' is a point of the circle C. Thus, in a given xy-plane, the circle C with center at the origin and with radius r is given by $C = \{(x, y) : x^2 + y^2 = r^2\}$.

We have proved the following theorem.

THEOREM 13.1 Let an xy-plane be given and let O be the origin and let r be a positive number. Let C be the circle in the xy-plane with center O and radius r. Then

$$C = \{(x, y) : x^2 + y^2 = r^2\}.$$

Definition 13.4 A chord of a circle or a sphere is a segment whose endpoints are points of the circle or sphere. A secant of a circle or sphere is a line containing a chord of the circle or sphere. A diameter of a circle or sphere is a chord containing the center of the circle or sphere. A radius of a circle or sphere is a segment with one endpoint at the center and the other endpoint on the circle or sphere.

In Figure 13-4, C is a circle with center P, and S is a sphere with center P. For the circle and the sphere, $\overline{AB}$ is a chord, $\overleftrightarrow{AB}$ is a secant, $\overline{ED}$ is a diameter (and also a chord), and $\overline{PQ}$ is a radius. The endpoint of a radius that is on a circle or a sphere (such as point Q in Figure 13-4) is often referred to as *the outer end* of that radius. Is $\overleftrightarrow{ED}$ in Figure 13-4 a secant? It follows from Definition 13.4 that a secant is a line that intersects a circle (or sphere) in two distinct points.

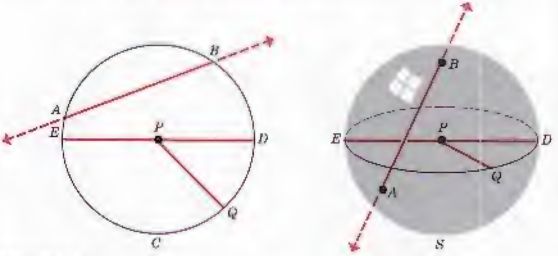


Figure 13-4

Note that, in connection with a circle or a sphere, the word "radius" is used in two different ways and the word "diameter" is used in two different ways: (1) each word is used to mean a certain segment and (2) each word is used to mean the positive number that is the length of a segment. This should not be confusing because the context in which the word is used should make it easy for you to decide which meaning is intended. For example, if we speak of *a* radius or *a* diameter of a circle or of a sphere, we mean *a segment*. If we speak of *the* radius

or *the* diameter, we mean *the number* that is the length of a segment. Thus a circle has infinitely many different radii if radius is interpreted to mean a segment; it has just one radius if radius means the length of a segment.

Just as we speak of congruent angles, or congruent segments, or congruent triangles, we often speak about *congruent circles* or *congruent spheres*. How would you define congruent circles or congruent spheres? Does your definition agree with the following one?

> ***Definition 13.5*** Two circles (distinct or not) are **congruent** if their radii are equal. Two spheres (distinct or not) are **congruent** if their radii are equal.

Using Definition 13.5, it is not hard to prove that congruence for circles (or spheres) is an equivalence relation; that is, it is reflexive, symmetric, and transitive. Thus if A, B, C are any three circles (or spheres), it is true that

1. $A \cong A$.	Reflexive Property
2. If $A \cong B$, then $B \cong A$.	Symmetric Property
3. If $A \cong B$ and $B \cong C$, then $A \cong C$.	Transitive Property

EXERCISES 13.2

In Exercises 1–10, refer to the circle with center P shown in Figure 13-5. Assume that all the points named in the figure are where they appear to be in the plane of the circle. Copy and replace the question marks with words or symbols that best name or describe the indicated parts.

1. $\overline{DE}$ is called a $\boxed{?}$ of the circle.
2. $\overline{PB}$ is called a $\boxed{?}$ of the circle.
3. $\overline{AB}$ is called a $\boxed{?}$ of the circle. $\overline{AB}$ could also be called a $\boxed{?}$ of the circle.
4. $\overleftrightarrow{FG}$ is called a $\boxed{?}$ of the circle.
5. $\overline{FG}$ is called a $\boxed{?}$ of the circle.
6. $\overleftrightarrow{DE}$ is called a $\boxed{?}$ of the circle.
7. C is the outer end of the radius $\boxed{?}$.
8. A is the $\boxed{?}$ of the radius $\boxed{?}$.
9. The points named in the figure that are points of the circle (that is, on the circle) are $\boxed{?}$.
10. The points named in the figure that are not points of the circle are $\boxed{?}$.

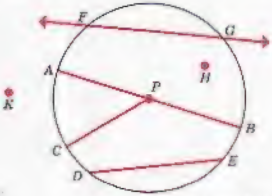


Figure 13-5

In Exercises 11–15, refer to the sphere with center Q shown in Figure 13-6. Copy and replace the question marks with words or symbols that best name or describe the indicated parts.

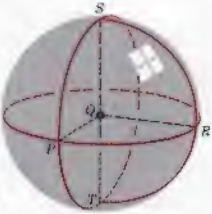


Figure 13-6

11. $\overline{QR}$ is a $\boxed{?}$ of the sphere.
12. If S, Q, T are collinear, then $\overline{ST}$ is a $\boxed{?}$ of the sphere.
13. $\overline{RS}$ is a $\boxed{?}$ of the sphere.
14. $\overleftrightarrow{RT}$ is a $\boxed{?}$ of the sphere.
15. Points $\boxed{?}$ are outer ends of given radii.

16. Prove that if two circles are congruent, then a diameter of one is congruent to a diameter of the other.
17. Prove that congruence of circles is an equivalence relation. (You will need to use Definition 13.5 and the theorem that congruence of segments (radii) is an equivalence relation.)

In Exercises 18–25, refer to the circle with center at the origin of the xy-coordinate system shown in Figure 13-7. $P(x, y)$ is a point on the circle.

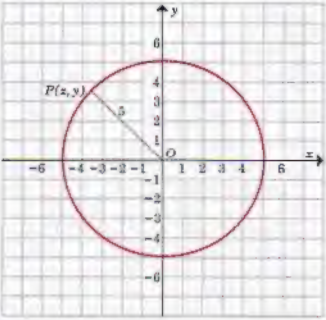
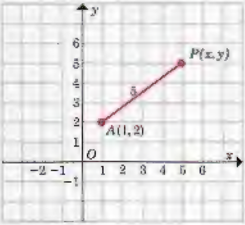


Figure 13-7

18. Use the Distance Formula and express the distance between O and P in terms of x and y.
19. Write an equation like that in Theorem 13.1 of the circle with center at the origin and radius 5.
20. Write the coordinates of the points on the x- and y-axes that are on the circle of Exercise 19. Check your answers by substituting the coordinates in the equation of the circle.
21. Is $A = (3, 4)$ a point of the circle of Exercise 19?
22. Is $B = (-4, 3)$ a point of the circle of Exercise 19?
23. Is $K = (1, 2\sqrt{6})$ a point of the circle of Exercise 19?
24. Is $R = (-2, 4.5)$ a point of the circle of Exercise 19?
25. Write the coordinates of two more points (different from those in Exercises 20–24) that lie on the circle of Exercise 19.

26. Given the xy-coordinate system shown in the figure with $A = (1, 2)$, $P = (x, y)$, and $AP = 5$, use the Distance Formula and express the distance between A and P in terms of x and y.



27. In Exercise 26, write an equation like that in Theorem 13.1 of the circle with center at A and radius $AP = 5$. Is the point $S = (4, 6)$ a point of this circle? Write the coordinates of at least three more points that are on this circle.
28. Which of the following are points of $C = \{(x, y) : x^2 + y^2 = 100\}$?
(a) $(0, 10)$ (b) $(-6, 8)$ (c) $(4, 9)$ (d) $(2\sqrt{5}, -2\sqrt{15})$
29. What is the radius of the circle of Exercise 28?
30. Find five more points of C in Exercise 28.

Exercises 31–34 concern the set C, where $C = \{(x, y) : x^2 + y^2 = 16\}$.

31. Is C a circle? Why?
32. Find x if $(x, 3)$ is a point of C. (There are two possible values for x.)
33. Find y if $(4, y)$ is a point of C.
34. Can you find x so that $(x, 5)$ is a point of C? Explain.

Exercises 35–37 concern the circle C, where $C = \{(x, y) : x^2 + y^2 = 36\}$.

35. What restrictions on x and y would give only the part of the circle in quadrant I?
36. What part of the circle would you be considering under the restriction $x^2 + y^2 = 36$ and $x \leq 0$?
37. What restrictions on x and y would give the intersection of C and quadrant III?

38. CHALLENGE PROBLEM. Let an xy-plane with $A = (h, k)$, $P = (x, y)$, $AP = r > 0$ be given. Write an equation in terms of x, y, h, k, r for the circle with center at A and radius $AP = r$.

13.3 TANGENT LINES

If you look at a drawing of a circle in a given plane, it is easy to see that the circle separates the points of the plane not on the circle into two sets. One of these sets consists of those points of the given plane that are "inside" the circle and the other set consists of those points of the given plane that are "outside" the circle.

Definition 13.6 (See Figure 13-8.) Let a circle with center O and radius r in plane α be given. The interior of the circle is the set of all points P in plane α such that $OP < r$. The exterior of the circle is the set of all points P in plane α such that $OP > r$.

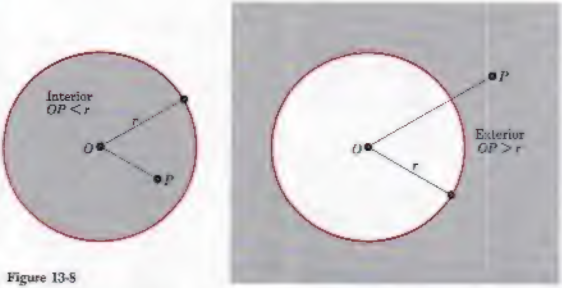


Figure 13-8

It is clear from Definitions 13.1 and 13.6 that if P is any point in the plane of a circle with center O and radius r, then P is on the circle ($OP = r$), or P is in the interior of the circle ($OP < r$), or P is in the ex-

terior of the circle ($OP > r$). We sometimes say "P is inside the circle" or "P is outside the circle" when we mean "P is in the interior of the circle" or "P is in the exterior of the circle," respectively.

Figure 13-9 shows an xy-coordinate system whose origin is the center of a given circle C with radius r. The figure also includes expressions for C, its interior and its exterior, in terms of coordinates using set-builder notation.

$$C = \{(x, y) : x^2 + y^2 = r^2\}$$
$$\text{Interior of } C = \{(x, y) : x^2 + y^2 < r^2\}$$
$$\text{Exterior of } C = \{(x, y) : x^2 + y^2 > r^2\}$$

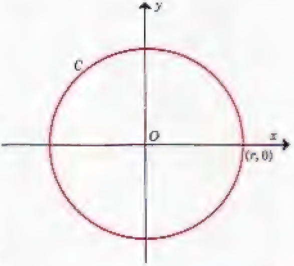


Figure 13-9

> ***Definition 13.7*** If a line in the plane of a circle intersects the circle in exactly one point, the line is called a **tangent** to the circle and the point is called the **point of tangency**, or the **point of contact**. We say that the line and the circle are **tangent** at this point. If a segment or a ray intersects a circle and if the line that contains that segment or ray is tangent to the circle, then the segment or ray is said to be **tangent** to the circle.

In Figure 13-10, if l is a line in the plane of circle C and if the intersection of l and C is just the one point P, then l is tangent to C at P.

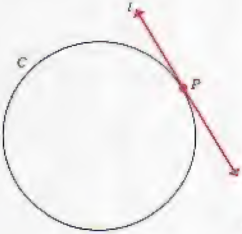


Figure 13-10

Given a circle and a line in the plane of the circle, what are the possibilities with regard to the intersection of the line and the circle? Figure 13-11 suggests that there are only these three possibilities: the intersection of the line and the circle may be the empty set, or it may be a set consisting of exactly one point, or it may be a set consisting of exactly two points.

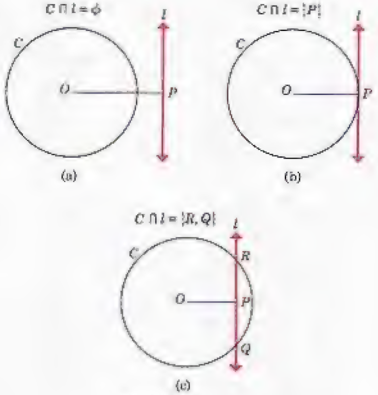


Figure 13-11

That these are the only three possibilities can be verified in the following way. Suppose that we are given a circle C with center O in plane α and that P is a point in plane α. Then P is outside the circle, as shown in Figure 13-11a, or P is on the circle, as shown in (b), or P is inside the circle as shown in (c).

If P is outside the circle, we shall show that the unique line l in plane α such that $\overline{OP} \perp l$ at P does not intersect the circle. If P is on the circle, we shall show that the unique line l in plane α such that $\overline{OP} \perp l$ at P intersects the circle in exactly one point (hence l is tangent to C at P). If P is inside the circle, then either $P = O$ or $P \neq O$. If $P \neq O$, we will show that the unique line l in plane α such that $\overline{OP} \perp l$ at P intersects the circle in exactly two points which are equidistant from P. Finally, if $P = O$, we will show that any line l which contains P intersects the circle in exactly two points. We are now ready for the following theorem.

THEOREM 13.2 Given a line l and a circle C in the same plane, let O be the center of the circle and let P be the foot of the perpendicular from O to line l.

1. Every point of l is outside C if and only if P is outside C.
2. l is tangent to C if and only if P is on C.
3. l is a secant of C if and only if P is inside C.

Proof: Let r be the radius of C and let $OP = a$. We select an xy-coordinate system in the plane of C and l with the origin at O, with the y-axis parallel to l and with P on the nonnegative x-axis. Then

$$P = (a, 0),$$
$$C = \{(x, y) : x^2 + y^2 = r^2\},$$

and

$$l = \{(x, y) : x = a\}.$$

Proof of 1: Suppose we are given that P is outside C as shown in Figure 13-12; then $a > r > 0$. Why? It follows that $a^2 > r^2$ and hence $a^2 + y^2 > r^2$. Therefore all points (a, y) are outside C. Since

$$l = \{(x, y) : x = a\} = \{(a, y) : y \text{ is real}\},$$

it follows that all points of l are outside C.

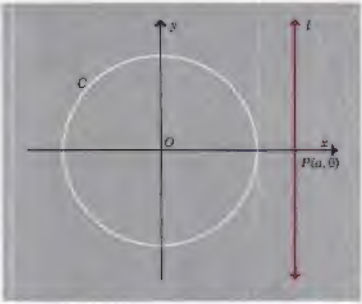


Figure 13-12

Now suppose that l is outside C; then every point of l, including P, is outside C and the proof of (1) is complete.

Proof of 2: Suppose we are given that P is on C as shown in Figure 13-13; then $a = r$. Why? The intersection of C and l is

$$\{(x, y) : x^2 + y^2 = r^2\} \cap \{(x, y) : x = a\} = \{(a, y) : y^2 = 0\}.$$

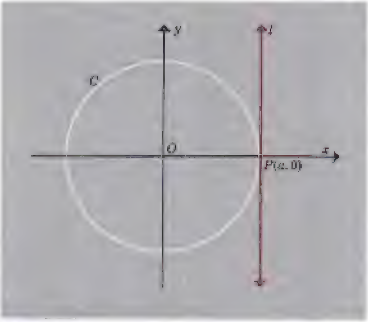


Figure 13-13

Since zero is the only number whose square is zero, it follows that $y = 0$. Therefore the only point of intersection of l and C is $P(a, 0)$. Therefore l is tangent to C at P. Why? This proves that l is tangent to C if P is on C.

Now suppose it is given that l is tangent to C. Then l and C have exactly one point in common. This means that $l \cap C$ is a set consisting of exactly one point. But

$$C = \{(x, y) : x^2 + y^2 = r^2\},$$
$$l = \{(x, y) : x = a\},$$

and

$$\begin{aligned} l \cap C &= \{(x, y) : x^2 + y^2 = r^2 \text{ and } x = a\} \\ &= \{(x, y) : a^2 + y^2 = r^2\}. \end{aligned}$$

If (a, y) with $y \neq 0$ is a point of $l \cap C$, then $(a, -y)$ is also a point of $l \cap C$ and there are two distinct points in $l \cap C$. Since there is only one point in $l \cap C$, it follows that $y = 0$ and $a^2 = r^2$. Since $a \geq 0$, it follows that $a = r$ and the one and only point in $l \cap C$ is the point $P(a, 0)$. This proves that l is tangent to C only if P is on C.

Proof of 3: Suppose we are given that P is inside C as shown in Figure 13-14a or b. Recall that O is the center of the given circle C. Then

$$P = (a, 0) \text{ where } 0 \leq a < r,$$

and

$$\begin{aligned} l \cap C &= \{(x, y) : x = a \text{ and } x^2 + y^2 = r^2\} \\ &= \{(a, \sqrt{r^2 - a^2}), (a, -\sqrt{r^2 - a^2})\}. \end{aligned}$$

Since $0 \leq a < r$, it follows that

$$r^2 - a^2 > 0, \qquad \sqrt{r^2 - a^2} \neq -\sqrt{r^2 - a^2},$$

and the points $(a, \sqrt{r^2 - a^2})$ and $(a, -\sqrt{r^2 - a^2})$ are distinct points. This proves that l is a secant of C if P is inside C.

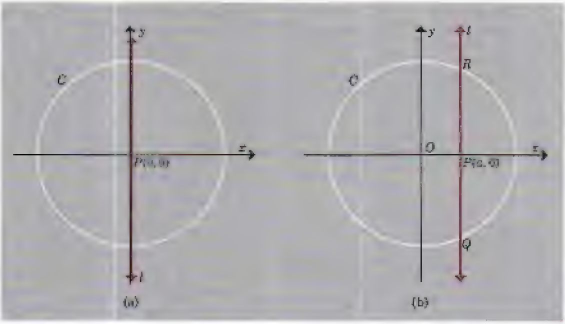


Figure 13-14

Suppose now that l is a secant of C; then l intersects C in two distinct points which we have just shown to be $(a, \sqrt{r^2 - a^2})$ and $(a, -\sqrt{r^2 - a^2})$. This means that $r^2 > a^2$. (Why is it that we cannot have $r^2 = a^2$ or $r^2 < a^2$?) Since $r > 0$ and $a \geq 0$, it follows that $r > a$ and hence that $OP < r$. Therefore $P(a, 0)$ is inside C and the proof is complete.

Now that we have proved Theorem 13.2 we proceed to state our first basic theorems on tangents and chords. To prove some of these theorems we need refer only to Theorem 13.2 and see which of parts (1), (2), or (3) apply.

THEOREM 13.3 Given a circle and a line in the same plane, if the line is tangent to the circle, then it is perpendicular to the radius whose outer end is the point of tangency.

Proof: Let C be the given circle with center O, let line l be tangent to C at R, and let P be the foot of the perpendicular from O to l. It follows from part 2 of Theorem 13.2 that P is on C. (See Figure 13-15.) The figure shows R and P to be distinct points. We shall prove that they are the same point.

If $P \neq R$, then $\overline{OR}$ is the hypotenuse of right triangle $\triangle OPR$ and $OR > OP$. But R and P are both on C. Therefore $OR = OP$. Since this is a contradiction, it follows that $P = R$. Since $l \perp \overline{OP}$ at P, it follows that $l \perp \overline{OR}$ at R, and the proof is complete.

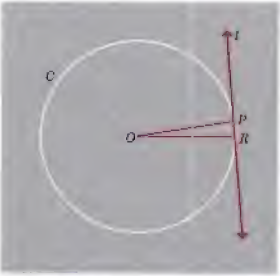


Figure 13-15

Our next theorem is the converse of Theorem 13.3.

THEOREM 13.4 Given a circle and a line in the same plane, if the line is perpendicular to a radius at its outer end, then the line is a tangent to the circle.

Proof: Assigned as an exercise.

THEOREM 13.5 A diameter of a circle bisects a chord of the circle other than a diameter if and only if it is perpendicular to the chord.

Proof: Let a circle C with radius r, center O, diameter $\overline{AB}$, and chord $\overline{RQ}$ be given. We must prove two things.

1. If $\overline{AB} \perp \overline{RQ}$, then $\overline{AB}$ bisects $\overline{RQ}$.
2. If $\overline{AB}$ bisects $\overline{RQ}$, then $\overline{AB} \perp \overline{RQ}$.

Proof of 1: Choose an xy-coordinate system in the plane of the given circle C such that the origin is at the center O, $A = (-r, 0)$, $B = (r, 0)$, and $\overline{RQ}$ is perpendicular to $\overline{AB}$ at $P(a, 0)$, where $0 < a < r$ as shown in Figure 13-16. Then $\overline{AB}$ is a diameter of the circle. Why?

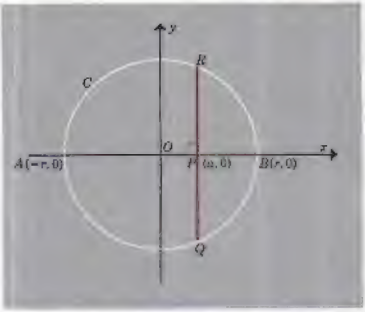


Figure 13-16

Suppose R and Q are named so that R is in the first quadrant. Then (see the proof of part 3 of Theorem 13.2) $R = (a, \sqrt{r^2 - a^2})$ and $Q = (a, -\sqrt{r^2 - a^2})$. Therefore

$$PR = |\sqrt{r^2 - a^2} - 0| = \sqrt{r^2 - a^2}$$

and

$$PQ = |0 - (-\sqrt{r^2 - a^2})| = \sqrt{r^2 - a^2}.$$

Therefore $PR = PQ$ and $\overline{AB}$ bisects $\overline{RQ}$ at P. This proves that if a diameter is perpendicular to a chord, then it bisects the chord.

Proof of 2: Choose an xy-coordinate system in the plane of the given circle C such that the origin O is the center of the circle, $A = (-r, 0)$, $B = (r, 0)$, and the chord $\overline{RQ}$ (not a diameter) intersects $\overline{AB}$ at $P(a, 0)$ with $0 < a < r$. (See Figure 13-17.)

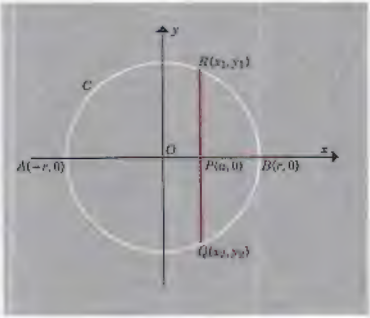


Figure 13-17

Suppose that P is the midpoint of $\overline{RQ}$. Then if $R = (x_1, y_1)$ and $Q = (x_2, y_2)$, we have

$$PR = PQ,$$
$$(x_1 - a)^2 + y_1{}^2 = (x_2 - a)^2 + y_2{}^2, \qquad \text{(Why?)}$$
$$x_1{}^2 - 2x_1a + a^2 + y_1{}^2 = x_2{}^2 - 2x_2a + a^2 + y_2{}^2.$$

But
$$x_1{}^2 + y_1{}^2 = r^2 = x_2{}^2 + y_2{}^2. \qquad \text{(Why?)}$$

Then $-2x_1a = -2x_2a$, $x_1 = x_2$, $\overleftrightarrow{RQ}$ is a vertical line, and $\overline{RQ} \perp \overline{AB}$. This proves that if a diameter bisects a chord that is not a diameter, then the diameter and the chord are perpendicular. This completes the proof of Theorem 13.5.

THEOREM 13.6 In the plane of a circle, the perpendicular bisector of a chord contains the center of the circle.

Proof: In the proof of part 1 of Theorem 13.5, $\overleftrightarrow{AB}$ is the perpendicular bisector of chord $\overline{RQ}$ in the plane of the given circle by the definition of the perpendicular bisector of a segment in a plane. Since $\overleftrightarrow{AB}$ contains O, the center of the given circle, Theorem 13.6 is proved.

THEOREM 13.7 Let a circle C and a line l in the plane of the circle be given. If l intersects the interior of C, then l intersects C in exactly two distinct points.

Proof: Theorem 13.7 follows from part 3 of Theorem 13.2. The details of the proof are assigned as an exercise.

THEOREM 13.8 Chords of congruent circles are congruent if and only if they are equidistant from the centers of the circles.

Proof: Let C and C' be the given congruent circles with centers P and P' and radii r and r', respectively. Then $r = r'$. Why? Let $\overline{AB}$ and $\overline{A'B'}$ be chords of the given circles C and C', respectively. Suppose, first, that the distance from $\overline{AB}$ to the center of circle C is zero, that is, $\overline{AB}$ is a diameter of C, and suppose that $\overline{AB} \cong \overline{A'B'}$. Then

$$AB = A'B' = 2r = 2r'$$

and $\overline{A'B'}$ is a diameter of circle C'. Therefore $\overline{AB}$ and $\overline{A'B'}$ are equidistant from P and P' (the distances being 0 in this case). Conversely, if the distances of $\overline{AB}$ and $\overline{A'B'}$ from P and P' are 0, then $\overline{AB}$ and $\overline{A'B'}$ are diameters of C and C'. It follows that $AB = 2r = 2r' = A'B'$; hence $\overline{AB} \cong \overline{A'B'}$.

Now suppose that $\overline{AB}$ is not a diameter of C. If $\overline{AB} \cong \overline{A'B'}$, it follows that $\overline{A'B'}$ is not a diameter of C'. Why? Let F be the foot of the perpendicular from P to $\overleftrightarrow{AB}$ and let F' be the foot of the perpendicular from P' to $\overleftrightarrow{A'B'}$ as shown in Figure 13-18. Then by Theorem 13.5,

$$AF = \tfrac{1}{2}AB \qquad \text{and} \qquad A'F' = \tfrac{1}{2}A'B'.$$

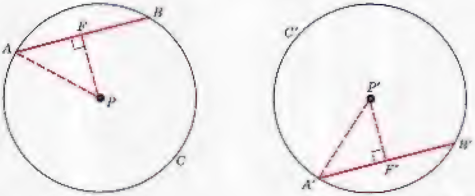


Figure 13-18

By hypothesis,

$$AB = A'B'.$$

Therefore

$$AF = A'F'.$$

We have

$$AP = A'P'. \qquad \text{(Why?)}$$

Since $\triangle APF$ and $\triangle A'P'F'$ are right triangles, it follows that $\triangle APF \cong \triangle A'P'F'$ by the Hypotenuse-Leg Theorem. Therefore $PF = P'F'$ and hence $\overline{AB}$ and $\overline{A'B'}$ are equidistant from P and P'.

Conversely, if $\overline{AB}$ and $\overline{A'B'}$ are equidistant from P and P', that is, if

$$PF = P'F' \neq 0,$$

it follows that $\triangle APF \cong \triangle A'P'F'$ by the Hypotenuse-Leg Theorem. (Show that $\triangle APF \cong \triangle A'P'F'$ if $PF = P'F'$.) Therefore $AF = A'F'$. But by Theorem 13.5,

$$AF = \tfrac{1}{2}AB \qquad \text{and} \qquad A'F' = \tfrac{1}{2}A'B'.$$

Therefore

$$AB = A'B' \qquad \text{and} \qquad \overline{AB} \cong \overline{A'B'}.$$

This completes the proof of Theorem 13.8.

It should be noted that the congruent circles of Theorem 13.8 could be the same circle, in which case the theorem still holds.

Figure 13-19 shows two different examples of two circles tangent to the same line at the same point. In Figure 13-19a the centers A and A' of the two circles are on the same side of the tangent line l and the circles are said to be *internally tangent*. In Figure 13-19b the centers B and B' of the two circles are on opposite sides of the tangent line n and the circles are said to be *externally tangent*.

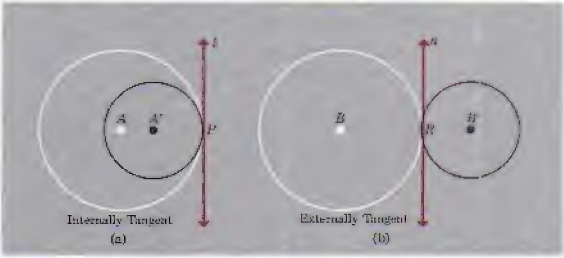


Figure 13-19

Our formal definition follows.

> ***Definition 13.8*** Two circles are **tangent** if and only if they are coplanar and tangent to the same line at the same point. If the centers of the tangent circles are on the same side of the tangent line, the circles are said to be **internally tangent**. If their centers are on opposite sides of the tangent line, the circles are said to be **externally tangent**.

EXERCISES 13.3

Exercises 1–10 refer to the circle $C = \{(x, y) : x^2 + y^2 = 64\}$. In each exercise, the coordinates of a point are given. Tell whether the point is on the circle, in the interior of the circle, or in the exterior of the circle.

1. $(0, -8)$	**4.** $(-4, 7)$	**7.** $(0, 8)$	**10.** $(6, -2\sqrt{7})$
2. $(3, 5)$	**5.** $(-4\sqrt{3}, 4)$	**8.** $(8, -1)$	
3. $(-7, 3)$	**6.** $(4\sqrt{3}, -4)$	**9.** $(-6, -6)$	

11. Find the endpoints of two distinct diameters of the circle

$$C = \{(x, y) : x^2 + y^2 = 64\}.$$

12. Use set-builder notation to express the set of points in the exterior E of the circle $C = \{(x, y) : x^2 + y^2 = 64\}$.

13. Use set-builder notation to express the set of points in the interior I of the circle $C = \{(x, y) : x^2 + y^2 = 64\}$.
14. Prove that the center of a circle is in the interior of the circle.

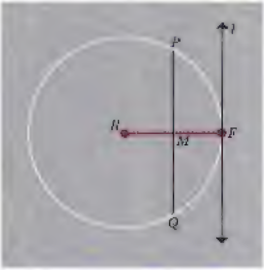
Exercises 15–20 refer to the circle C with center at $(0, 0)$ and passing through the point $P(-5, 12)$.

15. Find the radius of the circle.
16. Use set-builder notation to express the set of points on the circle.
17. Find eight distinct points that are on the circle.
18. Find two distinct points that are in the exterior of the circle.
19. Find two distinct points that are in the interior of the circle.
20. If l is the tangent line to C at $P(-5, 12)$, find an equation for l. (*Hint:* $l \perp \overline{OP}$ at P. Find the slope of l and use the Point-Slope Form of an equation.)

21. Let the circle $C = \{(x, y) : x^2 + y^2 = 36\}$ and the line $l = \{(x, y) : x = 3\}$ be given.
    (a) Does l intersect C?
    (b) If l intersects C, is l a tangent line or a secant line?
    (c) If l intersects C, find the coordinates of the point(s) of intersection.
22. Let the circle $C = \{(x, y) : x^2 + y^2 = 25\}$ and the lines $l_1 = \{(x, y) : x = -5\}$, $l_2 = \{(x, y) : y = 2x\}$, $l_3 = \{(x, y) : x = 7\}$ be given.
    (a) Which of the lines intersect the circle?
    (b) Which line is tangent to the circle? What are the coordinates of the point of tangency?
    (c) Which line is a secant? What are the coordinates of the points of intersection of the secant line and the circle?
23. Prove Theorem 13.4.
24. Prove Theorem 13.7.
25. Copy and complete: A tangent to a circle is $\boxed{?}$ to the radius drawn to the point of contact.
26. Copy and complete: If a diameter is perpendicular to a chord, then it $\boxed{?}$ the chord.
27. Copy and complete: If a diameter bisects a chord other than a diameter, then it is $\boxed{?}$.
28. Copy and complete: In the plane of a circle, the perpendicular bisector of a chord contains the $\boxed{?}$.
29. In a circle with radius 13 in., how long is a chord 5 in. from the center of the circle?
30. In a circle with diameter 12 cm., how long is a chord 4 cm. from the center of the circle?
31. Find the radius of a circle if a chord 8 in. long is 3 in. from the center of the circle.

32. How far from the center of a circle with a radius equal to 25 is a chord whose length is 30?

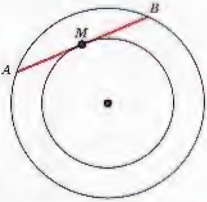
33. In the figure, $\overline{PQ}$ is parallel to l which is tangent to the circle at F. The center of the circle is R and $\overline{PQ}$ bisects $\overline{RF}$ at M. If $PQ = 18$, find RF. (*Hint:* Let $RF = 2x$. Then $RP = 2x$ and $RM = x$.)

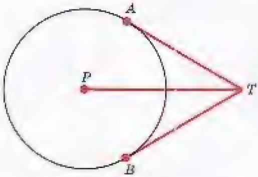


34. If $\overline{AB}$ is a diameter of a circle and if lines l_1 and l_2 are tangent to the circle at A and B, respectively, prove $l_1 \parallel l_2$.

35. Prove that the line containing the centers of two tangent circles contains the point of tangency. (See Figure 13-19.)

36. The figure below at left shows two concentric circles. $\overline{AB}$ is a chord of the larger circle and is tangent to the smaller circle at M. Prove that M is the midpoint of $\overline{AB}$.





37. In the figure above at right, T is a point in the exterior of the circle with center P. Two distinct tangents are drawn from T to the circle with points of contact A and B. Prove that $\overrightarrow{TP}$ bisects $\angle ATB$ and that $AT = BT$.

38. In Exercise 37, if the radius of the circle is 9 and $PT = 15$, find AT.

39. CHALLENGE PROBLEM. In Exercise 37 if the radius of the circle is 9 and $PT = 15$, find AB.

40. CHALLENGE PROBLEM. Show that the line $l = \{(x, y) : 3x + 4y = 25\}$ is tangent to the circle $C = \{(x, y) : x^2 + y^2 = 25\}$ and find the coordinates of the point of tangency.

41. CHALLENGE PROBLEM. Prove that no circle contains three collinear points. (*Hint:* Use Theorem 13.6.)

13.4 TANGENT PLANES

In Section 13.3 we studied relations between lines and circles in a plane. In this section we study relations between planes and spheres in space. There is a close analogy between the definitions and between the theorems of the two sections.

Our first theorem of this section is analogous to Theorem 13.1. In Chapter 12 you learned how to find the distance between two points in space by introducing a three-dimensional coordinate system (called an xyz-coordinate system). We shall use this Distance Formula to develop an equation for a sphere in an xyz-coordinate system.

Let O be a point and r a positive number. Let S be the sphere with center O and radius r. Suppose an xyz-coordinate system has been set up with O as the origin. (See Figure 13-20.) Then $P(x, y, z)$ is a point of S if and only if

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = r,$$
$$\sqrt{x^2 + y^2 + z^2} = r,$$
$$x^2 + y^2 + z^2 = r^2.$$

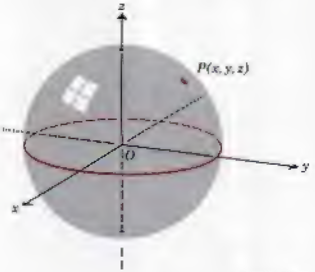


Figure 13-20

We have proved the following theorem.

THEOREM 13.9 Let O be a point, r a positive number, and S the sphere with center O and radius r. Given an xyz-coordinate system with origin O, $S = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$.

Definition 13.9 (See Figure 13-21.) Let a sphere with center O and radius r be given. The **interior** of the sphere is the set of all points P in space such that $OP < r$. The **exterior** of the sphere is the set of all points P in space such that $OP > r$.

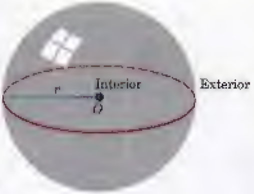


Figure 13-21

In view of Definition 13.9 and Theorem 13.9, if S is a sphere with radius r and center at the origin O of an xyz-coordinate system, then

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$$
$$I = \{(x, y, z) : x^2 + y^2 + z^2 < r^2\}$$

and

$$E = \{(x, y, z) : x^2 + y^2 + z^2 > r^2\}$$

where I is the interior of the sphere and E is its exterior.

Before reading Definition 13.10, try to form your own definition of a plane tangent to a sphere. See Figure 13-22.

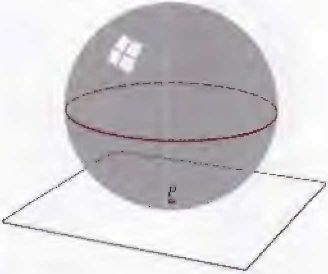


Figure 13-22

Definition 13.10 If a plane intersects a sphere in exactly one point, the plane is called a **tangent plane** to the sphere. The point is called the **point of tangency**, or the **point of contact**, and we say that the plane and the sphere are tangent at this point.

You have seen that there are three possibilities with regard to the intersection of a line and a circle in the same plane. Figure 13-23 suggests three possibilities with regard to the intersection of a plane and a sphere. For each part of Figure 13-23, S is a sphere with center O and P is the foot of the perpendicular from O to plane α. Figure 13-23a suggests that if P is in the exterior of the sphere, then all of plane α is in the exterior of the sphere and $S \cap \alpha = \varnothing$. Figure 13-23b suggests that if P is on the sphere, then α is tangent to the sphere at P and $S \cap \alpha = \{P\}$. Figure 13-23c suggests that if P is in the interior of the sphere, then the intersection of the sphere and plane α is a circle C with center P, that is, $S \cap \alpha = C$.

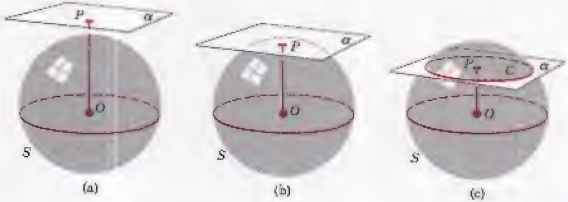


Figure 13-23

The following theorem concerning the intersection of a plane and a sphere is analogous to Theorem 13.2 concerning the intersection of a line and a circle in a plane. It can be proved using a three-dimensional coordinate system in much the same way that Theorem 13.2 was proved using a two-dimensional coordinate system. You will be asked to write a proof of Theorem 13.10 in the Exercises.

THEOREM 13.10 Given a sphere S with center O and a plane α which does not contain O, let P be the foot of the perpendicular from O to α.

1. Every point of α is in the exterior of S if and only if P is in the exterior of S.
2. α is tangent to S if and only if P is on S.
3. α intersects S in a circle with center P if and only if P is in the interior of S.

THEOREM 13.11 Let a sphere S with center O and radius r and a plane α be given. If the intersection of S with α contains the center O of the sphere, then the intersection is a circle whose center and radius are the same as those of the sphere.

Proof: Let a sphere S with center O and radius r be given as shown in Figure 13-24. Let α be a plane that contains O and intersects S. Since S is the set of all points of space whose distance from O is r, the intersection of S and α is the set of all points P of α such that $OP = r$. By definition, this set is the circle in plane α with center O and radius r. Thus the circle has the same center and radius as the sphere, and the proof is complete.

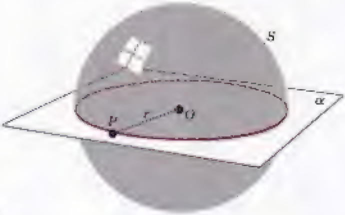


Figure 13-24

Definition 13.11 A circle that is the intersection of a sphere with a plane through the center of the sphere is called a great circle of the sphere.

The next two theorems can be easily proved using Theorem 13.11.

THEOREM 13.12 The perpendicular from the center of a sphere to a chord of the sphere bisects the chord.

Proof: The endpoints of the given chord and the center O of the given sphere determine a plane α. The intersection of plane α with the sphere is a *great circle* with center O and having the same chord as the given chord. It follows from Theorem 13.5 that the perpendicular from O to the given chord bisects the chord.

THEOREM 13.13 The segment joining the center of a sphere to the midpoint of a chord of the sphere is perpendicular to the chord.

Proof: Assigned as an exercise.

Our last theorem of this section is analogous to Theorems 13.3 and 13.4.

THEOREM 13.14 A plane is tangent to a sphere if and only if it is perpendicular to a radius of the sphere at its outer end.

Proof: Let a sphere S with center O and radius $\overline{OP}$ be given. Let α be the given plane. There are two parts to the proof.

1. If $\overline{OP} \perp \alpha$ at P, then α is tangent to S at P.
2. If α is tangent to S at P, then $\overline{OP} \perp \alpha$.

Proof of 1: We are given that $\overline{OP} \perp \alpha$ at P as shown in Figure 13-25. Let R be any point of α different from P; then $\overline{OP} \perp \overline{PR}$ (Why?) and $\triangle OPR$ is a right triangle with the right angle at P. Therefore $OP < OR$. Why? Therefore R is a point in the exterior of S. Why? It follows that P is the only point of α that belongs to both α and S; hence α is tangent to S at P.

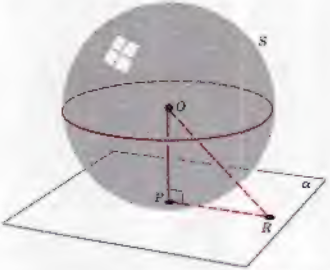


Figure 13-25

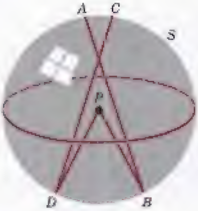
Proof of 2: We are given that α is tangent to S at P. We shall use an indirect proof to show that $\overline{OP} \perp \alpha$. Suppose, contrary to what we want to prove, that $\overline{OP}$ is not perpendicular to α. Let Q be the foot of the unique perpendicular from O to α. Then $\overline{OQ} < \overline{OP}$. Why? Therefore Q is in the interior of S. Why? It follows from part 3 of Theorem 13.10 that α intersects S in a circle. But this contradicts the hypothesis that α is tangent to S; that is, the intersection of α and S is exactly one point. Therefore our supposition that $\overline{OP}$ is not perpendicular to α is incorrect and we conclude that $\overline{OP} \perp \alpha$. This completes the proof of Theorem 13.14.

EXERCISES 13.4

1. Let a sphere with radius 10 in. be given. A plane 6 in. from the center of the sphere intersects the sphere in a circle. Find the radius of the circle.
2. Given a sphere S with center O, let α and β be two planes equidistant from O and such that α intersects S in a circle C and β intersects S in a circle C'. Prove that C is congruent to C'. (*Hint:* Let F be the foot of the perpendicular from O to α and let F' be the foot of the perpendicular from O to β. Let G be a point on C and G' be a point on C'. Prove that $\triangle OFG \cong \triangle OF'G'$.)
3. In Exercise 2, is it necessary for α and β to be parallel planes?
4. Prove that if the circles of intersection of two planes with a sphere are congruent, then the planes are equidistant from the center of the sphere.
5. Let a sphere with radius 12 be given. A segment from the center of the sphere to a chord and perpendicular to the chord has length 8. Find the length of the chord.
6. A sphere with center C is tangent to plane α at P. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines in plane α which contain P. In what way is $\overline{CP}$ related to $\overleftrightarrow{AB}$? To $\overleftrightarrow{CD}$? Draw a figure which illustrates the given information.
7. Prove Theorem 13.13. (*Hint:* Give a proof similar to that of Theorem 13.12.)
8. Given a sphere S with center P as shown in the figure, if $\overline{AB}$ and $\overline{CD}$ are chords of S which are equidistant from P, prove that

$$\overline{AB} \cong \overline{CD} \qquad \text{and} \qquad \angle ABP \cong \angle CDP.$$

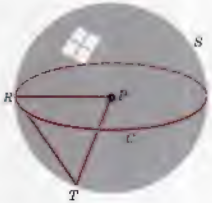
(*Hint:* Use Theorem 13.8.)



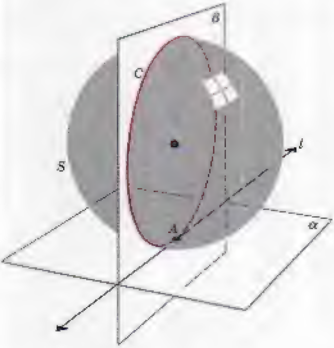
9. Given that $\overline{AC}$ and $\overline{BD}$ are perpendicular diameters of a sphere, prove that $ABCD$ is a square.

10. Given that $\overline{AC}$ and $\overline{BD}$ are distinct diameters of a sphere, prove that $ABCD$ is a rectangle.

11. Let a sphere S with center P as shown in the figure be given. C is a great circle of S. R is a point on C and T is a point on S, but T is not on C. If $m\angle RPT = 60$, prove that $\triangle RPT$ is equilateral.



12. Let a sphere S and a plane α tangent to S at point A be given. Let plane β be any plane other than α which contains A. (See the figure.)



(a) Prove that plane β intersects sphere S in a circle C.
(b) Prove that plane β intersects plane α in a line l.
(c) Prove that l is tangent to C at A.

(*Hint:* Suppose l intersects C in a second point Q. Then Q is on S (Why?), and hence α intersects S in a second point Q. Contradiction?)

Exercises 13–22 refer to the sphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 64\}.$$

In each exercise, given the coordinates of a point, tell whether the point is on the sphere, in the interior of the sphere, or in the exterior of the sphere.

13. $(0, 0, 8)$

14. $(0, -8, 0)$

15. $(4, 3, 5)$

16. $(7, 2, 3)$

17. $(-4, 5, 6)$

18. $(-6, -4, 2\sqrt{3})$

19. $(4, -4, -6)$

20. $(5, 0, -6)$

21. $(-2, 2\sqrt{15}, 0)$

22. $(4, 3\sqrt{5}, -2)$

23. Find the endpoints of three distinct diameters of the sphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 100\}.$$

Exercises 24–28 refer to the sphere with center at $(0, 0, 0)$ and containing the point $(4, 4, 2)$.

24. Find the radius of the sphere.

25. Use set-builder notation to express the set of points on the sphere.

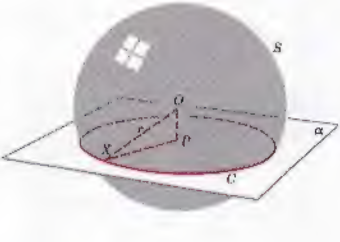
26. Find the coordinates of two distinct points that are on the sphere.

27. Find the coordinates of two distinct points that are in the exterior of the sphere.

28. Find the coordinates of two distinct points that are in the interior of the sphere.

29. CHALLENGE PROBLEM. Theorem 13.14 could be called a restatement of part 2 of Theorem 13.10. Complete the proof of the following restatement of part 3 of Theorem 13.10.

The intersection of a plane and a sphere is a circle whose center is the foot of the perpendicular from the center of the sphere to the plane if and only if the foot of the perpendicular is in the interior of the sphere.



Proof: Let a sphere S with center O and radius r and a plane α be given. Let P be the foot of the perpendicular from O to α as shown in the figure. There are two things to be proved.

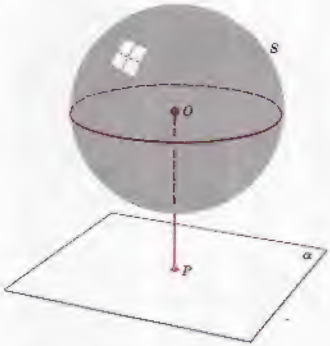
(a) If $OP < r$, then $\alpha \cap S$ is a circle C with center P.
(b) If $\alpha \cap S$ is a circle with center P, then $OP < r$.

Let X be any point of the intersection of α and S. To complete the proof of (a) you need to show that PX is a constant for all points X in the intersection of α and S. To complete the proof of (b) you need to show that if PX is a constant for all points X in the intersection of α and S, then $OP < r$.

30. CHALLENGE PROBLEM. Complete the proof of the following restatement of part 1 of Theorem 13.10.

The intersection of a plane and a sphere is the empty set if and only if the foot of the perpendicular from the center of the sphere to the plane is in the exterior of the sphere.

Proof: Let a sphere S with center O and radius r and a plane α be given. Let P be the foot of the perpendicular from O to α as shown in the figure.



There are two things to be proved.

(a) If $OP > r$, then all points of α are in the exterior of the sphere.
(b) If all points of α are in the exterior of the sphere, then $OP > r$.

31. CHALLENGE PROBLEM. See the proof of Theorem 13.2. Prove Theorem 13.10 in a similar way using an xyz-coordinate system.

13.5 CIRCULAR ARCS, ARC MEASURE

Thus far in this chapter we have treated circles and spheres in a similar manner. In the remainder of this chapter, we limit ourselves to the consideration of topics relating to circles only. The reason for this is that the treatment of the corresponding topics for spheres is too complicated to consider in a first course in geometry. We begin with some definitions.

Definition 13.12 An angle which is coplanar with a circle and has its vertex at the center of the circle is called a **central angle.**

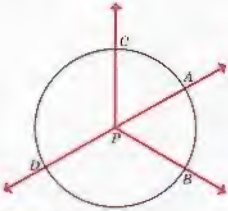


Figure 13-26

In Figure 13-26, P is the center of the given circle and $\overline{AD}$ is a diameter. $\angle APB$ is a central angle. Name three more central angles shown in the figure.

Definition 13.13 (See Figure 13-27.) If A and B are distinct points on a circle with center P and if A and B are not the endpoints of a diameter of the circle, then the union of A, B, and all points of the circle in the interior of $\angle APB$ is called a **minor arc** of the circle. The union of A, B, and all points of the circle in the exterior of $\angle APB$ is called a **major arc** of the circle. If A and B are the endpoints of a diameter of the circle, then the union of A, B, and all points of the circle in one of the two halfplanes, with edge $\overleftrightarrow{AB}$, lying in the plane of the circle is called a **semicircle.**

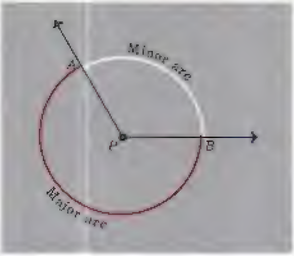


Figure 13-27

From Definition 13.13, an arc of a circle is either a minor arc, a major arc, or a semicircle. The points A and B in Definition 13.13 are called the endpoints of the arc.

Notation. We may denote an arc with endpoints A and B by the symbol $\widehat{AB}$, which is read "arc AB." However, it should be noted that the symbol $\widehat{AB}$ is ambiguous unless the word "minor" or the word "major" is used in connection with the symbol. For example, in Figure 13-27, we may speak of the minor $\widehat{AB}$ or the major $\widehat{AB}$. Also, semicircle $\widehat{AB}$ is ambiguous since there are two semicircles with endpoints A and B. One way to avoid confusion as to which arc is meant is by choosing an interior point of the arc in question (that is, a point of the arc other than its endpoints) and using this third point in naming the arc. Thus, in Figure 13-28, we may speak of minor $\widehat{AXB}$ or simply $\widehat{AXB}$. Similarly, we may speak of major $\widehat{AYB}$ or simply $\widehat{AYB}$, semicircle $\widehat{CXB}$ or semicircle $\widehat{CYB}$.

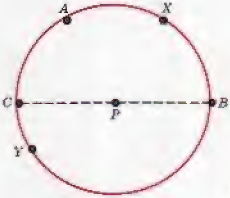


Figure 13-28

It is clear that for each pair of distinct points A, B on a circle there are two arcs which have these points as endpoints. If $\widehat{AB}$ is a minor arc, we sometimes say that major $\widehat{AB}$ is the *corresponding* major arc, or that major $\widehat{AB}$ *corresponds* to minor $\widehat{AB}$, or that minor $\widehat{AB}$ *corresponds* to major $\widehat{AB}$.

In Figure 13-28 name the major arc that corresponds to minor $\widehat{BX}$. Name the minor arc that corresponds to major $\widehat{ABY}$.

When we speak of the measure of a segment, we mean the number that is the length of the segment. However, this is not true of arcs. That is, when we speak of the measure of an arc, we do not mean the "length" of the arc, since length has not been defined for anything except segments. If $\widehat{AXB}$ is a minor arc of a circle with center P, we say that $\angle APB$ is the *associated* central angle with respect to $\widehat{AXB}$. The measure of a minor arc of a circle is related to the degree measure of its associated central angle. We make the following definitions.

Definition 13.14 If $\widehat{AXB}$ is any arc of a circle with center P, then its degree measure (denoted by $m\widehat{AXB}$) is given as follows:

1. If $\widehat{AXB}$ is a minor arc, then $m\widehat{AXB}$ is the measure of the associated central angle; that is,
$$m\widehat{AXB} = m\angle APB.$$
2. If $\widehat{AXB}$ is a semicircle then
$$m\widehat{AXB} = 180.$$
3. If $\widehat{AXB}$ is a major arc and $\widehat{AYB}$ is the corresponding minor arc, then
$$m\widehat{AXB} = 360 - m\widehat{AYB}.$$

Figure 13-29 shows a circle with center P. If

$$m\angle APB = 50,$$

then

$$m\widehat{AXB} = 50$$

and

$$m\widehat{AYB} = 360 - 50 = 310.$$

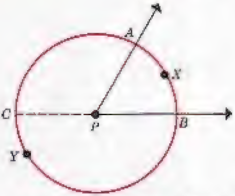


Figure 13-29

If $\overline{BC}$ is a diameter of the circle, what is $m\widehat{CXB}$ in the figure? What is $m\widehat{CYB}$?

Hereafter we shall call $m\widehat{AXB}$ simply the *measure* of arc $\widehat{AXB}$ with the understanding that we mean the *degree measure* of the arc. Is the measure of a minor arc always less than 180? Why? Is the measure of a major arc always greater than 180? Why? Can the measure of an arc be zero? Why?

Note that the measure of an arc does not depend on the size of the circle which contains the arc. Figure 13-30 shows concentric circles with center P, with

$$m\widehat{CYD} = m\widehat{AXB} = m\angle APB = m\angle CPD = 35.$$

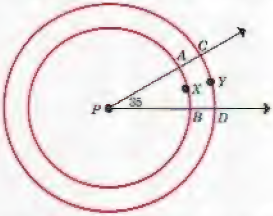


Figure 13-30

Given three points A, B, C such that B is between A and C, we know that

$$AB + BC = AC.$$

Suppose we are given an arc $\widehat{ABC}$ (that is, B is a point, but not an endpoint, of $\widehat{AC}$). It seems reasonable that

$$m\widehat{AB} + m\widehat{BC} = m\widehat{ABC}.$$

We state this as our next theorem.

THEOREM 13.15 **(*Arc Measure Addition Theorem*)** If A, B, C are distinct points on a circle, then

$$m\widehat{ABC} = m\widehat{AB} + m\widehat{BC}.$$

Proof: Following is a plan for a proof considering seven possible cases as shown in Figure 13-31. In each case, V is the center of the circle and the assertion of the theorem follows from the listed equations. This plans a complete proof since there are no other cases.

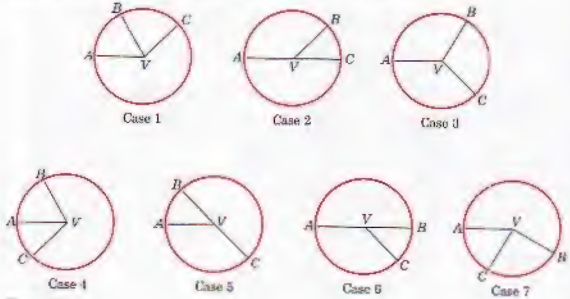


Figure 13-31

Case 1. $\widehat{ABC}$ is a minor arc. Then $\widehat{AB}$ and $\widehat{BC}$ are minor arcs,

$$m\widehat{ABC} = m\angle AVC$$
$$m\widehat{AB} = m\angle AVB$$
$$m\widehat{BC} = m\angle BVC$$
$$m\angle AVC = m\angle AVB + m\angle BVC.$$

Case 2. $\widehat{ABC}$ is a semicircle. Then $\widehat{AB}$ and $\widehat{BC}$ are minor arcs,

$$m\widehat{ABC} = 180$$
$$m\widehat{AB} = m\angle AVB$$
$$m\widehat{BC} = m\angle BVC$$
$$180 = m\angle AVB + m\angle BVC.$$

Case 3. $\widehat{ABC}$ is a major arc, and $\widehat{AB}$ and $\widehat{BC}$ are minor arcs. Then

$$m\widehat{ABC} = 360 - m\angle AVC$$
$$m\widehat{AB} = m\angle AVB$$
$$m\widehat{BC} = m\angle BVC$$
$$360 = m\angle AVB + m\angle BVC + m\angle CVA.$$

Case 4. $\widehat{ABC}$ is a major arc, $\widehat{AB}$ is a minor arc, and $\widehat{BC}$ is a major arc. Then

$$\begin{aligned} m\widehat{ABC} &= 360 - m\angle AVC \\ m\widehat{AB} &= m\angle AVB \\ m\widehat{BC} &= 360 - m\angle BVC \\ m\angle BVC &= m\angle BVA + m\angle AVC. \end{aligned}$$

Case 5. $\widehat{ABC}$ is a major arc, $\widehat{AB}$ is a minor arc, and $\widehat{BC}$ is a semicircle. Then

$$\begin{aligned} m\widehat{ABC} &= 360 - m\angle AVC \\ m\widehat{AB} &= m\angle AVB \\ m\widehat{BC} &= 180 \\ 180 &= m\angle AVB + m\angle AVC. \end{aligned}$$

Case 6. $\widehat{ABC}$ is a major arc, $\widehat{AB}$ is a semicircle, and $\widehat{BC}$ is a minor arc. Then

$$\begin{aligned} m\widehat{ABC} &= 360 - m\angle AVC \\ m\widehat{AB} &= 180 \\ m\widehat{BC} &= m\angle BVC \\ 180 &= m\angle AVC + m\angle BVC. \end{aligned}$$

Case 7. $\widehat{ABC}$ is a major arc, $\widehat{AB}$ is a major arc, and $\widehat{BC}$ is a minor arc. Then

$$\begin{aligned} m\widehat{ABC} &= 360 - m\angle AVC \\ m\widehat{AB} &= 360 - m\angle AVB \\ m\widehat{BC} &= m\angle BVC \\ m\angle AVB &= m\angle AVC + m\angle CVB. \end{aligned}$$

Note in the situation of Theorem 13.15 that if D is a point of the circle not on $\widehat{ABC}$, then

$$m\widehat{CDA} = 360 - m\widehat{ABC}$$

and

$$m\widehat{AB} + m\widehat{BC} + m\widehat{CDA} = m\widehat{ABC} + m\widehat{CDA} = 360.$$

In other words, if A, B, C are three distinct points on a circle that partitions the circle into arcs, $\widehat{AB}$, $\widehat{BC}$, $\widehat{CA}$, intersecting only at their endpoints, then

$$m\widehat{AB} + m\widehat{BC} + m\widehat{CA} = 360.$$

This idea may be extended to any number of points on a circle, as in the following corollary.

> *COROLLARY 13.15.1* If $A_1, A_2, \ldots, A_n$ are n distinct points on a circle that partition the circle into n arcs
>
> $$\overparen{A_1A_2} + \overparen{A_2A_3}, \ldots, \overparen{A_{n-1}A_n}, \overparen{A_nA_1}$$
>
> that intersect only at their endpoints, then
>
> $$m\overparen{A_1A_2} + m\overparen{A_2A_3} + \cdots + m\overparen{A_{n-1}A_n} + m\overparen{A_nA_1} = 360.$$

Proof: Following is the proof for $n = 5$. The proofs for other values of n are similar. By repeated application of Theorem 13.15, we get the following:

$$\begin{aligned} m\overparen{A_1A_2} + m\overparen{A_2A_3} + m\overparen{A_3A_4} + m\overparen{A_4A_5} + m\overparen{A_5A_1} &= \\ m\overparen{A_1A_2A_3} + m\overparen{A_3A_4} + m\overparen{A_4A_5} + m\overparen{A_5A_1} &= \\ m\overparen{A_1A_2A_4} + m\overparen{A_4A_5} + m\overparen{A_5A_1} &= \\ m\overparen{A_1A_2A_5} + m\overparen{A_5A_1} &= 360. \end{aligned}$$

EXERCISES 13.5

Exercises 1–7 are on the proof of Theorem 13.15. In each exercise, show how to derive the assertion of the theorem from the listed equations for the given case.

1. Case 1
2. Case 2
3. Case 3
4. Case 4
5. Case 5
6. Case 6
7. Case 7

8. In the figure, $m\overparen{ABC} = 240$ and $m\overparen{BXC} = 100$. Find $m\overparen{AZB}$ and $m\overparen{AYC}$.

Exercises 9–20 refer to Figure 13-32. In the figure, P is the center of the circle; A, B, C, D are points on the circle; $A\text{–}P\text{–}B$; $m\widehat{BC} = 50$; and $mBD = 110$.

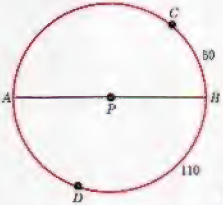


Figure 13-32

9. Name five minor arcs determined by points labeled in the figure.
10. Find the measure of each of the minor arcs named in Exercise 9.
11. Name the five major arcs corresponding to the five minor arcs named in Exercise 9.
12. Find the measure of each of the major arcs named in Exercise 11.
13. Name two semicircles.
14. Which theorem justifies the conclusion that $m\widehat{CBD} = 160$?
15. Find $m\angle BPC$.
16. Find $m\angle CPD$.
17. Copy Figure 13-32 and draw $\overline{AC}$, $\overline{BC}$, and $\overline{PC}$. Find $m\angle BAC$. How does $m\angle BAC$ compare with $m\widehat{BC}$?
18. Show that $m\angle ACB = 90$.
19. Draw $\overline{AD}$, $\overline{BD}$, and $\overline{PD}$ on your copy of Figure 13-32. Show that $m\angle BAD = \frac{1}{2}m\widehat{BD}$.
20. Find $m\angle ADB$.

Exercises 21–23 refer to Figure 13-33. In the figure, P is the center of the circle; A, B, C are points on the circle; and $A\text{–}P\text{–}B$.

21. Copy Figure 13-33 and draw $\overline{PC}$. Prove that $m\angle BAC = \frac{1}{2}m\widehat{BC}$.
22. Prove that $m\angle ABC = \frac{1}{2}m\widehat{AC}$.
23. Prove that $\triangle ABC$ is a right triangle.

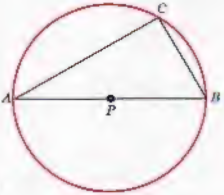


Figure 13-33

24. CHALLENGE PROBLEM. In the figure, P is the center of the circle; A, B, C, D are points on the circle; and A–P–B. Prove that

$$m\angle CAD = \tfrac{1}{2}m\widehat{CBD}.$$

13.6 INTERCEPTED ARCS, INSCRIBED ANGLES, ANGLE MEASURE

In the discussions that follow we shall be concerned with angles *inscribed* in an arc of a circle and about arcs of a circle that are *intercepted* by certain angles. In Figure 13-34a, $\angle ABC$ is inscribed in $\widehat{ABC}$ and $\angle \widehat{ABC}$ intercepts $\widehat{AXC}$. In Figure 13-34b, $\angle PQR$ is inscribed in $\widehat{PQR}$ and $\angle PQR$ intercepts $\widehat{PYR}$.

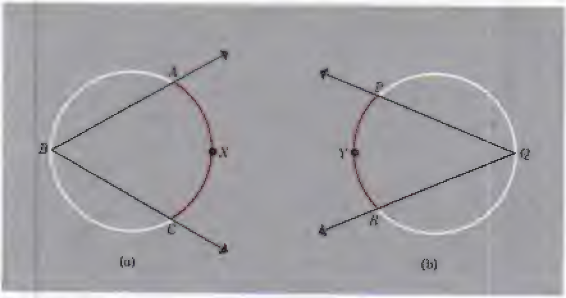


Figure 13-34

None of the angles shown in Figure 13-35 is an inscribed angle, but each angle intercepts one or more arcs of a circle. In Figure 13-35a, $\angle APB$ intercepts $\widehat{AXB}$. In Figure 13-35b, $\angle RST$ intercepts $\widehat{RYS}$. In Figure 13-35c, $\angle AVB$ intercepts $\widehat{AKB}$ and also $\widehat{CMD}$. In Figure 13-35d, $\angle GEF$ intercepts $\widehat{GHF}$. In Figure 13-35e, $\angle AVB$ intercepts $\widehat{ARB}$ and also $\widehat{CSB}$. In Figure 13-35f, $\angle AVB$ intercepts $\widehat{AFB}$ and also $\widehat{AGB}$.

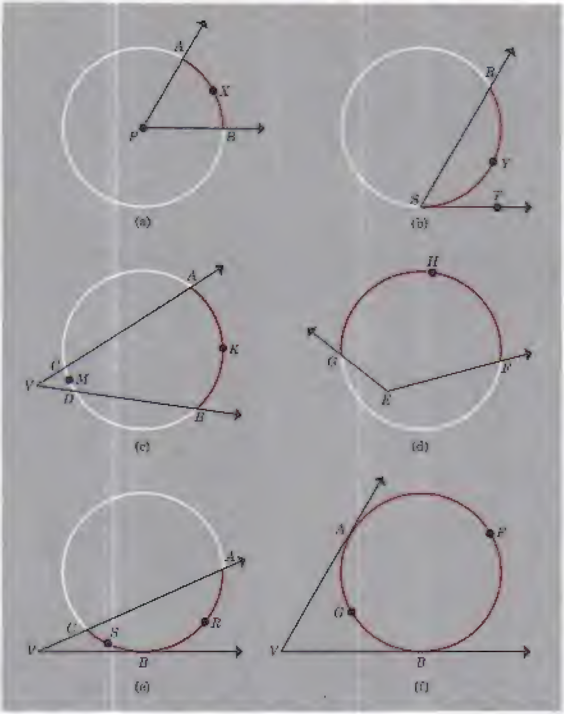


Figure 13-35

On the other hand, $\angle LKP$ shown in Figure 13-36 does not intercept an arc of the circle.

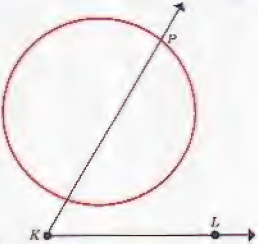


Figure 13-36

Our formal definitions of an inscribed angle and of an intercepted arc follow.

Definition 13.15 An angle is said to be **inscribed** in an arc of a circle and is called an **inscribed angle** if and only if both of the following conditions are satisfied:

1. Each side of the angle contains an endpoint of the arc.
2. The vertex of the angle is a point, but not an endpoint, of the arc.

Explain why each of the angles shown in Figure 13-35 fails to be an inscribed angle. Explain why each of the angles shown in Figure 13-34 is an inscribed angle. Draw a picture of an angle inscribed in a semicircle.

Definition 13.16 An angle is said to **intercept an arc** of a circle and the arc is called an **intercepted arc** of the angle if and only if all three of the following conditions are satisfied:

1. The endpoints of the arc lie on the angle.
2. Each side of the angle contains at least one endpoint of the arc.
3. Each point of the arc, except its endpoints, lies in the interior of the angle.

You should check to see that each of the figures shown in Figure 13-35 satisfies all three of the conditions stated in Definition 13.16. Which of the three conditions stated in Definition 13.16 are not satisfied by the figure shown in Figure 13-36?

Figure 13-37 shows two angles inscribed in the same arc. (Of course, the angles intercept the same arc.) Name the arc in which both angles, $\angle ABD$ and $\angle ACD$, are inscribed. Name the arc that both of these angles intercept. It appears that $\angle ABD$ and $\angle ACD$ in the figure are congruent. (Measure each of them with your protractor.) That they actually are congruent is a corollary of our next theorem.

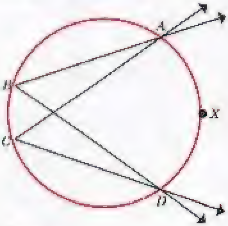


Figure 13-37

THEOREM 13.16 The measure of an inscribed angle is one-half the measure of its intercepted arc.

Proof: Let a circle with center V be given and let $\angle ABC$ be inscribed in $\widehat{ABC}$. Then the intercepted arc is $\widehat{AC}$. We must prove $m\angle ABC = \frac{1}{2}m\widehat{AC}$. There are three possible cases as suggested in Figure 13-38.

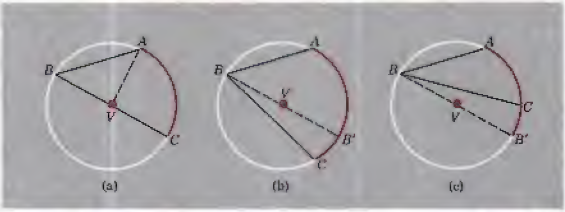


Figure 13-38

Case 1. V is on $\angle ABC$.
Case 2. V is in the interior of $\angle ABC$.
Case 3. V is in the exterior of $\angle ABC$.

Proof of Case 1: Since V is on $\angle ABC$, then either $\overline{AB}$ or $\overline{BC}$ is a diameter of the given circle. Suppose $\overline{BC}$ is a diameter as in Figure 13-38a. Draw $\overline{VA}$. Then $\triangle AVB$ is isosceles (Why?) and $m\angle A = m\angle ABC$. $\angle AVC$ is an exterior angle of $\triangle AVB$, so by Theorem 7-30,

$$m\angle AVC = m\angle A + m\angle ABC.$$

Since $m\angle A = m\angle ABC$,

$$m\angle AVC = m\angle ABC + m\angle ABC,$$

or

$$2m\angle ABC = m\angle AVC.$$

But

$$m\angle AVC = m\widehat{AC}. \qquad \text{(Why?)}$$

Therefore

$$2m\angle ABC = m\widehat{AC}$$

and

$$m\angle ABC = \tfrac{1}{2}m\widehat{AC}.$$

This completes the proof of Case 1.

Proof of Case 2: V is in the interior of $\angle ABC$ as shown in Figure 13-38b, so $\overrightarrow{BV}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Let B' be the point where *opp* $\overrightarrow{VB}$ intersects the circle; then $\overline{BB'}$ is a diameter and B' is an interior point

of the intercepted arc $\widehat{AC}$. It follows from Case 1 that

$$m\angle ABB' = \tfrac{1}{2}m\widehat{AB'} \qquad \text{and} \qquad m\angle B'BC = \tfrac{1}{2}m\widehat{B'C}.$$

Adding, we get

$$m\angle ABB' + m\angle B'BC = \tfrac{1}{2}m\widehat{AB'} + \tfrac{1}{2}m\widehat{B'C},$$

or

$$m\angle ABB' + m\angle B'BC = \tfrac{1}{2}(m\widehat{AB'} + m\widehat{B'C}).$$

But

$$m\angle ABB' + m\angle B'BC = m\angle ABC \qquad \text{(Why?)}$$

and

$$m\widehat{AB'} + m\widehat{B'C} = m\widehat{AC}. \qquad \text{(Why?)}$$

Therefore

$$m\angle ABC = \tfrac{1}{2}m\widehat{AC}$$

and the proof of Case 2 is complete. The proof of Case 3 is assigned as an exercise.

Recall how congruent segments and congruent angles are defined in terms of their measures. How would you define congruent arcs? Write what you think is a good definition of congruent arcs. Turn back to Figure 13-30. Does your definition allow the conclusion that arcs $\widehat{AXB}$ and $\widehat{CYD}$ of Figure 13-30 are congruent? If it does, you should reword your definition so that it excludes this conclusion. Compare your definition with the following.

Definition 13.17 Two arcs (not necessarily distinct) are **congruent** if and ony if they have the same measure and are arcs of congruent circles.

Notation: If arcs $\widehat{AB}$ and $\widehat{CD}$ are congruent, we write $\widehat{AB} \cong \widehat{CD}$.

Note that we cannot say that two arcs are congruent unless we know that they have the same measure *and* we know that they are in the same circle or that they are in congruent circles.

The following three corollaries are important consequences of Theorem 13.16. Their proofs are easy and are assigned as exercises.

COROLLARY 13.16.1 Angles inscribed in the same arc are congruent.

COROLLARY 13.16.2 An angle inscribed in a semicircle is a right angle.

COROLLARY 13.16.3 Congruent angles inscribed in the same circle or in congruent circles intercept congruent arcs.

Figure 13-39 shows two distinct parallel lines intersecting a circle. We call an arc whose endpoints are on the lines, one endpoint on each of the lines and all of whose interior points are between the lines, an **intercepted** arc. Thus, in Figure 13-39, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intercept arcs $\widehat{AXC}$ and $\widehat{BYD}$. Are these two arcs congruent? They are congruent by our next theorem.

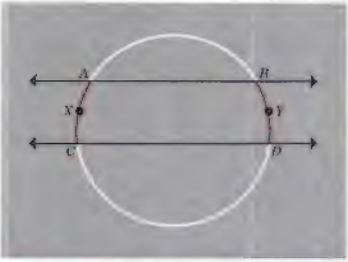


Figure 13-39

THEOREM 13.17 If two distinct parallel lines in the plane of a circle intersect that circle, they intercept congruent arcs.

Proof: Let distinct parallel lines l and m intersect a given circle in arcs $\widehat{AXC}$ and $\widehat{BYD}$. There are three possible cases as suggested in Figure 13-40.

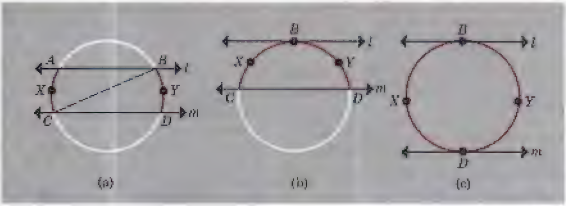


Figure 13-40

Case 1. Both l and m are secant lines as shown in Figure 13-40a.
Case 2. The line l is a tangent line and the other line m is a secant line as shown in Figure 13-40b.
Case 3. Both l and m are tangent lines as shown in Figure 13-40c.

Proof of Case 1: Since l and m are secant lines, l intersects the circle in distinct points A and B, and m intersects the circle in distinct points C and D. (See Figure 13-40a.) We must prove that $\widehat{AXC} \cong \widehat{BYD}$. Draw $\overline{BC}$. Then

$$m\angle BCD = \tfrac{1}{2}m\widehat{BYD}$$

and

$$m\angle ABC = \tfrac{1}{2}m\widehat{AXC}. \qquad \text{Why?}$$

But

$$m\angle ABC = m\angle BCD. \qquad \text{Why?}$$

Therefore

$$\tfrac{1}{2}m\widehat{AXC} = \tfrac{1}{2}m\widehat{BYD}$$

or

$$m\widehat{AXC} = m\widehat{BYD}.$$

Therefore

$$\widehat{AXC} \cong \widehat{BYD}. \qquad \text{Why?}$$

This completes the proof of Case 1 of Theorem 13.17. The proofs of Cases 2 and 3 are assigned as exercises.

EXERCISES 13.6

Exercises 1–17 refer to Figure 13-41. In the figure, P is the center of the circle; A, B, C, D are points on the circle; $\overline{AC}$ is a diameter; $m\widehat{AD} = 100$; and $m\widehat{BC} = 40$.

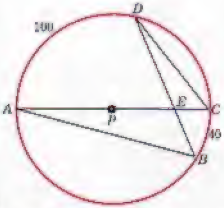


Figure 13-41

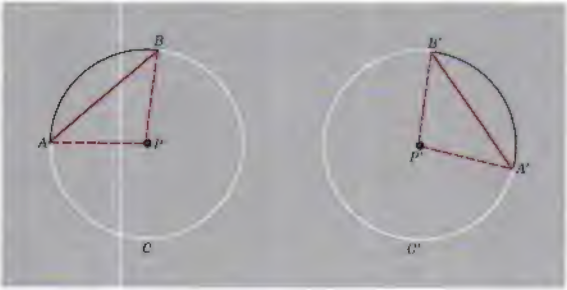
1. Name four inscribed angles and name the arc each intercepts.
2. Name the five minor arcs determined by points labeled in the figure.
3. Name the five major arcs corresponding to the five minor arcs named in Exercise 2.
4. Name two semicircles.
5. Find the measure of each of the minor arcs named in Exercise 2.
6. Find the measure of each of the major arcs named in Exercise 3.

7. Find the measure of each of the angles $\angle A$, $\angle B$, $\angle C$, and $\angle D$.
8. Find the measure of $\angle CPD$.
9. Find the measure of $\angle ABC$.
10. Name an angle that is congruent to $\angle B$.
11. Name an angle that is congruent to $\angle D$.
12. Prove that $\overline{AD} \perp \overline{CD}$.
13. Find the measure of $\angle DCB$.
14. If $\overline{AC}$ intersects $\overline{BD}$ at E, name a pair of similar triangles.
15. If $\overline{AC}$ intersects $\overline{BD}$ at E, prove that $m\angle DEC = \frac{1}{2}(m\widehat{CD} + m\widehat{AB})$.
16. Prove that $\triangle ADE \sim \triangle BCE$.
17. Prove that $DE \cdot EB = AE \cdot EC$.

18. With each chord of a circle which is not a diameter there are two associated arcs of the circle. One of the arcs is a minor arc and the other arc is a major arc. The endpoints of the chord are the endpoints of the arcs. Complete the proof of the following theorem.

THEOREM 13.18 In the same circle, or in congruent circles, two chords that are not diameters are congruent if and only if their associated minor arcs are congruent.

Proof: Using the notation of the figure, we are given two congruent circles, C and C', with centers P and P', respectively. $\overline{AB}$ is a chord of C and $\overline{A'B'}$ is a chord of C'. There are two things to prove.



(a) If $\overline{AB} \cong \overline{A'B'}$, then $\widehat{AB} \cong \widehat{A'B'}$.
(b) If $\widehat{AB} \cong \widehat{A'B'}$, then $\overline{AB} \cong \overline{A'B'}$.
(*Hint:* In proving (a), show that $\triangle APB \cong \triangle A'P'B'$ by the S.S.S. Postulate. In proving (b), show that $\triangle APB \cong \triangle A'P'B'$ by the S.A.S. Postulate.)

19. Does Theorem 13.18 of Exercise 18 still hold if we replace "minor arcs" with "major arcs" in the statement of the theorem?
20. Prove Case 3 of Theorem 13.16. (Refer to Figure 13-38c.)
21. Prove Corollary 13.16.1.
22. Prove Corollary 13.16.2.
23. Prove Corollary 13.16.3.
24. Figure 13-42 shows three cases of a chord $\overline{AV}$ of a circle with center P and a tangent $\overleftrightarrow{VT}$ to the same circle intersecting at the point of tangency V. Using the notation of the figure, the angle whose sides are rays $\overrightarrow{VT}$ and $\overrightarrow{VA}$ is sometimes called a **tangent-chord angle.** Complete the proof of the following theorem.

THEOREM 13.19 **The measure of a tangent-chord angle is one-half the measure of its intercepted arc.**

Proof: Using the notation of Figure 13-42, P is the center of the given circle, $\widehat{AXV}$ is the intercepted arc, $\overline{AV}$ is the given chord, and $\overleftrightarrow{VT}$ is the given tangent. There are three cases to consider.

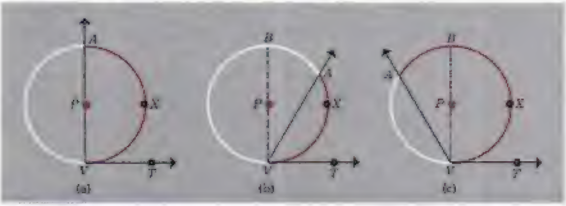


Figure 13-42

Case 1. P is on $\overline{VA}$ as shown in Figure 13-42a.
Case 2. P is in the exterior of $\angle AVT$ as shown in Figure 13-42b.
Case 3. P is in the interior of $\angle AVT$ as shown in Figure 13-42c.

Proof of Case 1: (See Figure 13-42a.) $\overline{AV}$ is a diameter; hence $m\angle AVT = 90$ (Why?) and $m\widehat{AXV} = 180$. Thus $m\angle AVT = \frac{1}{2}m\widehat{AXV}$.

Proof of Case 2: (See Figure 13-42b.) Draw diameter $\overline{VB}$.

$$\begin{aligned} m\angle BVT &= \tfrac{1}{2}m\widehat{BXV}, && \text{Why?} \\ &= \tfrac{1}{2}(m\widehat{BA} + m\widehat{AXV}), && \text{Why?} \\ &= \tfrac{1}{2}m\widehat{BA} + \tfrac{1}{2}m\widehat{AXV}. \end{aligned}$$

Since $m\angle BVT = m\angle BVA + m\angle AVT$, we have

$$m\angle BVA + m\angle AVT = \tfrac{1}{2}m\widehat{BA} + \tfrac{1}{2}m\widehat{AXV}$$

by the Substitution Property of Equality. But

$$m\angle BVA = \tfrac{1}{2}m\widehat{BA}. \qquad \text{Why?}$$

Therefore, by the Addition Property of Equality, we get

$$m\angle AVT = \tfrac{1}{2}m\widehat{AXV}.$$

Complete the proof of Theorem 13.19 by proving Case 3.

25. Prove Case 2 of Theorem 13.17.

26. Prove Case 3 of Theorem 13.17. (*Hint:* In Figure 13-40c, prove that $\overline{BD}$ is a diameter. It will then follow that $\widehat{BXD}$ and $\widehat{BYD}$ are semicircles and that $\widehat{BXD} \cong \widehat{BYD}$.)

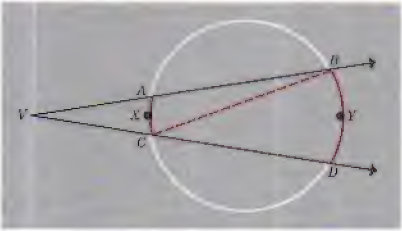
27. A quadrilateral is said to be **inscribed in a circle** and is called an **inscribed quadrilateral** if all of its vertices are on the circle. Prove that the opposite angles of an inscribed quadrilateral are supplementary.

28. If the diagonals of an inscribed quadrilateral are diameters, prove that the quadrilateral is a rectangle. (See Exercise 27.)

29. Prove that the midray of a central angle of a circle bisects the arc intercepted by the angle.

30. The figure shows two secants intersecting in an exterior point V of a circle. The rays $\overrightarrow{VA}$ and $\overrightarrow{VC}$ are sometimes called secant-rays and the angle whose sides are these rays is called a secant-secant angle. Complete the proof of the following theorem.

THEOREM 13.20 The measure of a secant-secant angle is one-half the difference of the measures of the intercepted arcs.



Using the notation of the figure, we must prove

$$m\angle V = \tfrac{1}{2}(m\widehat{BYD} - m\widehat{AXC}).$$

We have

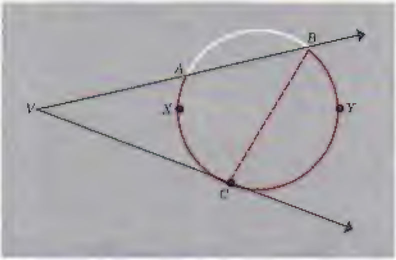
$$m\angle V + m\angle ABC = m\angle BCD. \qquad \text{Why?}$$

Therefore

$$m\angle V + \tfrac{1}{2}m\widehat{AXC} = \tfrac{1}{2}m\widehat{BYD}. \qquad \text{Why?}$$

Complete the proof.

31. An angle whose sides are a secant-ray and a tangent-ray from an exterior point of a circle is called a **secant-tangent angle.** Prove that the measure of a secant-tangent angle is one-half the difference of the measures of the intercepted arcs.

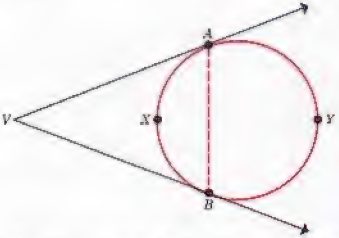


(*Hint:* Using the notation of the figure you must prove that

$$m\angle V = \tfrac{1}{2}(m\widehat{BYC} - m\widehat{AXC}).$$

Use Theorem 13.19 stated in Exercise 24. Also see Exercise 30.)

32. An angle whose sides are two tangent-rays from an exterior point of a circle is called a **tangent-tangent angle.** Prove that the measure of a tangent-tangent angle is one-half the difference of the measures of the intercepted arcs.



(*Hint:* Using the notation of the figure you must prove that

$$m\angle V = \tfrac{1}{2}(m\widehat{AYB} - m\widehat{AXB})$$

Use Theorem 13.19 stated in Exercise 24. Also see Exercise 30.)

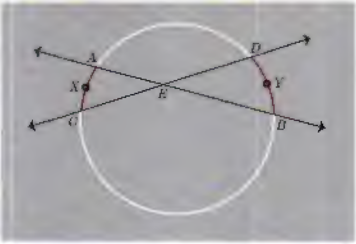
33. Copy and complete the following theorem which combines the statements of Theorem 13.20 (see Exercise 30) and Exercises 31 and 32 into a single statement.

THEOREM 13.21 If an angle has its vertex in the exterior of a circle and if its sides consist of two secant-rays, or a secant-ray and a tangent-ray, or two tangent-rays to the circle, then the measure of the angle is $\boxed{?}$.

34. Complete the proof of the following theorem.

THEOREM 13.22 The measure of an angle whose vertex is in the interior of a circle and whose sides are contained in two secants is one-half the sum of the measures of the intercepted arcs.

(An angle such as the one described in Theorem 13.22 is called a **chord-chord angle.**



Using the notation of the figure, it is given that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are secants intersecting at E, a point in the interior of the circle. Prove that

$$m\angle DEB = \tfrac{1}{2}(m\widehat{DYB} + m\widehat{AXC}).$$

Exercises 35–44 refer to Figure 13-43. In the figure, $\overrightarrow{VA}$ and $\overrightarrow{VC}$ are secant-rays of the circle with center P. $\overrightarrow{VE}$ is a tangent-ray, chords $\overline{AD}$ and $\overline{BC}$ intersect at F, and B-P-C. If the measures of arcs $\widehat{AB}$, $\widehat{BD}$, $\widehat{DE}$, and AC are as shown, use the results of Exercises 30, 31, 32, and 34 and other theorems proved in this section to find the indicated measure in each exercise.

35. $m\angle AVC$
36. $m\widehat{CE}$
37. $m\angle CVE$
38. $m\angle AVE$
39. $m\angle BFD$
40. $m\angle AFB$
41. $m\angle DEG$
42. $m\angle BEG$
43. $m\angle CAB$
44. $m\angle BEA$

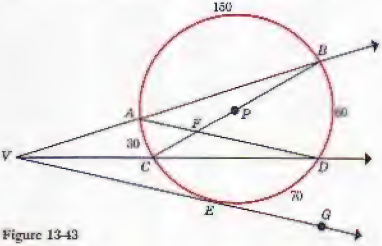
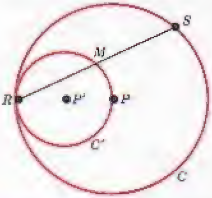


Figure 13-43

45. CHALLENGE PROBLEM. In the figure, circles C and C' with centers P and P', respectively, are tangent internally at R and circle C' contains P. If $\overline{RS}$ is any chord (with one endpoint at R) of C and if $\overline{RS}$ intersects C' at M, prove that M is the midpoint of $\overline{RS}$.



13.7 SEGMENTS OF CHORDS, TANGENTS, AND SECANTS

If two distinct chords of a circle intersect at an interior point of the circle, the point of intersection together with the endpoints of the chords determine four distinct segments (other than the chords) which are subsets of the chords. For example, in Figure 13-44, chords $\overline{AB}$ and $\overline{CD}$ intersect at P. The four segments to which we refer are segments $\overline{AP}$, $\overline{PB}$, $\overline{CP}$, and $\overline{PD}$. It is easy to prove that the product of the lengths of the two segments on $\overline{AB}$ is equal to the product of the lengths of the two segments on $\overline{CD}$.

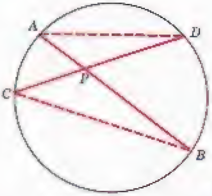


Figure 13-44

THEOREM 13.23 If two chords of a circle intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other.

Proof: Using the notation of Figure 13-44, we are given a circle with chords $\overline{AB}$ and $\overline{CD}$ intersecting at P. We are to prove that

$$AP \cdot PB = CP \cdot PD.$$

We draw $\overline{AD}$ and $\overline{BC}$. Then

$$\angle A \cong \angle C \qquad \text{(Why?)}$$

and

$$\angle D \cong \angle B. \qquad \text{(Why?)}$$

Therefore

$$\triangle ADP \sim \triangle CBP$$

by the A.A. Similarity Theorem. It follows that

$$(AP, PD) \underset{p}{=} (CP, PB)$$

and that

$$AP \cdot PB = CP \cdot PD.$$

This completes the proof of Theorem 13.23.

Figure 13-45 shows a line $\overleftrightarrow{VT}$ tangent to a circle at T and a secant $\overleftrightarrow{PA}$ intersecting the circle in points A and B. In the statements of our next three theorems, we refer to segments such as $\overline{VT}$, $\overline{PB}$, and $\overline{PA}$.

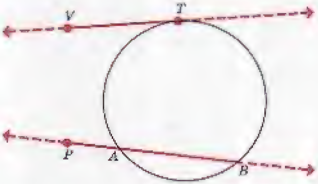


Figure 13-45

Therefore it is convenient to have a name for each of them. We make the following definition.

Definition 13.18 If V and T are distinct points and if the line $\overleftrightarrow{VT}$ is tangent to a circle at T, then the segment $\overline{VT}$ is called a tangent-segment from V to the circle. If secant $\overleftrightarrow{PA}$ intersects a circle in points A and B such that A is between P and B, then the segment $\overline{PB}$ is called a secant-segment from P to the circle and the segment $\overline{PA}$ is called an external secant-segment from P to the circle.

The proof of our next theorem is Exercise 37 of Exercises 13.3.

THEOREM 13.24 The two distinct tangent-segments to a circle with center O from an external point P are congruent and the angle whose vertex is P and whose sides contain the two tangent-segments is bisected by the ray $\overrightarrow{PO}$.

Proof: Using the notation of Figure 13-46, given a circle with center O, tangent-segments $\overline{PA}$ and $\overline{PB}$, and ray $\overrightarrow{PO}$, we are to prove that $\overline{PA} \cong \overline{PB}$ and that $\overrightarrow{PO}$ bisects $\angle APB$. Give reasons for the statements in the proof when asked.

$\angle OAP$ and $\angle OBP$ are right angles. Why? Therefore $\triangle OAP$ and $\triangle OBP$ are right triangles.

$\overline{OA} \cong \overline{OB}$ (Why?)

and

$\overline{PO} \cong \overline{PO}$.

Therefore

$\triangle OAP \cong \triangle OBP$. (Why?)

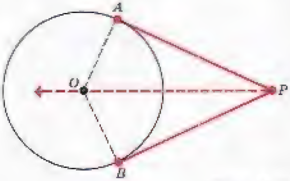


Figure 13-46

It follows that $\overline{PA} \cong \overline{PB}$ and that $\angle APO \cong \angle BPO$. To complete the proof we need to show that O is in the interior of $\angle APB$.

The union of the given circle and its interior is a convex set. Call it S. If Q is any point of S, then $OQ \leq OA$. If R is any point of $\overline{PA}$, except A, then $OR > OA$. Therefore S does not intersect $\overleftrightarrow{PA}$ except at A. Therefore all of S except A lies on one side of $\overleftrightarrow{PA}$. In particular, O lies on the B-side of $\overleftrightarrow{PA}$. Similarly, it may be shown that O lies on the A-side of $\overleftrightarrow{PB}$. Therefore O is in the interior of $\angle APB$ and since $\angle APO \cong \angle BPO$, it follows that $\overrightarrow{PO}$ bisects $\angle APB$.

THEOREM 13.25 The product of the length of a secant-segment from a given exterior point of a circle and the length of its external secant-segment is the same for any secant to the given circle from the given exterior point.

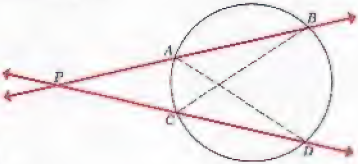


Figure 13-47

Using the notation of Figure 13-47, given secant-segments $\overline{PB}$ and $\overline{PD}$ and external secant-segments $\overline{PA}$ and $\overline{PC}$, we are to prove that $PB \cdot PA = PD \cdot PC$. Plan: Prove $\triangle PCB \sim \triangle PAD$.

Proof: Assigned as an exercise.

Our next theorem gives us still another relation between the products of the lengths of certain segments. Figure 13-48 shows an exterior point P of a given circle and a tangent to the circle at C which contains the exterior point P. If l is any secant to the given circle which contains P and intersects the circle in points A and B, we can prove that the product $PA \cdot PB$ is the same as the product $PC \cdot PC$, or $(PC)^2$.

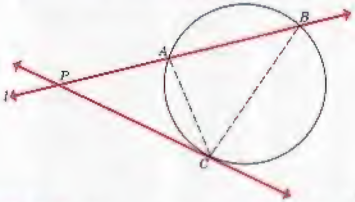


Figure 13-48

THEOREM 13.26 Given a tangent-segment $\overline{PC}$ from P to a circle at C and a secant through P intersecting the given circle in points A and B, then

$$PA \cdot PB = (PC)^2.$$

Proof: Using the notation of Figure 13-48, we are given a tangent-segment $\overline{PC}$ from an exterior point P to the circle at C. $\overline{PB}$ is any secant-segment from P and intersecting the given circle in points A and B. We are to prove that $PA \cdot PB = (PC)^2$.

Statement	Reason
1. $m\angle PCA = \frac{1}{2}m\widehat{AC}$	1. The measure of a tangent-chord angle is one-half the measure of the intercepted arc.
2. $m\angle B = \frac{1}{2}m\widehat{AC}$	2. Why?
3. $m\angle PCA = m\angle B$	3. Why?
4. $\angle PCA \cong \angle B$	4. Why?
5. $\angle P \cong \angle P$	5. Why?
6. $\triangle PCA \sim \triangle PBC$	6. Why?
7. $(PC, PA) \underset{p}{=} (PB, PC)$	7. Why?
8. $PA \cdot PB = (PC)^2$	8. Why?

It follows from Theorem 13.23 that if P is any point in the interior of a circle, the product $PA \cdot PB$ remains unchanged for any chord $\overline{AB}$ which contains P. Also, if P is a point in the exterior of a circle as in Theorems 13.25 and 13.26, the product $PA \cdot PB$ remains unchanged for any secant-segment from P and intersecting the circle in points A and B, or for any tangent-segment from P to the circle. Theorems 13.25 and 26 suggest that the value of the "unchanged product" is the square of the length of the tangent-segment from P to the circle. This seems to have no significance for Theorem 13.23. Figure 13-49 suggests (with a bit of help from Pythagoras) that $(OP)^2 - r^2$ exists even when P is inside the circle. As we shall see, the idea of $(OP)^2 - r^2$ as the value of the "unchanged product" is what relates Theorem 13.23 to Theorems 13.25 and 13.26.

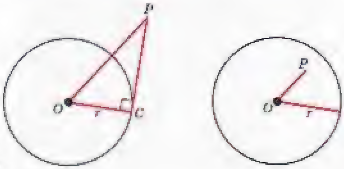


Figure 13-49

> ***Definition 13.19*** Given a circle S with center O and radius r, and a point P in the same plane as S, the **power** of P with respect to S is $(OP)^2 - r^2$.

Given a circle S with center O and radius r, a point P coplanar with S is in the interior of S, on S, or in the exterior of S, according to whether $OP < r$, $OP = r$, or $OP > r$, hence according to whether $(OP)^2 < r^2$, $(OP)^2 = r^2$, or $(OP)^2 > r^2$, and hence according to whether the power of P with respect to S is negative, zero, or positive. If an xy-coordinate system is set up in the plane of S with its center O as origin, then the power of P with respect to S is

$$(OP)^2 - r^2 = x^2 + y^2 - r^2$$

and

$$\begin{aligned}\text{interior of } S &= \{(x, y) : x^2 + y^2 - r^2 < 0\}\\ S &= \{(x, y) : x^2 + y^2 - r^2 = 0\}\\ \text{exterior of } S &= \{(x, y) : x^2 + y^2 - r^2 > 0\}.\end{aligned}$$

Compare this with the representations using set-builder notation in Figure 13-9.

We are now ready for the theorem that relates Theorem 13.23 to Theorems 13.25 and 13.26.

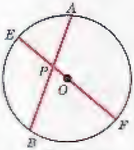
THEOREM 13.27 Let a circle S with center O and radius r be given. Let P be a point in the plane of S and let p be the power of P with respect to S. If a line through P intersects S in points A and B, then

1. $PA \cdot PB = -p$ if P is inside S, and
2. $PA \cdot PB = p$ if P is on S or outside S.

Proof: Our proof is by cases. In Cases 1 and 3, $\overline{EF}$ is the diameter of S such that $\overleftrightarrow{EF}$ contains P.

Case 1.

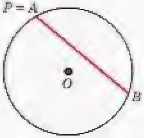
$$\begin{aligned} PA \cdot PB &= PE \cdot PF \qquad \text{(Why?)} \\ &= (r - OP)(r + OP) \\ &= r^2 - (OP)^2 \\ &= -((OP)^2 - r^2) \\ &= -p. \end{aligned}$$



Case 1

Case 2a.

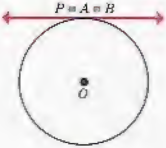
$$\begin{aligned} PA \cdot PB &= 0 \cdot PB \\ &= 0 \\ &= (OP)^2 - r^2 \\ &= p. \end{aligned}$$



Case 2a

Case 2b.

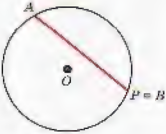
$$\begin{aligned} PA \cdot PB &= 0 \cdot 0 \\ &= 0 \\ &= (OP)^2 - r^2 \\ &= p. \end{aligned}$$



Case 2b

Case 2c.

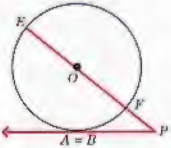
$$\begin{aligned} PA \cdot PB &= PA \cdot 0 \\ &= 0 \\ &= (OP)^2 - r^2 \\ &= p. \end{aligned}$$



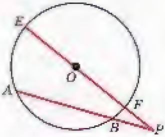
Case 2c

Cases 3a and 3b.

$$\begin{aligned} PA \cdot PB &= PE \cdot PF \qquad \text{(Why?)} \\ &= (OP + r)(OP - r) \\ &= (OP)^2 - r^2 \\ &= p \end{aligned}$$



Case 3a



Case 3b

The theorems we have proved in this section enable us to do many numerical problems.

Example 1 In Figure 13-50, $CD = 38$, $CE = 20$, and $BE = 24$. Find DE and AE.

Solution: $DE = CD - CE$
$= 38 - 20 = 18.$

Let $AE = x$; then, by Theorem 13.23,

$$x \cdot BE = CE \cdot DE.$$

Thus

$$x \cdot 24 = 20 \cdot 18$$

or

$$24x = 360$$

and

$$x = 15.$$

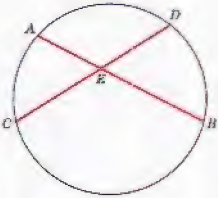


Figure 13-50

What is the power of E in this example? Is the power of E a positive number or a negative number?

Example 2 In Figure 13-51, $PB = 70$ and $PC = 40$. Find PA.

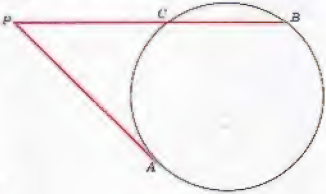


Figure 13-51

Solution: By Theorem 13.26, $(PA)^2 = PB \cdot PC$. Therefore

$$(PA)^2 = 70 \cdot 40.$$

Thus

$$(PA)^2 = 2800,$$
$$(PA)^2 = 400 \cdot 7,$$

and

$$PA = \sqrt{400 \cdot 7},$$
$$PA = 20\sqrt{7}.$$

What is the power of P in Example 2? Is the power of P a positive number or a negative number?

Example 3 In Figure 13-52, $PB = 78$, $AB = 26$, and $PD = 82$. Find CD.

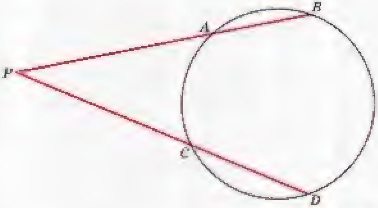


Figure 13-52

Solution: First, we need to find PA and PC. Why? We have

$$PA = PB - AB = 78 - 26 = 52.$$

By Theorem 13.25,

$$PD \cdot PC = PB \cdot PA.$$

Therefore

$$82 \cdot PC = 78 \cdot 52$$

and

$$PC = \frac{78 \cdot 52}{82} = 49\frac{19}{41}.$$

Thus

$$CD = PD - PC = 82 - 49\tfrac{19}{41} = 32\tfrac{22}{41}.$$

What is the power of P in Example 3?

EXERCISES 13.7

1. Prove Theorem 13.25. (Refer to Figure 13-47.)

Exercises 2–5 refer to Figure 13-53 which shows two intersecting chords of a circle.

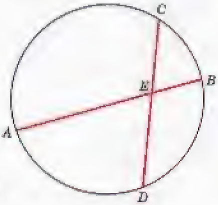


Figure 13-53

2. If $CE = 7$ and $DE = 9$, find the power of E.
3. If $AB = 24$, $BE = 8$, and $CE = 12$, find DE and the power of E.
4. If $CE = x$, $DE = 8$, $AE = 12$, and $BE = x - 2$, find CE, BE, and the power of E.
5. If $AE = x$, $BE = x - 4$, $CE = 4$, and $ED = 16 - x$, find AE, BE, ED, and the power of E. How are $\overline{AB}$ and $\overline{CD}$ related?

Exercises 6–14 refer to Figure 13-54 which shows two secant-rays and a tangent-ray from an exterior point of a circle. If an answer is an irrational number, put it in simplest radical form. (For example, $\sqrt{32} = 4\sqrt{2}$ in simplest radical form.)

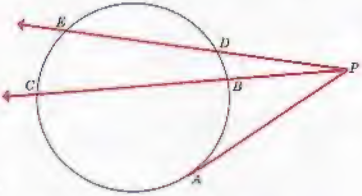
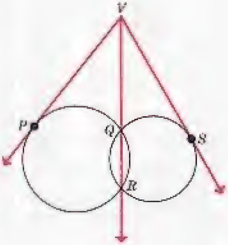


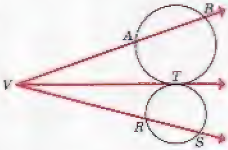
Figure 13-54

6. If $PE = 16$, $PD = 10$, and $PB = 8$, find PC.
7. Find the power of P in Exercise 6.
8. Find PA in Exercise 6.
9. If $ED = 9$, $DP = 12$, and $PC = 18$, find BC.

10. Find the power of P in Exercise 9.
11. Find PA in Exercise 9.
12. If $PA = 16$, $PB = 10$, and $PE = 24$, find DE, PC, BC, and the power of P.
13. If $PA = x$, $PE = 50$, $PD = 32$, and $PC = x + 20$, find PA, PC, PB, and BC.
14. Find the power of P in Exercise 13.
15. The figure shows two circles intersecting at Q and R, a tangent-ray from V to the larger circle at P, a tangent-ray from V to the smaller circle at S, and a secant-ray from V intersecting the two circles in points Q and R. Prove that $VP = VS$. (The line $\overleftrightarrow{QR}$ in the figure is called the **radical axis** of the two circles. It is the set of all points of like power with respect to both circles.)

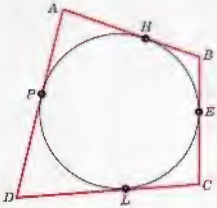


16. The figure shows two circles tangent externally at T and $\overleftrightarrow{VT}$ is a *common tangent*. A secant-ray from V intersects the larger circle in points A and B, and a secant-ray from V intersects the smaller circle in points R and S. Prove $VA \cdot VB = VR \cdot VS$.

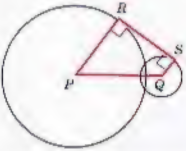


17. Given the figure in Exercise 16. Prove that $\triangle VAR \sim \triangle VSB$.
18. Given that the sides of quadrilateral $ABCD$ are tangent to a circle at points H, E, L, P, as shown in the figure, prove that

$$AB + CD = AD + BC.$$



19. If a common tangent of two circles does not intersect the segment joining their centers, it is called a **common external tangent.** If it does intersect the segment joining their centers, it is called a **common internal tangent.** In the figure, $\overline{RS}$ is a common external tangent of the two circles with centers P and Q, $PR = 21$, $QS = 6$, and $PQ = 25$. Find RS. (*Hint:* Draw $\overline{QT} \perp \overline{PR}$ at T.)

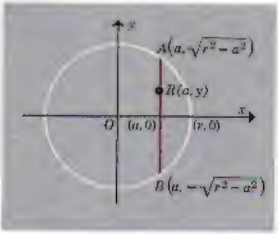


20. Give reasons for steps 2 to 8 in the proof of Theorem 13.26.
21. Complete the proof of the following theorem.

THEOREM If A and B are distinct points on a circle and if R is any point between A and B, then R is in the interior of the circle.

Proof: Set up an xy-coordinate system with the origin at the center of the given circle and with the x-axis the perpendicular bisector of $\overline{AB}$ as shown in the figure. Let r be the radius of the circle. Then there is a number a such that $-r < a < r$ and

$$\overleftrightarrow{AB} = \{(x, y) : x = a\}.$$



Then the endpoints of $\overline{AB}$ are $(a, \sqrt{r^2 - a^2})$ and $(a, -\sqrt{r^2 - a^2})$, and a point $R(a, y)$ is between A and B

$$\text{if and only if} \qquad -\sqrt{r^2 - a^2} < y < \sqrt{r^2 - a^2},$$
$$\text{if and only if} \qquad y^2 < r^2 - a^2.$$

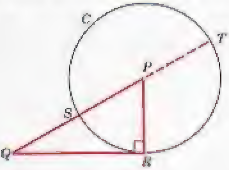
Complete the proof by showing that $OR < r$ and hence that R is in the interior of the circle.

22. Let line l be a tangent to a circle at T. Prove that all points of the circle, except T, are on one side of l in the plane of the circle. (*Hint:* Let A and B be any two distinct points of the circle such that $A \neq T$ and $B \neq T$, and suppose that A and B are on opposite sides of l. Then there is a point of l between A and B. Why? Therefore l intersects the interior of the circle. Why? See Exercise 21. Contradiction?)

23. CHALLENGE PROBLEM. Given a right triangle, $\triangle PQR$, with the right angle at R, let C be the circle with center at P and radius PR as shown in the figure. Then $\overline{QR}$ is a tangent-segment (Why?) and the circle intersects $\overleftrightarrow{QP}$ in two points S and T. Why? Use Theorem 13.26 and prove that

$$(QP)^2 = (QR)^2 + (RP)^2.$$

Thus you will have given another proof of the Pythagorean Theorem.



CHAPTER SUMMARY

The following terms and phrases were defined in this chapter. Be sure that you know the meaning of each of them.

CIRCLE
SPHERE
RADIUS OF CIRCLE (SPHERE)
DIAMETER OF CIRCLE (SPHERE)
CONCENTRIC CIRCLES (SPHERES)
CHORD OF CIRCLE (SPHERE)
TANGENT LINE TO CIRCLE
TANGENT CIRCLES
TANGENT PLANE TO SPHERE
SECANT
CONGRUENT CIRCLES (SPHERES)
INTERIOR OF CIRCLE (SPHERE)
EXTERIOR OF CIRCLE (SPHERE)
GREAT CIRCLE
CENTRAL ANGLE OF CIRCLE
MINOR ARC OF CIRCLE
MAJOR ARC OF CIRCLE
SEMICIRCLE
DEGREE MEASURE OF ARCS
INSCRIBED ANGLE
INTERCEPTED ARCS
CONGRUENT ARCS
TANGENT—SEGMENT
SECANT—SEGMENT
EXTERNAL SECANT—SEGMENT
POWER OF A POINT

There were 27 numbered theorems in this chapter. You should read them again and study them so that you understand what they say. The following are a list of a few of the more important theorems concerning circles. You should know the corresponding theorems with regard to spheres where applicable.

THEOREMS 13.3 and 13.4 Let a circle and a line in the same plane be given. The line is tangent to the circle if and only if it is perpendicular to a radius at the outer end of the radius.

THEOREM 13.5 A diameter of a circle bisects a chord of the circle other than a diameter if and only if it is perpendicular to the chord.

THEOREM 13.8 Chords of congruent circles are congruent if and only if they are equidistant from the centers of the circles.

THEOREM 13.16 The measure of an inscribed angle is one-half the measure of its intercepted arc.

REVIEW EXERCISES

In Exercises 1–22, refer to the circle C with center P shown in Figure 13-55. Assume that all points in the figure are where they appear to be in the plane of the circle. Copy the statements, replacing the question marks with words or symbols that best name or describe the indicated parts.

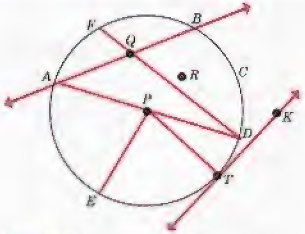


Figure 13-55

1. $\overline{AB}$ is called a [?] of the circle.
2. $\overline{PE}$ is called a [?] of the circle.
3. $\overline{AD}$ is called a [?] of the circle. $\overline{AD}$ could also be called a [?] of the circle.
4. $\overleftrightarrow{AB}$ is called a [?].
5. If $\overleftrightarrow{TK} \cap C = \{T\}$, then $\overleftrightarrow{TK}$ is called a [?] to the circle at T.

6. If $\overleftrightarrow{TK} \cap C = \{T\}$, then $\overleftrightarrow{TK}$ is $\boxed{?}$ to $\overline{PT}$ at T.
7. Those points named in the figure and which are on the circle are $\boxed{?}$.
8. Those points named in the figure and which are points in the interior of the circle are $\boxed{?}$.
9. $\boxed{?}$ is a point in the exterior of the circle.
10. $\angle ADF$ is an $\boxed{?}$ angle.
11. $\angle BAD$ is $\boxed{?}$ in arc $\widehat{BAD}$.
12. $\angle ADF$ intercepts arc $\boxed{?}$.
13. $\widehat{DTE}$ is a $\boxed{?}$ arc of the circle.
14. $\widehat{DAF}$ is a $\boxed{?}$ arc of the circle.
15. $\widehat{ABD}$ is a $\boxed{?}$.
16. $m\angle BAD$ is $\boxed{?}$ $m\widehat{BD}$.
17. $\angle EPT$ is a $\boxed{?}$ angle.
18. $m\widehat{ET} = \boxed{?}$.
19. The power of Q is $\boxed{?}$.
20. The power of K is $\boxed{?}$.
21. The power of A is $\boxed{?}$.
22. $\triangle AFQ$ is similar to $\triangle\boxed{?}$.

In Exercises 23–29 refer to the sphere S with center Q shown in Figure 13-56. Copy the statements, replacing the question marks with words or symbols that best name or describe the indicated parts.

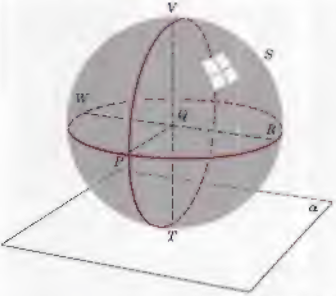


Figure 13-56

23. $\overline{QR}$ is a $\boxed{?}$ of the sphere.
24. If V, Q, T are collinear, then $\overline{VT}$ is a $\boxed{?}$ of the sphere.
25. If W, Q, R are collinear, then the circle with center Q and containing points W, P, R is a $\boxed{?}$ of the sphere.
26. $\overline{RV}$ is a $\boxed{?}$ of the sphere.
27. $\overleftrightarrow{RP}$ is a $\boxed{?}$.
28. If $\alpha \cap S = \{T\}$, then α is $\boxed{?}$ to S at T.
29. If $\alpha \cap S = \{T\}$, then $\overline{QT}$ is $\boxed{?}$ to α at T.

30. Find the radius of a circle if one of its chords 16 in. long is 6 in. from the center of the circle.

31. How far from the center of a circle with radius 16 is a chord whose length is 24?

32. A circle C and a line l in an xy-plane are given by

$$C = \{(x, y) : x^2 + y^2 = 36\}$$

and

$$l = \{(x, y) : y = x\}.$$

(a) Write the coordinates of six points that are on the circle.
(b) Write the coordinates of three points that are on the line.
(c) Write the coordinates of two points that are on the circle and on the line.

33. A sphere S in an xyz-coordinate system is given by

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 81\}.$$

(a) Write the coordinates of eight points that are on the sphere.
(b) Write the coordinates of two points that are in the interior of the sphere.
(c) Write the coordinates of two points that are in the exterior of the sphere.
(d) Write the coordinates of two points on the sphere that are endpoints of a diameter of the sphere and are not on any of the three coordinate axes.

34. Write an equation of a sphere with center at the origin of an xyz-coordinate system and which contains the point $(4, -2, 5)$. What is the radius of this sphere?

35. Let a sphere with radius 10 be given. A segment from the center of the sphere to a chord and perpendicular to the chord has length 4. Find the length of the chord.

In Exercises 36–45 refer to the circle with center P shown in Figure 13–57. Given the notation of the figure and the degree measures labeled in the figure, find the measure asked for in each exercise. You may need to refer to the theorems stated in Exercises 24, 30, 31, 32, and 34 of Exercises 13.6.

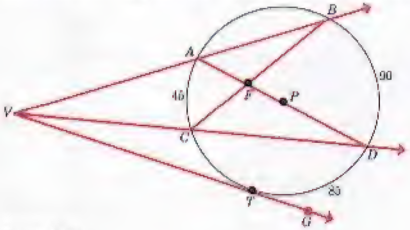


Figure 13-57

36. $m\widehat{AB}$
37. $m\widehat{CT}$
38. $m\angle BCD$
39. $m\angle CPT$
40. $m\angle AVC$
41. $m\angle CVT$
42. $m\angle BTC$
43. $m\angle BED$
44. $m\angle ABD$
45. $m\angle PTV$

In Exercises 46–50 refer to the circle with center P shown in Figure 13-58. In the figure, $\overrightarrow{VA}$ and $\overrightarrow{VC}$ are secant-rays intersecting the circle in points A, B and C, D, respectively. $\overleftrightarrow{VT}$ is a tangent to the circle at T. Chords $\overline{AD}$ and $\overline{BT}$ intersect at E.

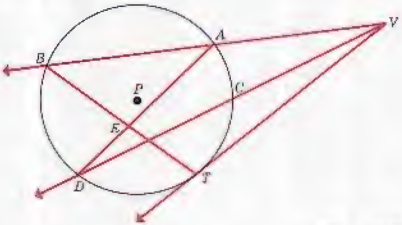


Figure 13-58

46. If $VA = 12$, $VB = 20$, and $VC = 14$, find VD and the power of V.
47. If $VA = 16$, and $AB = 9$, find VT and the power of V.
48. If $AD = 24$, $AE = 18$, and $ET = 8$, find BT and the power of E.
49. If $BE = 12$, $ET = x$, $AE = 18$, and $ED = x - 2$, find x, BT, AD, and the power of E.
50. If $BE = 9$, $EA = x$, $AE = x + 12$, and $ED = x - 3$, find x, AE, ED, and the power of E.

Chapter 14

Brooks/Monkmeyer

Circumferences and Areas of Circles

14.1 INTRODUCTION

In the first part of this chapter, we consider some of the properties of regular polygons that are useful in developing the formula for the circumference of a circle and the formula for the area of a circular region. We usually say "the area of a circle" as an abbreviation for the phrase "the area of a circular region," or "the area enclosed by a circle."

In the last part of the chapter, we depart from our formal geometry and present an intuitive approach to the development of the formulas for the circumference of a circle, the area of a circle, the length of an arc, and the area of a sector. We appeal to your intuition in developing these formulas because a formal treatment involves the use of limits, a topic that you would study in a mathematics subject called "calculus." We use the idea of a limit to make the formulas seem plausible.

14.2 POLYGONS

In this section we investigate some of the angle measure properties of convex polygons. We also consider some of the properties of a certain subset of convex polygons called regular polygons.

Recall the definition of a convex polygon in Chapter 4. We say that a polygon is a convex polygon if and only if each of its sides lies on the edge of a halfplane which contains all of the polygon except that one side. In this chapter all the polygons with which we are concerned are convex polygons. Therefore, when we speak of a polygon, we mean a convex polygon.

Recall that two vertices of a polygon that are endpoints of the same side are called consecutive vertices, or *adjacent vertices*. Two sides of a polygon that have a common endpoint are called consecutive sides, or *adjacent sides*. An angle determined by two adjacent sides of a polygon is called *an angle of the polygon*. Two angles of a polygon are called *adjacent angles* of the polygon if their vertices are adjacent vertices of the polygon.

For the polygon $ABCDE$ shown in Figure 14-1, A and B are adjacent vertices, $\overline{AB}$ and $\overline{AE}$ are adjacent sides, and $\angle A$ and $\angle B$ are adjacent angles of the polygon. Vertices such as A and C, or such as A and D, or such as B and D, and so on, are called **nonadjacent vertices** of the polygon. A segment whose endpoints are nonadjacent vertices of a polygon is called a **diagonal** of the polygon. In Figure 14-1, $\overline{AC}$ is a diagonal of the polygon. Name three more diagonals of the polygon $ABCDE$. How many distinct diagonals does this polygon have altogether? How many distinct diagonals are there that have a given vertex as an endpoint? In the work that follows we are going to be concerned with determining the number of diagonals from an arbitrary vertex of a given polygon. Note that a triangle has no diagonals since each pair of vertices in a triangle is a pair of adjacent vertices.

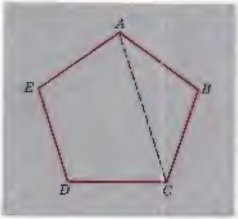


Figure 14-1

In Chapter 7 we proved that the sum of the measures of the angles of a triangle is 180. We then used this theorem to prove that the sum of the measures of the angles of a convex quadrilateral is 360. (See Theorem 7.33 and its proof.) Let us now review the ideas of this proof with the aid of the new terminology introduced in the study of areas in Chapter 9.

If the quadrilateral $ABCD$ shown in Figure 14-2 is a convex quadrilateral, then the diagonal $\overline{AC}$ partitions the polygonal region $ABCD$ into two triangular regions, ABC and ADC. The union of these two triangular regions is the polygonal region $ABCD$. In the proof of Theorem 7.33, we showed that the sum of the measures of the four angles of the quadrilateral is the same as the sum of the measures of a certain set of six angles, three from each of the two triangles. In this way we obtained $2 \cdot 180$, or 360, as the sum of the measures of the angles of a convex quadrilateral.

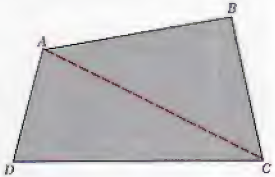


Figure 14-2

Now let us extend this idea to convex polygons of 5, 6, 7, 8, or n sides, where n is a positive integer greater than 4. Figure 14-3 shows pictures of polygons with 5, 6, 7, and 8 sides. The names of these polygons are pentagon, hexagon, heptagon, and octagon, respectively.

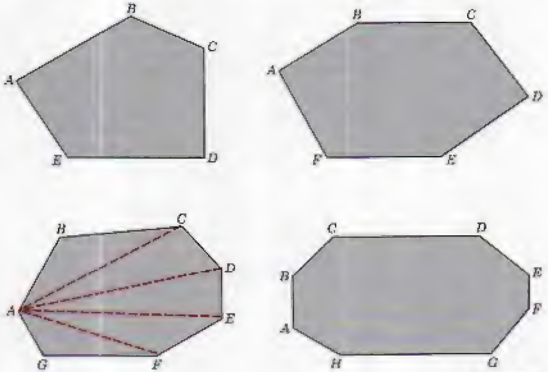


Figure 14-3

On a sheet of paper draw four convex polygons, a pentagon, a hexagon, a heptagon, and an octagon. In each polygon, label one vertex A and draw all distinct diagonals from A. The diagonals partition each polygonal region into a certain number of nonoverlapping triangular regions whose union is the given polygonal region. By a procedure similar to the one we used with the quadrilateral, find the sum of the measures of the angles of each polygon and summarize the results in a table like the one shown in Figure 14-4. Complete the last two rows of the table on the basis of your computations for the first five rows. The entries in the last row should be formulas involving n.

Number of Sides of Convex Polygon	Number of Diagonals from A	Number of Triangular Regions	Sum of Measures of the Angles of the Polygon
4	1	2	$2 \cdot 180 = 360$
5	2	?	?
6	3	?	?
7	4	?	?
8	?	?	?
9	?	?	?
n	?	?	?

Figure 14-4

On the basis of the results of the computations suggested, it seems reasonable to conclude that each convex polygonal region of n sides can be partitioned into $n - 2$ triangular regions by drawing the diagonals from one vertex. Since the sum of the measures of the angles of each triangle that bounds a triangular region is 180, the formula $(n - 2)180$ appears to be a correct formula for determining the sum of the measures of the angles of a convex polygon of n sides. We state this result as a theorem.

THEOREM 14.1 The sum of the measures of the angles of a convex polygon of n sides is $(n - 2)180$.

Proof: Let A be any vertex of the given convex polygon with n sides and let the polygon be $ABCD \ldots MN$ as suggested in Figure 14-5. Since a diagonal exists from A to each of the n vertices of the polygon except the vertices A, B, N (Why?), there are $n - 3$ diagonals from the vertex A. Match $\triangle ABC$ with $\overline{AC}$, $\triangle ACD$ with $\overline{AD}$, ... This es-

tablishes a one-to-one correspondence between the set of $n - 3$ diagonals and the set of all triangular regions with the exception of AMN.

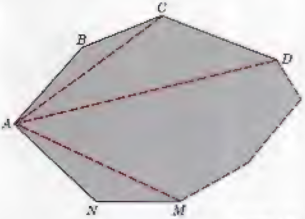


Figure 14-5

Therefore there are $(n - 3) + 1$, or $n - 2$, triangular regions. The union of these $n - 2$ triangular regions, ABC, ACD, . . . , AMN, is the polygonal region

$$ABCD \ldots MN.$$

The sum of the measures of all the angles of the triangles that bound these triangular regions is $(n - 2)180$. It follows from the Angle Measure Addition Theorem that the sum of the measures of all the angles of the triangles

$$\triangle ABC, \triangle ACD, \ldots, \triangle AMN$$

is the same as the sum of the measures of all the angles of the polygon $ABCD \ldots MN$. This completes the proof.

An important subset of the set of all convex polygons is the set of polygons whose sides are all congruent and whose angles are all congruent. We call such polygons *regular polygons*. Figure 14-6 shows a regular pentagon $ABCDE$ and a regular hexagon $ABCDEF$.

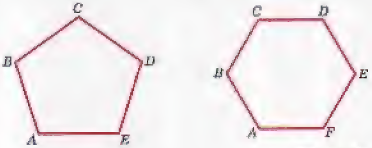


Figure 14-6

Definition 14.1 A regular polygon is a convex polygon all of whose sides are congruent and all of whose angles are congruent.

What do we call a regular polygon of three sides? Of four sides? Note that a rhombus (Figure 14-7a) which is not a square has all of its sides congruent, but it is not a regular polygon. Why? Similarly, a rectangle (Figure 14-7b) which is not a square has all of its angles congruent, but it is not a regular polygon. Why?

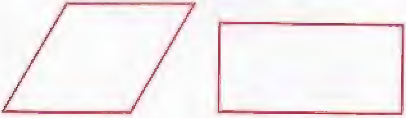


Figure 14-7 (a) Rhombus (b) Rectangle

Since a polygon of n sides has n vertices and therefore n angles, we have an important corollary of Theorem 14.1 that applies to a regular polygon of n sides.

COROLLARY 14.1.1 The measure of each angle of a regular polygon of n sides is

$$\frac{(n-2)180}{n}.$$

Proof: A regular polygon of n sides has n angles and each of these angles has the same measure as every other angle of the polygon. Since the sum of the measures of the angles is $(n-2)180$, it follows that each angle has a measure of $\dfrac{(n-2)180}{n}$.

In Chapter 6 we defined an exterior angle of a triangle. We now extend the definition to convex polygons of more than three sides.

Definition 14.2 Each angle of a convex polygon is called an interior angle of the polygon. An angle which forms a linear pair with an interior angle of a convex polygon is called an exterior angle of the polygon. Each exterior angle is said to be adjacent to the interior angle with which it forms a linear pair.

Note that there are two exterior angles at each vertex of a polygon as suggested in Figure 14-8 and that, being vertical angles, they are congruent to each other. It follows that a polygon of n sides has $2n$ exterior angles.

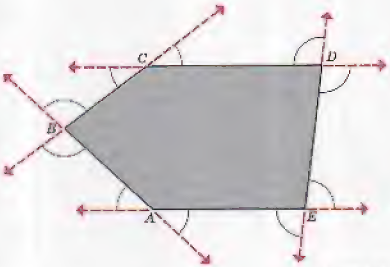


Figure 14-8

Suppose that we are given a convex polygon of n sides and suppose that we choose one of the two exterior angles at each vertex. The chosen exterior angle and the adjacent interior angle of the polygon at that vertex are supplementary. Why? Therefore the sum of their measures is 180. Since there are n vertices, the sum of the measures of all the interior angles and the chosen exterior angles is $n \cdot 180$. But we have shown that the sum of the measures of all the interior angles of the polygon is $(n - 2)180$. Therefore the sum of the measures of all the chosen exterior angles (one at each vertex) is

$$\begin{aligned} n(180) - (n - 2)180 &= 180n - 180n + 360 \\ &= 360. \end{aligned}$$

We have proved the following theorem.

THEOREM 14.2 The sum of the measures of the exterior angles, one at each vertex, of a convex polygon of n sides is 360.

This means that the sum of the measures of the exterior angles of a polygon is independent of the number of sides the polygon may have. Suppose we consider a particular regular polygon, such as a regular hexagon. We know that the sum of the measures of all the interior angles of the hexagon is $(n - 2)180$ and that the measure of each angle of a regular hexagon is $\frac{(n - 2)180}{n}$. For a hexagon, $n = 6$.

Therefore the measure of each angle of a regular hexagon is

$$\frac{(6-2)180}{6} = 120.$$

It follows that the measure of each exterior angle of a regular hexagon is 60 and that the sum of the measures of the exterior angles, one at each vertex, is $6 \cdot 60 = 360$.

Another way to calculate the measure of each exterior angle of a regular hexagon is to use Theorem 14.2. Since supplements of congruent angles are congruent, it follows that all the exterior angles of a regular polygon are congruent. If we choose just one exterior angle at each vertex, there are 6 chosen exterior angles of a regular hexagon and each of them has a measure of $\frac{1}{6} \cdot 360 = 60$. This leads us to the following corollary of Theorem 14.2.

COROLLARY 14.2.1 The measure of each exterior angle of a regular polygon of n sides is $\frac{360}{n}$.

Proof: Assigned as an exercise.

EXERCISES 14.2

In Exercises 1–6, use the formula of Corollary 14.1.1 to find the measure of each interior angle of the indicated regular polygon.

1. Pentagon	3. Octagon	5. 24-gon (24 sides)
2. Heptagon (7 sides)	4. Decagon (10 sides)	6. 180-gon

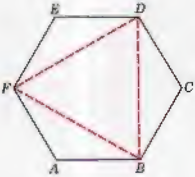
7. Find the measure of each exterior angle of the regular polygons of Exercises 1–6 in two different ways.

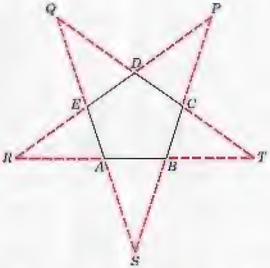
Copy and complete the statements in Exercises 8 and 9 with the word *increases* or the word *decreases*.

8. As the number of sides of a regular polygon increases, the measure of each interior angle of the polygon $\boxed{?}$.
9. As the number of sides of a regular polygon increases, the measure of each exterior angle of the polygon $\boxed{?}$.

10. The measure of each interior angle of a regular n-gon is 140. Find n. (*Hint:* Solve the equation $\frac{(n-2)180}{n} = 140$ for n.)
11. The measure of each interior angle of a regular n-gon is 150. Find n. (See Exercise 10.)

12. Can the measure of each interior angle of a regular polygon be 136? Explain.
13. Find the measure of each exterior angle of a regular 12-gon.
14. The measure of each exterior angle of a regular n-gon is 12. Find n.
15. Can the measure of each exterior angle of a regular polygon be 50? Explain.
16. Given a regular hexagon $ABCDEF$ as shown in the figure below at left, prove that $\triangle BDF$ is equilateral.





17. The figure above at right shows a regular pentagon $ABCDE$ whose sides have been extended to form a five-pointed star. Find the measure of each vertex angle of the star (that is, $\angle P$, $\angle Q$, $\angle R$, $\angle S$, $\angle T$).
18. If the sides of a regular hexagon were extended to form a six-pointed star, what would be the measure of each vertex angle of the star?
19. The sum of the measures of 14 angles of a polygon of 15 sides is 2184.
    (a) What is the measure of the remaining angle?
    (b) Could the polygon be a regular polygon?
    (c) Is there enough information to decide whether it is a regular polygon?
20. The sum of the measures of 8 angles of a 9-gon is 1140.
    (a) What is the measure of the remaining angle?
    (b) Could the polygon be a regular polygon? Explain.
21. Prove Corollary 14.2.1.
22. Given a regular pentagon $ABCDE$, prove that diagonal $\overline{AD}$ is parallel to side $\overline{BC}$.
23. Given a regular hexagon $ABCDEF$, prove that diagonal $\overline{AD}$ is parallel to side $\overline{BC}$.
24. CHALLENGE PROBLEM. Given a regular polygon $ABCD \ldots N$ of n sides, prove that diagonal $\overline{AD}$ is parallel to side $\overline{BC}$ if $n \geq 5$.
25. CHALLENGE PROBLEM. Determine the maximum number of acute angles a convex polygon can have.

14.3 REGULAR POLYGONS AND CIRCLES

We know that three noncollinear points determine exactly one triangle. That is, given a set of three noncollinear points there is exactly one triangle which has these three given points as its vertices. We now proceed to show that, given a set of three noncollinear points, there is exactly one circle that contains the three given points. In this sense, we say that three noncollinear points *determine* a circle.

Let three noncollinear points D, E, F be given as shown in Figure 14-9. Let α be the unique plane that contains D, E, F. In plane α, let l_1 be the unique perpendicular bisector of $\overline{DE}$ and let l_2 be the unique perpendicular bisector of $\overline{FE}$. Let P be the unique point of intersection of l_1 and l_2. (How do we know that l_1 intersects l_2?) Then P is equidistant from D and E because it lies on the perpendicular bisector of $\overline{DE}$, and P is equidistant from E and F because it lies on the perpendicular bisector of $\overline{EF}$. Therefore P is equidistant from D, E, and F. It follows that P is the center of a circle C with radius

$$r = PD = PE = PF.$$

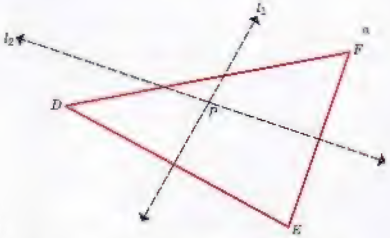


Figure 14-9

Therefore C is a circle which contains D, E, F. Furthermore, C is the only circle which contains D, E, F. For if C' were any other circle with center P' and radius r' and containing points D, E, F in plane α, then

$$r' = P'D = P'E = P'F$$

and P' would be equidistant from D, E, and F. Therefore P' would lie on l_1 and l_2, the perpendicular bisectors of $\overline{DE}$ and $\overline{EF}$ in plane α. Since there is only one point that lies on both of these lines, $P' = P$, $r' = r$, and hence $C' = C$. Therefore there is exactly one circle which contains any given set of three noncollinear points. It follows that there is exactly one circle that contains the vertices of any given triangle.

We have proved Theorem 14.3.

Definition 14.3 The circle which contains the three vertices of a given triangle is called the **circumscribed circle**, or **circumcircle**, of the triangle and we say that it **circumscribes** the triangle. The triangle is said to be **inscribed** in the circle and is called an **inscribed triangle** of the circle.

THEOREM 14.3 A given triangle has exactly one circumcircle.

We now extend the statement of Theorem 14.3 to include any regular polygon. We want to prove that, given any regular polygon, there is exactly one circle that contains all the vertices of the polygon (that is, the given regular polygon has exactly one circumcircle).

THEOREM 14.4 A given regular polygon has exactly one circumcircle.

Proof: Let a regular polygon of n sides be given. (In Figure 14-10, we have shown a polygon of 6 sides.) Let Q be the unique circle with center P and radius r which contains A, B, and C. (How do we know this circle exists and is unique?) We shall prove that $PD = r$ and hence that D lies on circle Q. A similar argument could be given to show that each of the vertices of the given polygon lies on Q.

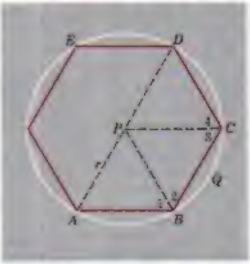


Figure 14-10

Statement	Reason
1. $PA = PB = PC = r$	1. Definition of circumcircle
2. $\angle 2 \cong \angle 3$	2. Why?
3. $m\angle ABC = m\angle BCD$	3. Why?
4. $m\angle 2 = m\angle 3$	4. Why?
5. $m\angle 1 = m\angle 4$	5. Angle Measure Addition Theorem (3, 4)
6. $\angle 1 \cong \angle 4$	6. Why?
7. $\overline{AB} \cong \overline{DC}$	7. Why?
8. $\overline{PB} \cong \overline{PC}$	8. Why?
9. $\triangle PBA \cong \triangle PCD$	9. S.A.S. Postulate (7, 6, 8)
10. $\overline{PA} \cong \overline{PD}$	10. Why?
11. $PD = PA = r$	11. Statements 10 and 1

Thus D is on Q and this completes the proof for point D.

Definition 14.4 (See Figure 14-11.) The **circumcenter** of a regular polygon is the center of its circumscribed circle. A **circumradius** of a regular polygon is a segment (or its length) joining the center of the polygon to one of the vertices of the polygon. An **inradius** of a regular polygon is a segment (or its length) whose endpoints are the center of the polygon and the foot of the perpendicular from the center of the polygon to a side of the polygon. A **central angle** of a regular polygon is an angle whose vertex is at the center of the polygon and whose sides contain adjacent vertices of the polygon.

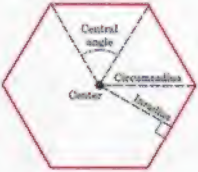


Figure 14-11

Some authors use *radius* instead of circumradius, and *apothem* instead of inradius. We prefer the more descriptive terms, circumradius and inradius. They are the radius of the circumscribed circle and the radius of the inscribed circle of a regular polygon. We know that a regular polygon has a circumscribed circle (Theorem 14.4) and we shall see that a regular polygon has an inscribed circle (Theorem 14.7).

Note that, in connection with a regular polygon, each of the words *circumradius* and *inradius* is used in two different ways: (1) as one of a set of segments and (2) as a positive number. For example, *the* inradius of a regular polygon means the number that is the length of a segment —any one of the segments defined as an inradius in Definition 14.4. We shall see in Theorem 14.6 that all these segments have the same length. On the other hand, *an* inradius of a regular polygon is a segment, usually one of the segments defined as an inradius in Definition 14.4, but it might also mean any radius of its inscribed circle. Similarly, *the* circumradius of a regular polygon means a number, whereas *a* circumradius of a regular polygon usually means any one of the segments joining the center of the polygon to a vertex of the polygon, but it might also refer to any radius of the circumscribed circle. The context in which the word is used should make it easy to decide which meaning is intended.

It follows from the definition of a central angle that a regular polygon of n sides has n central angles.

Suppose that we are given a regular polygon $ABCD \ldots MN$ of n sides. (Figure 14-12 shows a regular polygon of 6 sides.) Let P be the circumcenter of the given polygon. It follows that all of the triangles, $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, . . . , $\triangle PNA$, are congruent by the S.S.S. Postulate.

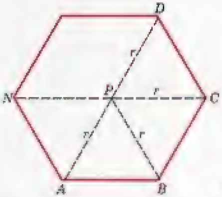


Figure 14-12

(Prove that $\triangle PAB \cong \triangle PBC$.) Let us agree to call each of these triangles (that is, a triangle whose vertices are the center and the endpoints of a side of the polygon) a central triangle of the regular polygon. It follows from the definition of congruent triangles that all of the central angles of a given regular polygon are congruent. We combine these results into the statement of our next theorem.

> *THEOREM 14.5* Let a regular polygon of n sides be given. Then all the central triangles of the given polygon are congruent and all of the central angles of the given polygon are congruent.

Again, suppose that we are given a regular polygon $ABCD \ldots MN$ of n sides. (Figure 14-13 shows a regular polygon of 6 sides.)

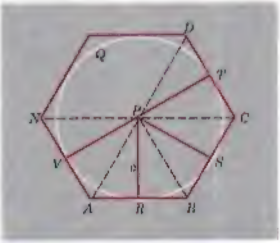


Figure 14-13

Let P be the circumcenter of the given polygon and let $\overline{PR}$, $\overline{PS}$, $\overline{PT}$, . . . , $\overline{PV}$ be the n inradii of the polygon. By the definition of an inradius, the

segments $\overline{PR}, \overline{PS}, \overline{PT}, \ldots, \overline{PV}$ are the altitudes to the bases $\overline{AB}, \overline{BC}, \overline{CD}, \ldots, \overline{NA}$, respectively, of the central triangles, $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\ldots$, $\triangle PNA$, of the given polygon. By Theorem 14.5 these central triangles are all congruent. Since corresponding altitudes of congruent triangles are congruent, it follows that all the inradii of a given regular polygon are congruent. This means that, in the plane of the given polygon, the points $R, S, T, \ldots, V$ lie on a circle Q whose center is P and whose radius, a, is the inradius (a number) of the polygon. Since $\overline{PR} \perp \overline{AB}, \overline{PS} \perp \overline{BC}, \overline{PT} \perp \overline{CD}, \ldots, \overline{PV} \perp \overline{NA}$, it follows that each of the sides $\overline{AB}, \overline{BC}, \overline{CD}, \ldots, \overline{NA}$ of the given regular polygon is tangent to the circle Q and that the points $R, S, T, \ldots, V$ are their respective points of tangency.

Definition 14.5 A circle is said to be **inscribed** in a polygon and is called an **inscribed circle**, or **incircle**, of the polygon if each of the sides of the polygon is tangent to the circle. We also say that the polygon **circumscribes** the circle. The center of an incircle of a polygon is an **incenter** of the polygon.

We now show that there is exactly one circle inscribed in a given regular polygon. Let a regular polygon such as the one in Figure 14-13 be given.

We have demonstrated that the circle Q with center P and radius a is an inscribed circle of the given polygon. It can be shown that a point which is equidistant from the sides of a regular polygon is also equidistant from its vertices and hence must be the circumcenter. Since there is only one circumcenter, it follows that there is only one incenter. Therefore Q is the only inscribed circle of the polygon. We have proved the following two theorems.

THEOREM 14.6 All the inradii of a given regular polygon are congruent.

THEOREM 14.7 There is exactly one circle that is inscribed in a given regular polygon.

We can now think of the **center** of a regular polygon as its incenter or its circumcenter.

Recall from Chapter 10 that two polygons are similar if there is a correspondence between their vertices such that corresponding angles are congruent and lengths of corresponding sides are proportional. It

follows from the definition of similar polygons that if two regular polygons are similar, then they have the same number of sides. Is the converse statement true? That is, is it true that if two regular polygons have the same number of sides, then they are similar? This brings us to our next theorem.

THEOREM 14.8 Two regular polygons are similar if they have the same number of sides.

Proof: Let two regular polygons $ABCD \ldots MN$ and $A'B'C'D' \ldots M'N'$, each having n sides, be given as suggested in Fig-

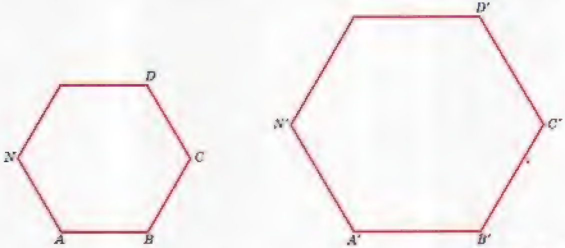


Figure 14-14

ure 14-14. We want to prove that the correspondence

$$ABCD \ldots MN \longleftrightarrow A'B'C'D' \ldots M'N'$$

is a similarity. By the definition of regular polygon and Corollary 14.1.1, we have

$$m\angle A = m\angle B = m\angle C = m\angle D = \cdots = m\angle N = \frac{(n-2)180}{n}$$

and

$$m\angle A' = m\angle B' = m\angle C' = m\angle D' = \cdots = m\angle N' = \frac{(n-2)180}{n}.$$

Therefore

$$m\angle A = m\angle A', m\angle B = m\angle B', \ldots, m\angle N = m\angle N',$$

and the corresponding angles of the two polygons are congruent.

It follows from the definition of regular polygon that there are two positive numbers s and s' such that

$$AB = BC = CD = \cdots = NA = s$$

and

$$A'B' = B'C' = C'D' = \ldots = N'A' = s'.$$

Then

$$AB = s = \frac{s}{s'} \cdot s' = \frac{s}{s'} \cdot A'B', \quad BC = s = \frac{s}{s'} \cdot s' = \frac{s}{s'} \cdot B'C', \quad \text{etc.}$$

It follows that

$$(AB, BC, CD, \ldots, NA) \underset{p}{=} (A'B', B'C', C'D', \ldots, N'A')$$

with proportionality constant $k = \frac{s}{s'}$. Therefore the lengths of the corresponding sides of the two given polygons are proportional and

$$ABCD \ldots MN \sim A'B'C'D' \ldots M'N'.$$

This completes the proof of Theorem 14.8.

> *THEOREM 14.9* The perimeters of two regular polygons, with the same number of sides, are proportional to the lengths of their circumradii, or their inradii.

Proof: Let two regular polygons $ABCD \ldots MN$ and $A'B'C'D' \ldots M'N'$, each having n sides, be given as suggested in Figure 14-15.

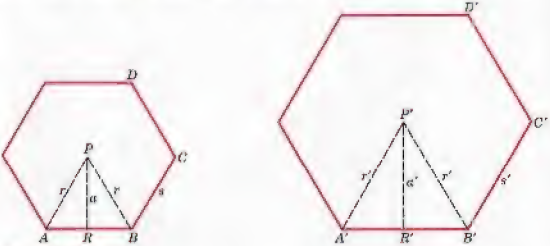


Figure 14-15

Let p and p', s and s', r and r', a and a' be their respective perimeters, lengths of sides, circumradii, and inradii. Let P and P' be the centers of

the two polygons and consider the two central triangles

$$\triangle APB \text{ of } ABCD \ldots MN$$

and

$$\triangle A'P'B' \text{ of } A'B'C'D' \ldots M'N'.$$

Since

$$m\angle APB = m\angle A'P'B' = \frac{360}{n}$$

(you will be asked to show this in the Exercises), it follows that $\triangle APB \sim \triangle A'P'B'$ by the S.A.S. Similarity Theorem. Therefore

$$(AB, r, s) \underset{p}{=} (A'B', r', s'). \qquad \text{Why?}$$

Since corresponding altitudes of similar triangles are proportional to the lengths of any two corresponding sides, we have

$$(s, r, a) \underset{p}{=} (s', r', a').$$

Finally,

$$p = ns \text{ for } ABCD \ldots MN \quad \text{and} \quad p' = ns' \text{ for } A'B'C'D' \ldots M'N'.$$

Since

$$ps' = (ns)s' = (ns')s = p's,$$

it follows that

$$(p, s) \underset{p}{=} (p', s')$$

and hence

$$(p, s, r, a) \underset{p}{=} (p', s', r', a').$$

This completes the proof of Theorem 14.9.

Since all central triangles of a given regular polygon of n sides are congruent, and since they partition the given polygonal region into n nonoverlapping triangular regions, it is easy to prove that the area of a regular polygon is equal to one-half the product of its inradius and perimeter. We state this as our next theorem.

THEOREM 14.10 The area of a regular polygon is equal to one-half the product of its inradius and perimeter, that is, $S = \frac{1}{2}ap$.

Proof: Assigned as an exercise.

The last theorem of this section follows easily from Theorems 14.9 and 14.10.

> *THEOREM 14.11* The areas of two regular polygons with the same number of sides are proportional to the squares of their circumradii (or the squares of their inradii, or the squares of their side lengths, or the squares of their perimeters).

Proof: Assigned as an exercise.

As we said in the introduction to this chapter, the idea of a limit is important in developing formulas for the circumference and the area of a circle. We conclude this section with a brief discussion of sequences and limits.

An infinite sequence of numbers, denoted by $\{x_n\}$, is a sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ in which, for every positive integer n, x_n is a number. In some applications it is convenient to start counting with some integer other than 1. Thus $x_3, x_4, x_5, \ldots, x_n, \ldots$ is an infinite sequence.

Example 1 Let the nth term of a sequence be given by $x_n = 2n$. Write the first five terms and the 40th term of the sequence $\{2n\}$.

Solution:

$$
\begin{aligned}
x_1 &= 2 \cdot 1 = 2.\\
x_2 &= 2 \cdot 2 = 4.\\
x_3 &= 2 \cdot 3 = 6.\\
x_4 &= 2 \cdot 4 = 8.\\
x_5 &= 2 \cdot 5 = 10.\\
x_{40} &= 2 \cdot 40 = 80.
\end{aligned}
$$

Therefore the first five terms of the sequence are 2, 4, 6, 8, 10 and the 40th term is 80.

Example 2 Let the nth term of a sequence be given by

$$x_n = \frac{n+1}{n}.$$

Write the first five terms, the 100th term, and the 1000th term of the sequence $\left\{\frac{n+1}{n}\right\}$.

Solution:

$$x_1 = \frac{1+1}{1} = 2. \qquad x_5 = \frac{5+1}{5} = \frac{6}{5}.$$

$$x_2 = \frac{2+1}{2} = \frac{3}{2}. \qquad x_{100} = \frac{100+1}{100} = \frac{101}{100}.$$

$$x_3 = \frac{3+1}{3} = \frac{4}{3}. \qquad x_{1000} = \frac{1000+1}{1000} = \frac{1001}{1000}.$$

$$x_4 = \frac{4+1}{4} = \frac{5}{4}.$$

In Example 2, as n gets larger and larger, the terms of the sequence get closer and closer to some particular number. What number is it?

In Example 1, as n gets larger and larger, do the terms of the sequence $\{2n\}$ get closer and closer to some particular number? If so, what number is it?

If the terms of a sequence $\{x_n\}$ get arbitrarily close (as close as we desire) to a particular number L as n gets larger and larger, we then say that the nth term of the sequence is approaching L (denoted by $x_n \to L$) and we call L the limit of the sequence. Thus, in Example 2,

$$\frac{n+1}{n} \to 1,$$

and we call 1 the limit of the sequence $\left\{\frac{n+1}{n}\right\}$. In Example 1, the sequence $\{2n\}$ has no limit. Find the limit (if any) of the sequence $\left\{\frac{1}{n}\right\}$.

EXERCISES 14.3

1. It follows from Definition 14.4 that a regular polygon of n sides has n central angles. Prove that the measure of each central angle of a regular polygon is $\frac{360}{n}$.
2. Prove that each central triangle of a regular polygon is an isosceles triangle.
3. Prove that each central triangle of a regular hexagon is an equilateral triangle.
4. Prove that the length of a side of a regular hexagon is equal to the circumradius of the circumcircle.

5. Prove that an inradius of a regular polygon bisects a side of the polygon and hence lies on the perpendicular bisector (in the plane of the polygon) of the side of the polygon.
6. Prove that the bisector rays of the interior angles of a regular polygon are concurrent at the center of the polygon.
7. Prove that the measure of a central angle of a regular polygon is equal to the measure of an exterior angle of the polygon.
8. Two regular pentagons have sides of length 4 in. and 5 in., respectively. What is the ratio of the perimeter of the smaller pentagon to the perimeter of the larger one? What is the ratio of their circumradii? Of their inradii? Of their areas?
9. The area of a regular 12-gon is $\frac{25}{9}$ times the area of a second regular 12-gon. What is the ratio of their perimeters? Of their circumradii? Of their inradii?
10. A regular hexagon has twice the area of another regular hexagon. What is the ratio of the perimeter of the smaller hexagon to the perimeter of the larger one? What is the ratio of the lengths of their sides? Of their circumradii?
11. Two regular polygons of the same number of sides have perimeters of 36 in. and 48 in., respectively. The inradius of the first polygon is $3\sqrt{3}$ in. What is the inradius of the second polygon? What is the area of each of the polygons?
12. Find the circumradius, inradius, and area of an equilateral triangle each of whose sides is of length $2\sqrt{3}$.
13. Find the circumradius, inradius, and area of a regular hexagon each of whose sides is of length 12.
14. Show that the inradius of a regular hexagon is $\frac{\sqrt{3}}{2} \cdot s$, where s is the length of a side of the hexagon.
15. Derive a formula for the area S of a regular hexagon in terms of the length s of its side. (*Hint:* See Exercise 14.)
16. Use the formula you derived in Exercise 15 to find the area of a regular hexagon each of whose sides is of length 12. Does your answer for the area agree with that of Exercise 13?
17. If a regular hexagon and a regular triangle are inscribed in the same circle, prove that the length of the side of the hexagon is twice the inradius of the triangle.
18. A square is inscribed in a circle of radius 1. A second square is circumscribed about the same circle. Find the area and the perimeter of each square.
19. Repeat Exercise 18 using regular hexagons instead of squares.
20. The length of each side of a regular hexagon is $8\sqrt{3}$. Find the area of the hexagon in two different ways.

21. The circumradius of a regular pentagon is r and the length of each of its sides is s. Find the area of the pentagon in terms of r and s.
22. In Exercise 21, if the circumradius of a second regular pentagon is $2r$, find the area of the second pentagon in terms of r and s.
23. Prove Theorem 14.10.
24. Prove Theorem 14.11.
25. Write reasons for statements (2), (3), (4), (6), (7), (8), and (10) in the proof of Theorem 14.4.
26. What is the 1000th term of the sequence $\left\{\frac{1}{n}\right\}$? What is the millionth term? Which of these two terms is closer to zero? Could we make $\frac{1}{n}$ as close to zero as we desire by choosing n large enough?
27. Find the first five terms, the 100th term, and the 1000th term of the sequence $\left\{\frac{2n}{n+3}\right\}$.
28. What is the millionth term of the sequence $\left\{\frac{2n}{n+3}\right\}$ of Exercise 27? Find the limit (if any) of the sequence.

In Exercises 29–37, a formula for x_n is given. Find x_1, x_2, x_3, and x_{10}. What is the limit (if any) of the sequence $\{x_n\}$?

29. $x_n = 1 + \frac{1}{2n}$
30. $x_n = 1 - \frac{1}{2n}$
31. $x_n = n + \frac{1}{2n} = \frac{2n^2+1}{2n}$
32. $x_n = 2^n$
33. $x_n = \frac{1}{2^n}$
34. $x_n = \frac{n+100}{2n} = \frac{1}{2} + \frac{50}{n}$
35. $x_n = \frac{2n+100}{n}$
36. $x_n = \frac{n^2+1}{2n^2}$
37. $x_n = \frac{n^2+1}{2n}$

38. Let a circle be given. For each n, $n \geq 3$, let a regular polygon of n sides be inscribed in the circle. Let p_n be the perimeter of the n-gon. Does $p_3, p_4, p_5, \ldots, p_n, \ldots$ define a sequence? Do you think $\{p_n\}$ has a limit? If so, how would you describe the limit?
39. Let a circle be given. For each n, $n \geq 3$, let a regular polygon of n sides be inscribed in the circle. Let S_n be the area of the polygon. Does $S_3, S_4, S_5, \ldots, S_n, \ldots$ define a sequence? Do you think $\{S_n\}$ has a limit? If so, how would you describe the limit?
40. CHALLENGE PROBLEM. The sequence $\left\{\frac{n^2+7n+12}{2n(n+4)}\right\}$ has a limit. What is it?

14.4 THE CIRCUMFERENCE OF A CIRCLE

Thus far, in our formal geometry, length has not been defined for anything except segments. If a path from one point to a second point is such that every point of the path lies on the same segment, then the length of that path is, of course, the length of the segment joining the two points. However, if the path is a circular arc, what is the distance from the first point to the second point along the circular arc, that is, what is the length of the arc? The degree measure of the arc would not be a satisfactory way of describing its length since it is possible for two arcs to have the same degree measure and to have different lengths as suggested in Figure 14-16.

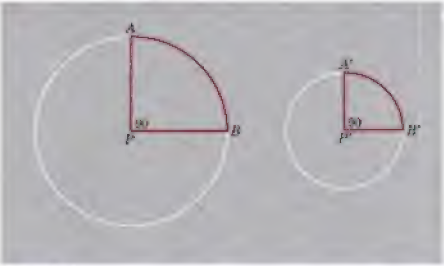


Figure 14-16

Each of the arcs $\widehat{AB}$ and $\widehat{A'B'}$ shown in the figure has a degree measure of 90. But it certainly seems reasonable to think of the arc $\widehat{AB}$ as having a greater length than the arc $\widehat{A'B'}$. We start by explaining what we mean by the length of circular arcs and then deriving ways of finding such lengths. We first proceed informally, referring to the physical world.

You may have been asked, at one time or another in your study of informal geometry, to wrap a string around a circular object, then pull it out straight, and measure its length. By doing this you are able to arrive at an approximation to the distance around the object. However, we cannot describe the process of wrapping a string around a circular object in our formal geometry.

We call the "distance around a circle" the circumference of the circle and denote it by C. A more sophisticated approach to finding an approximation to the circumference of a circle is in terms of the perimeters of regular polygons inscribed in the circle. There was no difficulty

in defining the perimeter of a polygon because the sides of a polygon are segments and each of these segments has a length. But a circle contains no segment of a line, and thus we cannot define its circumference (or perimeter) so simply. It seems reasonable to suppose that if we want to find the circumference of a circle approximately, we can do it by inscribing in the circle a regular polygon with a large number of sides and then measuring or computing the perimeter of the polygon.

Given a circle, let p_n be the perimeter of a regular polygon of n sides inscribed in the circle. Then as n gets larger and larger, the number p_n increases, that is, each term in the sequence $\{p_n\}$ is greater than the preceding term. For example, we can inscribe a square in a circle. By bisecting the central angles of the square we obtain a regular octagon inscribed in the same circle as shown in Figure 14-17. Using the Triangle Inequality Theorem, it is easy to show that the perimeter p_8 of the regular octagon is greater than the perimeter p_4 of the square. If we continue to bisect the central angles, we obtain a regular 16-gon, a regular 32-gon, and so on. The perimeters of these regular polygons $p_4, p_8, p_{16}, p_{32}, \ldots$ form an infinite sequence of numbers, and each term in the sequence is greater than the preceding term.

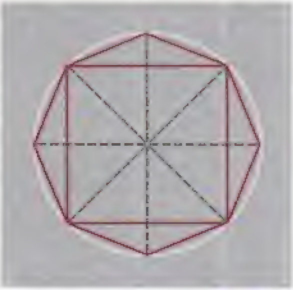


Figure 14-17

It is shown in the calculus that if a sequence of numbers is increasing (that is, if each term in the sequence is greater than the preceding term), and if the sequence is bounded (that is, if there is a number that is equal to or greater than any term in the sequence), then the sequence has a limit. The squence $p_4, p_8, p_{16}, p_{32}, \ldots$ described above in connection with the circle shown in Figure 14-17 can be shown to be bounded. In fact, it can be shown that any square that circumscribes the given circle has a perimeter that is greater than any of the terms in the sequence.

Let a sequence of perimeters of regular polygons of n sides inscribed in a given circle be denoted by $\{p_n\}$. Let the limit of this sequence be denoted by C, that is, $\lim p_n = C$. We are now ready for our formal definition of circumference as the limit of the p_n.

> ***Definition 14.6*** The circumference of a circle is the limit of the sequence of perimeters p_n of the inscribed regular polygons (that is, $C = \lim p_n$).

Note that we are forced to use limits in defining the circumference of a circle. In order to derive the formulas for the circumference and for the area of a circle, we need a theorem about limits which we state here without proof. We can treat the theorem as a postulate, although it is not a postulate concerning our formal geometry. Rather, it is a postulate concerning the real number system.

THEOREM 14.12 Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers with nth terms x_n and y_n, respectively.

1. If the limit of x_n is L_1 and the limit of y_n is L_2, then the sequence whose nth term is $x_n y_n$ has a limit, and
$$\lim x_n y_n = L_1 \cdot L_2.$$
2. If the limit of x_n is L_1 and the limit of y_n is $L_2 \neq 0$, then the sequence whose nth term is $\frac{x_n}{y_n}$ has a limit and $\lim \frac{x_n}{y_n} = \frac{L_1}{L_2}$.
3. If $x_n = y_n$ for every positive integer $n \geq 1$ and if $\{x_n\}$ and $\{y_n\}$ each has a limit, then $\lim x_n = \lim y_n$.
4. If k is a real number and if $x_n = k$, for every $n \geq 1$, then $\lim x_n = k$.

Example 1 If $\lim \frac{2n-3}{n} = 2$ and $\lim \frac{3n}{n+2} = 3$, then

$$\lim \left(\frac{2n-3}{n} \cdot \frac{3n}{n+2}\right) = 2 \cdot 3 = 6$$

and

$$\lim \left(\frac{2n-3}{n} \div \frac{3n}{n+2}\right) = 2 \div 3 = \tfrac{2}{3}.$$

Example 2 Find $\lim x_n$ if $x_n = 2$ for every $n \geq 1$.

Solution: If $x_n = 2$ for every $n \geq 1$, then $\{x_n\}$ is a sequence of numbers whose every term is 2; that is, $\{x_n\} = 2, 2, 2, \ldots$. By part 4 of Theorem 14.12, $\lim x_n = 2$.

Before we derive a formula for the circumference of a circle, we need to know that the number $\frac{C}{d}$, where C is the circumference of a circle and d is its diameter, is the same for all circles. That the number $\frac{C}{d}$ is the same for all circles is a corollary of our next theorem.

THEOREM 14.13 If C and C' are the circumferences of any two circles with diameters d and d', respectively, then

$$(C, d) \underset{p}{=} (C', d').$$

Proof: Let K, K' be any two circles with circumferences C, C' and radii r, r', respectively, as shown in Figure 14-18.

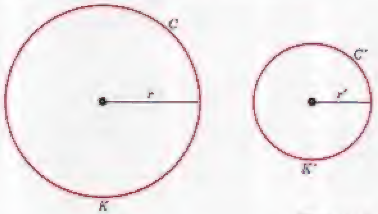


Figure 14-18

Let $\{p_n\}$ be the sequence of perimeters of regular polygons of n sides inscribed in circle K with radius r and let $\{p'_n\}$ be the sequence of perimeters of regular polygons of n sides inscribed in circle K' with radius r'. By Theorem 14.9,

$$(p_n, r) \underset{p}{=} (p'_n, r').$$

It follows that

$$(p_n, 2r) \underset{p}{=} (p'_n, 2r');$$

hence that

$$(p_n, d) \underset{p}{=} (p'_n, d'), \tag{1}$$

where d and d' are the diameters of the circles K and K', respectively.

By Definition 14.6,

$$\lim p_n = C$$

and

$$\lim p'_n = C'.$$

It follows from Equation (1) that

$$\frac{p_n}{p'_n} = \frac{d}{d'}. \tag{2}$$

By part 3 of Theorem 14.12, we get from Equation (2)

$$\lim \frac{p_n}{p'_n} = \lim \frac{d}{d'}. \tag{3}$$

By part 1 of Theorem 14.12,

$$\lim \frac{p_n}{p'_n} = \frac{C}{C'},$$

and, since d and d' are the same for all n, it follows from part 4 of Theorem 14.12 that

$$\lim \frac{d}{d'} = \frac{d}{d'}.$$

Substituting these last two results in Equation (3), we have that

$$\frac{C}{C'} = \frac{d}{d'}$$

or that

$$(C, d) \underset{p}{=} (C', d'),$$

and the proof is complete.

COROLLARY 14.13.1 If C and d are the circumference and diameter, respectively, of a circle, then the number $\dfrac{C}{d}$ is the same for all circles.

Proof: Let K, K' be any two circles with circumferences C, C' and diameters d, d', respectively. By Theorem 14.13, we know that

$$(C, d) \underset{p}{=} (C', d').$$

By alternation, we get

$$(C, C') \underset{p}{=} (d, d').$$

Therefore $\frac{C}{d} = \frac{C'}{d'}$. This proves that the number $\frac{C}{d}$ is the same for any two circles.

> ***Definition 14.7*** If C is the circumference of a circle and d is its diameter, then the number $\frac{C}{d}$, which is the same for all circles, is denoted by the Greek letter π.

It follows from Corollary 14.13.1 and Definition 14.7 that

$$C = \pi d$$

for any circle with circumference C and diameter d. Since $d = 2r$, where r is the radius of the circle with diameter d, we have

$$C = 2\pi r$$

as another formula for the circumference of a circle.

It has been proved that π is not a rational number, that is, π cannot be represented by $\frac{a}{b}$ where a and b are integers with $b \neq 0$. However, we can approximate π as closely as we desire by means of rational numbers. Some of the more common rational number approximations to π are 3.14, $\frac{22}{7}$, and 3.1416. It has been shown that π, to ten decimal places, is 3.1415926535.

EXERCISES 14.4

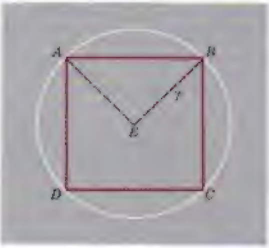
1. Show that $\frac{355}{113}$ is a closer approximation to π than is $\frac{22}{7}$.
2. Show that $\frac{22}{7} \simeq 3.14$ to the nearest hundredth. (We read "$\simeq$" as "is approximately equal to.")

In Exercises 3–10, C, r, and d represent the circumference, radius, and diameter, respectively, of a given circle. Express answers in exact form, in terms of π if necessary.

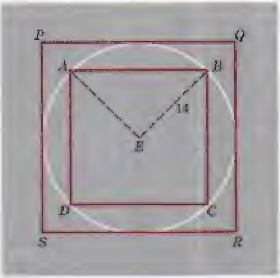
3. If $r = 7$, find C.
4. If $C = 63\pi$, find r.
5. If $d = 12.5$, find C.

6. If $r = 36.4$ in., find C.
7. If $C = 36.4$ in., find r.
8. If $C = 2\pi$, find r.
9. If $r = 3\pi$, find C.
10. If $C = 14.26\pi$, find d.

11. In Exercise 6, use $\pi = \frac{22}{7}$ and find C to the nearest inch.
12. In Exercise 6, use $\pi = 3.14$ and find C to the nearest inch. Compare your answer with that of Exercise 11.
13. In Exercise 7, use $\pi = \frac{22}{7}$ and find r to the nearest hundredth of an inch.
14. In Exercise 7, use $\pi = 3.14$ and find r to the nearest hundredth of an inch. Compare your answer with that of Exercise 13.
15. Prove that the circumferences of two circles are proportional to their radii.
16. The circumference of one circle is two-thirds the circumference of a second circle. What is the ratio of the radius of the first circle to the radius of the second circle?
17. Two circles have radii of 15 and 25. What is the ratio of the circumference of the smaller circle to the circumference of the larger circle? What is the ratio of the diameters of the two circles?
18. A tire on a wheel of a car has a diameter of 28 in. If the wheel makes 12 revolutions per second, what is the approximate speed of the car in miles per hour? (Use $\pi = \frac{22}{7}$.)
19. Given the same car as in Exercise 18, how many revolutions per second would the wheel make if the car were traveling 50 miles per hour? (Use $\pi = \frac{22}{7}$.)
20. A square $ABCD$ is inscribed in a circle with center E as shown in the figure. If the radius of the circle is 7, find the perimeter of the square. (*Hint:* Show that $\triangle EAB$ is an isosceles right triangle.)



21. A square $ABCD$ is inscribed in a circle with center E. Another square $PQRS$ is circumscribed about the same circle as shown in the figure. If the radius of the given circle is 14 in., find the following:
    (a) The perimeter of each square to the nearest inch.
    (b) The area of each square to the nearest square inch.
    (c) The circumference of the circle to the nearest inch. (Use $\pi = \frac{22}{7}$.)



22. The perimeter of a square is equal to the circumference of a circle. If the radius of the circle is 1, find the area of the square in terms of π.

23. Show that if the radius of a circle is increased by 1 unit, the circumference of the circle is increased by 2π units.

24. Show that if the circumference of a circle is increased by 1 unit, the radius of the circle is increased by $\dfrac{1}{2\pi}$ units.

25. Assume that the surface of the earth is a sphere. Imagine that a steel band is fit snugly around the equator (a great circle of the earth). Suppose that one foot is added to the length of the band and that it is raised a uniform amount all the way around the earth. To the nearest inch, how many inches will the new band be above the earth? (See Exercise 24.)

26. Assume that the surface of an orange is a sphere. Imagine that a steel band is placed around a great circle of the orange so that it just fits. Suppose that one foot is added to the length of the band and that it is raised a uniform amount all the way around the orange. To the nearest inch, how many inches will the new band be above the orange? (See Exercises 24 and 25.)

Exercises 27–36 refer to Figure 14-19. $ABCD$ is a square inscribed in the circle with center P. The bisector rays of the four central angles of square $ABCD$ intersect the circle in points E, F, G, H, respectively. The polygon $AEBFCGDH$ is a regular octagon inscribed in the circle, and $TQRS$ is a square circumscribed about the circle with points of tangency E, F, G, H. The radius of the circle is 8. If an answer to an exercise is an irrational number, give a rational approximation to the nearest tenth.

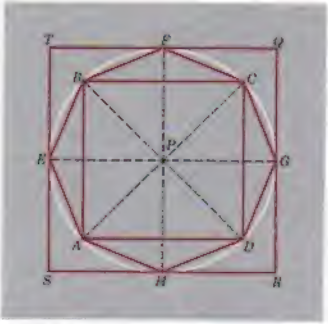


Figure 14-19

27. Find the length s of a side of square $ABCD$.
28. Find the perimeter of square $ABCD$.
29. Find the area of square $ABCD$.
30. Find the length s' of a side of the regular octagon $AEBFCGDH$.
31. Find the perimeter of the octagon.
32. Find the area of the octagon.
33. Find the length s'' of a side of the square $TQRS$.
34. Find the perimeter of square $TQRS$.
35. Find the area of square $TQRS$.
36. Find the circumference of the circle.

37. Imagine an infinite number of regular polygons inscribed in a circle with radius r. The first polygon has 3 sides, the second polygon has 4 sides, the third has 5 sides, and so on. For every n, $n \geq 3$, let p_n, a_n, s_n, and S_n be the perimeter, inradius, side length, and area, respectively, of the regular inscribed polygon with n sides. What is $\lim p_n$? $\lim a_n$? $\lim s_n$? $\lim S_n$?

14.5 AREAS OF CIRCLES; ARC LENGTH; SECTOR OF A CIRCLE

In Chapter 9 we considered areas of polygonal regions. Recall that a polygonal region is the union of a polygon and its interior. In this section we are concerned with areas of circular regions. We make the following definition.

> *Definition 14.8* A **circular region** is the union of a circle and its interior.

As we said at the beginning of the chapter, "the area of a circle" is an abbreviation for "the area of a circular region," or for "the area enclosed by a circle."

We now proceed to get a formula for the area of a circle. We already have a formula for the area of a regular polygon of n sides which is

$$S_n = \tfrac{1}{2}a_n p_n,$$

where a_n is the inradius of the polygon, p_n is the perimeter of the polygon, and S_n is the area of the polygon.

If P_n is a regular polygon of n sides inscribed in a circle with center Q and radius r, as shown in Figure 14-20 (in the figure, $n = 8$), we observe that the area of the inscribed polygon is less than the area of the circle.

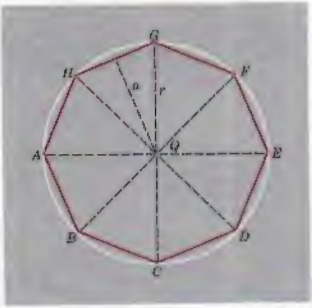


Figure 14-20

For the expression

$$S_n = \tfrac{1}{2}a_n p_n,$$

where $n = 3, 4, 5, \ldots$, there are three sequences involved:

$$\{S_n\}, \{a_n\}, \text{ and } \{p_n\}.$$

Let us consider each of these sequences separately.

1. The sequence $\{S_n\}$. As noted above, for each n, S_n is always less than S, the area of the circle. The difference between S_n and S can be made arbitrarily small by taking n large enough. It seems reasonable then to say that

$$\lim S_n = S.$$

Definiton 14.9 The area of a circle is the limit of the sequence of areas of the inscribed regular polygons.

Thus $\lim S_n = S$, by definition.

2. The sequence $\{a_n\}$. Since the length of a leg of a right triangle is less than the length of the hypotenuse of the right triangle, we observe that the inradius a_n is always less than the radius r of the circle for each particular value of n. However, the difference between a_n and r can be made arbitrarily small by choosing n large enough. Thus it seems reasonable to say that

$$\lim a_n = r,$$

and we accept this fact without proof.

3. The sequence $\{p_n\}$. By the definition of the circumference C of a circle,

$$\lim p_n = C.$$

Now we have

$$S_n = \tfrac{1}{2}a_n p_n.$$

By parts 3 and 4 of Theorem 14.12, we get

(1) $$\lim S_n = \tfrac{1}{2}\lim a_n p_n.$$

By definition 14.9,

(2) $$\lim S_n = S.$$

By part 1 of Theorem 14.12, we get

(3) $$\lim a_n p_n = \lim a_n \cdot \lim p_n$$

But

(4) $$\lim a_n = r$$

and

(5) $$\lim p_n = C.$$

Substituting the results of (2), (3), (4), and (5) in Equation (1), we obtain

(6) $$S = \tfrac{1}{2}rC$$

as a formula for the area of a circle. Since

$$C = 2\pi r,$$

we get, by substitution into Equation (6),

$$S = \tfrac{1}{2}r \cdot 2\pi r$$

or

$$S = \pi r^2$$

as a formula for the area of a circle—a formula that should be familiar to all of you.

We now state this result formally as a theorem.

THEOREM 14.14 The area S of a circle with radius r is πr^2, that is,

$$S = \pi r^2.$$

COROLLARY 14.14.1 The areas of two circles are proportional to the squares of their radii.

Proof: Assigned as an exercise.

We have defined the circumference of a circle to be the limit of the sequence of the perimeters of the inscribed regular polygons. We now proceed to define the length of an arc of a circle as a certain limit.

Consider an arc $\widehat{AB}$ of a circle with center V as shown in Figure 14-21. For $k \geq 2$, let $P_1, P_2, P_3, \ldots, P_{k-1}$ be points on $\widehat{AB}$ such that each of the k angles

$$\angle AVP_1,\ \angle P_1VP_2,\ \angle P_2VP_3, \ldots, \angle P_{k-1}VB$$

has a measure of $\frac{1}{k} m\widehat{AB}$. (In Figure 14-21, $k = 4$.)

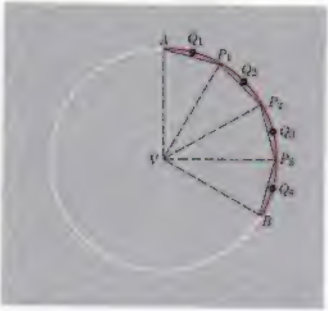


Figure 14-21

Let

$$A_k = AP_1 + P_1P_2 + P_2P_3 + \cdots + P_{k-1}B.$$

Thus, in Figure 14-21,

$$A_4 = AP_1 + P_1P_2 + P_2P_3 + P_3B.$$

If we bisect each of the central angles

$$\angle AVP_1,\ \angle P_1VP_2, \ldots, \angle P_{k-1}VB,$$

we obtain k more points $Q_1, Q_2, \ldots, Q_k$ on $\widehat{AB}$ such that

$$A_{2k} = AQ_1 + Q_1P_1 + P_1Q_2 + Q_2P_2 + \cdots + Q_kB.$$

In Figure 14-21,

$$A_8 = (AQ_1 + Q_1P_1) + (P_1Q_2 + Q_2P_2) + (P_2Q_3 + Q_3P_3) + (P_3Q_4 + Q_4B).$$

Now

$$\begin{aligned} AQ_1 + Q_1P_1 &> AP_1, \\ P_1Q_2 + Q_2P_2 &> P_1P_2, \\ P_2Q_3 + Q_3P_3 &> P_2P_3, \\ P_3Q_4 + Q_4B &> P_3B. \end{aligned} \qquad \text{Why?}$$

It follows that $A_8 > A_4$. If we continue to bisect the central angles with vertex V, we obtain a sequence of sums

$$A_4, A_8, A_{16}, \ldots, A_n$$

(where $n = k, 2k, 4k, 8k, \ldots$) in which each term in the sequence is greater than the preceding term. Also, this is a bounded sequence. Each number in it is less than the circumference of the circle. Therefore the sequence $\{A_n\}$ has a limit and we define that limit to be the length of arc $\widehat{AB}$.

Definition 14.10 The length of arc $\widehat{AB}$ (denoted by $l\widehat{AB}$) is the limit of $\{A_n\}$ where

$$A_n = AP_1 + P_1P_2 + \cdots + P_{n-1}B$$

and where $P_1, P_2, \ldots, P_{n-1}$ are $n - 1$ distinct points of $\widehat{AB}$ subtending congruent angles at the center V of the circle containing $\widehat{AB}$.

We now have two types of measure for arcs of a circle: their degree measures and their lengths.

Definition 14.11 In the same circle or in congruent circles, two arcs are congruent if and only if they have the same length.

Thus, if arcs $\widehat{AB}$ and $\widehat{A'B'}$ are arcs of congruent circles and if $\widehat{AB} \cong \widehat{A'B'}$, we have

(1) $$l\widehat{AB} = l\widehat{A'B'}.$$

Also, from Definition 13.17, if $\widehat{AB} \cong \widehat{A'B'}$, we have

(2) $$m\widehat{AB} = m\widehat{A'B'}.$$

Hence, if $\widehat{AB}$ and $\widehat{A'B'}$ are congruent arcs of congruent circles, then

$$(l\widehat{AB}, l\widehat{A'B'}) \underset{p}{=} (m\widehat{AB}, m\widehat{A'B'}).$$

In other words, lengths of congruent arcs of congruent circles are proportional to their degree measures. This is a trivial assertion, of course. Suppose, for example, that $l\widehat{AB} = 100$ and $m\widehat{AB} = 20$. What we are asserting is that

$$(100, 100) \underset{p}{=} (20, 20).$$

It is also true (not trivial and we shall not prove it here) that the lengths of any two arcs of congruent circles are proportional to their degree measures. We state this as our next theorem.

THEOREM 14.15 The lengths of arcs of congruent circles are proportional to their degree measures.

Thus, if K and K' (as shown in Figure 14-22) are congruent circles and if $\widehat{AB}$ is an arc of K and $\widehat{A'B'}$ is an arc of K', we have

$$(l\widehat{AB}, l\widehat{A'B'}) \underset{p}{=} (m\widehat{AB}, m\widehat{A'B'}).$$

Suppose that $\widehat{A'B'}$ in Figure 14-22 is a semicircle. Then

$$m\widehat{A'B'} = 180 \qquad \text{and} \qquad l\widehat{A'B'} = \pi r. \qquad \text{Why?}$$

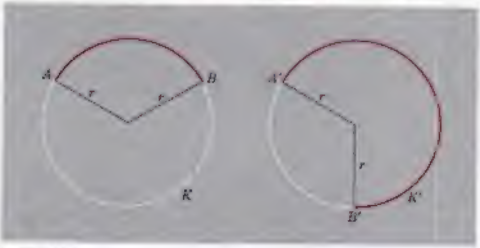


Figure 14-22

Let $l\widehat{AB}$ be denoted by L and $m\widehat{AB}$ be denoted by M. It follows from Theorem 14.15 that

$$(L, \pi r) \underset{p}{=} (M, 180).$$

Therefore

$$180L = \pi r M$$

and

$$L = \left(\frac{M}{180}\right)\pi r.$$

We have proved the following theorem.

THEOREM 14.16 The length L of an arc of degree measure M contained in a circle with radius r is $\left(\frac{M}{180}\right)\pi r$, that is,

$$L = \left(\frac{M}{180}\right)\pi r.$$

Example 1 Find the length of an arc of a circle with radius 12 if the degree measure of the arc is 60.

Solution: Use the formula of Theorem 14.16 with $M = 60$ and $r = 12$. Therefore

$$L = \left(\frac{M}{180}\right)\pi r = \left(\frac{60}{180}\right)\pi \cdot 12 = 4\pi.$$

Example 2 Find the degree measure of an arc of a circle with radius 2 if the length of the arc is $\frac{4\pi}{3}$.

Solution: Solving the formula given in Theorem 14.16 for M, we obtain

$$M = \left(\frac{180}{\pi r}\right)L. \qquad \text{(Show this.)}$$

We are given that $L = \frac{4\pi}{3}$ and that $r = 2$. Therefore

$$M = \frac{180}{\pi \cdot 2} \cdot \frac{4\pi}{3} = 120.$$

Note that if $M = 360$ in the formula of Theorem 14.16, we get

$$L = \left(\frac{360}{180}\right)\pi r = 2\pi r.$$

Thus L equals the circumference of the circle as it should.

Figure 14-23 shows a portion R of a circular region which is bounded by two radii, $\overline{PA}$, $\overline{PB}$, and an arc $\widehat{AB}$ of the circle. We call R a *sector* of the circle. A more precise definition follows.

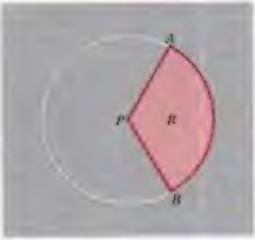


Figure 14-23

Definition 14.12 (See Figure 14-24.) Given a circle of radius r with center P and an arc $\widehat{AB}$ of this circle, the union of all segments $\overline{PQ}$ such that Q is a point on arc $\widehat{AB}$ is called a **sector**. We call $\widehat{AB}$ the **arc of the sector** and we call r the **radius of the sector.**

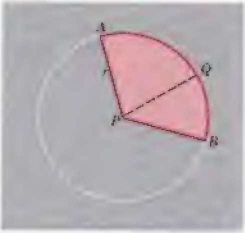


Figure 14-24

Suppose that we are given a sector of a circle with radius r and center V as shown in Figure 14-25. Let $\widehat{AB}$ be the arc of the sector. For $n \geq 2$, let $P_1, P_2, \ldots, P_{n-1}$ be $n - 1$ points on $\widehat{AB}$ such that each of the n angles

$$\angle AVP_1,\ \angle P_1VP_2, \ldots,\ \angle P_{n-1}VB$$

has a measure of $\frac{1}{n} \cdot m\widehat{AB}$. (In Figure 14-25, $n = 4$.)

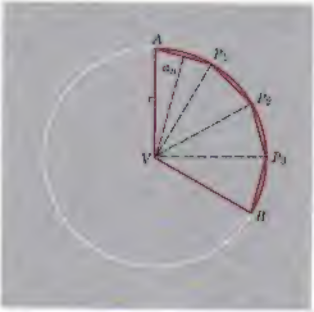


Figure 14-25

Let

(1) $$A_n = AP_1 + P_1P_2 + \cdots + P_{n-1}B.$$

Let S_n be the area of the polygonal region $VAP_1P_2 \ldots P_{n-1}B$; then

(2) $$S_n = \tfrac{1}{2}a_n(AP_1) + \tfrac{1}{2}a_n(P_1P_2) + \cdots + \tfrac{1}{2}a_n(P_{n-1}B),$$

where a_n is the altitude to each of the bases $\overline{AP_1}, \overline{P_1P_2}, \ldots, \overline{P_{n-1}B}$ of triangles $\triangle AVP_1, \triangle P_1VP_2, \ldots, \triangle P_{n-1}VB$, respectively. From Equation (2) we get

(3) $$S_n = \tfrac{1}{2}a_n(AP_1 + P_1P_2 + \cdots + P_{n-1}B);$$

hence

(4) $$S_n = \tfrac{1}{2}a_nA_n,$$

by substitution from Equation (1) into Equation (3).

We can make the difference between S_n and S (the area of the sector) arbitrarily small by choosing n large enough. Similarly, the difference between a_n and r can be made arbitrarily small by choosing n large enough. It seems reasonable, then, that

$$\lim S_n = S \qquad \text{and} \qquad \lim a_n = r.$$

By Definition 14.10, $\lim A_n = L$, the length of arc $\widehat{AB}$. By applying parts 3 and 4 of Theorem 14.12 to Equation (4), we obtain

(5) $$\lim S_n = \tfrac{1}{2}\lim a_nA_n.$$

From equation (5), it follows that $S = \frac{1}{2}rL$ is a formula for the area of a sector with radius r and arc length L. We state this result as our next theorem.

THEOREM 14.17 The area S of a sector is one-half the product of its radius r and the length L of its arc, that is,

$$S = \tfrac{1}{2}rL.$$

If we combine the results of Theorems 14.16 and 14.17, we get the following theorem.

THEOREM 14.18 If M is the degree measure of the arc of a sector with radius r, then the area S of the sector is $\left(\frac{M}{360}\right)\pi r^2$, that is,

$$S = \left(\frac{M}{360}\right)\pi r^2.$$

Proof: Assigned as an exercise.

Example 3 Find the area of a sector with radius 10 if the degree measure of the arc of the sector is 72.

Solution: If we use the formula of Theorem 14.18 with $r = 10$ and $M = 72$, we obtain

$$S = \left(\frac{M}{360}\right)\pi r^2,$$
$$S = \left(\frac{72}{360}\right)\cdot \pi \cdot (10)^2 = 20\pi.$$

Note that, in the formula of Theorem 14.17, if L is the circumference C of the circle, then

$$L = 2\pi r$$

and

$$S = \tfrac{1}{2}r \cdot 2\pi r = \pi r^2,$$

as it should for a circle. Similarly, in the formula of Theorem 14.18, if the given arc is the circumference of the circle, then $M = 360$ and

$$S = (\tfrac{360}{360})\pi r^2 = \pi r^2,$$

as it should for a circle.

EXERCISES 14.5

In working the exercises of this set, do not use an approximation for π unless instructed to do so.

1. Prove Corollary 14.14.1. Let S_1 and S_2 be the areas of two circles with radii r_1 and r_2, respectively. Prove that
$$(S_1, S_2) \underset{p}{=} (r_1{}^2, r_2{}^2).$$
2. The area of one circle is $\frac{4}{9}$ times the area of a second circle. What is the ratio of the radius of the first circle to the radius of the second circle?
3. The radii of two circles are r and $2r$. How does the area of the larger circle compare with the area of the smaller circle? How does the circumference of the larger circle compare with the circumference of the smaller circle?
4. Show that if $r < 2$, the area of a circle is less than the circumference of the circle. (Of course, the area units and the length units are in different systems. If the length units are inches and the area units are square inches, and if $r < 2$, then the statement to be proved asserts that the number of square inches in the area is less than the number of inches in the circumference.)

In Exercises 5–11, r is the radius of a circle and S is its area.

5. If $r = 8$, find S.
6. If $S = 108\pi$, find r.
7. If $S = 154$, find r.
8. If $r = 3\pi$, find S.
9. If $r = \sqrt{41}$, find S.
10. If $r = \sqrt{17\pi}$, find S.
11. If $S = 75.36$, find r.

12. In Exercise 7, use $\pi = \frac{22}{7}$ and find r.
13. In Exercise 9, use $\pi = 3.14$ and find S to the nearest tenth.
14. In Exercise 11, use $\pi = 3.14$ and find r to the nearest tenth.

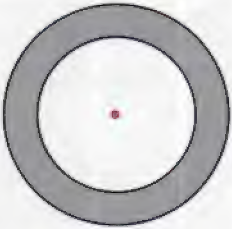
In Exercises 15–18, the area of a circle is given. In each exercise, find the circumference of the given circle.

15. 81π
16. 49π
17. 81
18. 49

In Exercises 19–22, the circumference of a circle is given. In each exercise, find the area of the given circle.

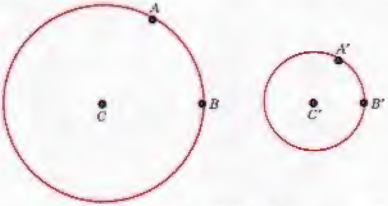
19. 40π
20. 16π
21. 40
22. 16

23. The figure shows two concentric circles with radii 7 cm. and 10 cm. The union of the two circles and the shaded portion between them is sometimes called an **annulus**. Find the area of the annulus.



24. A square and a circle have the same area.
    (a) Find the perimeter of the square in terms of the radius r of the circle.
    (b) Find the circumference of the circle in terms of the length s of the side of the square.
25. Find the areas of the inscribed and circumscribed circles of a square with side length 12.
26. In Exercise 25, find the area of the annulus (see Exercise 23) between the two circles.

27. The radius of the circle with center C is 24. The radius of the circle with center C' is 12. If $m\widehat{AB}=60$ and $m\widehat{A'B'}=60$, show $l\widehat{AB}=2l\widehat{A'B'}$.



28. For the circles of Exercise 27, if $l\widehat{AB} = l\widehat{A'B'}$ and $m\widehat{AB} = 60$, find $m\widehat{A'B'}$.

Given a circle with radius r and an arc $\widehat{AB}$ of the circle, use the given information in Exercises 29–35 to find the indicated measure.

29. If $r = 12$ and $m\widehat{AB} = 45$, find $l\widehat{AB}$.

30. If $r = 1$ and $l\widehat{AB} = \frac{5\pi}{4}$, find $m\widehat{AB}$.

31. If $l\widehat{AB} = \frac{11\pi}{6}$ and $m\widehat{AB} = 60$, find r.

32. If $l\widehat{AB} = \frac{3\pi}{2}$ and $m\widehat{AB} = 270$, find r.

33. If $r = 18$ and $m\widehat{AB} = 240$, find $l\widehat{AB}$.

34. If $r = 9$ and $m\widehat{AB} = 132$, find $l\widehat{AB}$.

35. If $r = 4$ and $l\widehat{AB} = 6\pi$, find $m\widehat{AB}$.

36. Given a circle with radius $r = 1$, find the degree measure of each of the following arcs of the circle.

(a) $\widehat{AB}$ if $l\widehat{AB} = \frac{\pi}{6}$

(b) $\widehat{AC}$ if $l\widehat{AC} = \frac{\pi}{4}$

(c) $\widehat{AD}$ if $l\widehat{AD} = \frac{\pi}{3}$

(d) $\widehat{AE}$ if $l\widehat{AE} = \frac{\pi}{2}$

(e) $\widehat{AF}$ if $l\widehat{AF} = \frac{2\pi}{3}$

(f) $\widehat{AG}$ if $l\widehat{AG} = \frac{3\pi}{4}$

(g) $\widehat{AH}$ if $l\widehat{AH} = \frac{5\pi}{6}$

(h) $\widehat{AI}$ if $l\widehat{AI} = \pi$

(i) $\widehat{AJ}$ if $l\widehat{AJ} = \frac{7\pi}{6}$

(j) $\widehat{AK}$ if $l\widehat{AK} = \frac{5\pi}{4}$

(k) $\widehat{AL}$ if $l\widehat{AL} = \frac{4\pi}{3}$

(l) $\widehat{AM}$ if $l\widehat{AM} = \frac{3\pi}{2}$

(m) $\widehat{AN}$ if $l\widehat{AN} = \frac{5\pi}{3}$

(n) $\widehat{AP}$ if $l\widehat{AP} = \frac{7\pi}{4}$

(o) $\widehat{AQ}$ if $l\widehat{AQ} = \frac{11\pi}{6}$

37. Prove Theorem 14.18. (*Hint:* Combine the results of Theorems 14.16 and 14.17.)
38. The radius of a circle is 12 in. Find the area of a sector with the following arc length:
(a) 4π (b) 2.7π (c) 8π (d) $\frac{\pi}{12}$
39. The radius of a circle is 15 in. Find the area of a sector with an arc whose degree measure is
(a) 60 (b) 144 (c) 1 (d) 330
40. Let a circle with center P and radius r be given. Points A and B are points of the circle such that $m\angle APB = 120$ and the area of sector APB is 12π. Find r and $l\widehat{AB}$.
41. In an xy-plane, let sets C and l be defined as follows:

$$C = \{(x, y) : x^2 + y^2 = 64\},$$
$$l = \{(x, y) : y = x\}.$$

Let P be the point where l intersects C in the first quadrant, let O be the origin of the given xy-plane, and let B be the point on the positive x-axis where C intersects the x-axis. Find the area of the sector POB.

Exercises 42–44 refer to Figure 14-26 which shows a circle with center V and chord $\overline{AB}$. The shaded portion bounded by the chord $\overline{AB}$ and the arc $\widehat{AB}$ is called a **segment of the circle**. Let h be the altitude to $\overline{AB}$ of $\triangle AVB$, let $AB = s$, let $VA = r = VB$, and let $L = l\widehat{AB}$.

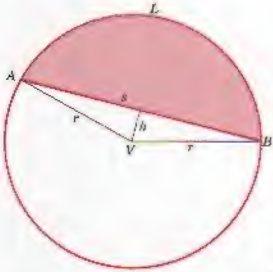
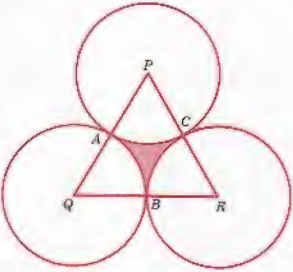


Figure 14-26

42. CHALLENGE PROBLEM. Derive a formula for h in terms of r and s.
43. CHALLENGE PROBLEM. Derive a formula for the area S of the segment of the circle in terms of r, s, and L.
44. See Exercise 43. Find S if $r = 5$, $s = 6$, and $L = \frac{10\pi}{3}$. (Use $\pi = 3.14$ and round your answer to the nearest tenth.)

45. CHALLENGE PROBLEM. The figure shows three congruent circles with centers P, Q, R. Each of the circles is tangent to the other two and the points of tangency are A, B, C.



(a) Show that $\triangle PQR$ is equilateral.
(b) If the radius of each of the circles is 12, find the area S of the shaded region bounded by arcs $\widehat{AB}$, $\widehat{BC}$, and $\widehat{AC}$. (Use $\pi = 3.14$ and round your answer to the nearest tenth.)

CHAPTER SUMMARY

In this chapter we defined the following terms and phrases. Be sure that you know the meaning of each of them.

INTERIOR ANGLE OF A POLYGON
EXTERIOR ANGLE OF A POLYGON
REGULAR POLYGON
CIRCUMSCRIBED CIRCLE (CIRCUMCIRCLE)
INSCRIBED POLYGON
INSCRIBED CIRCLE (INCIRCLE)
CIRCUMSCRIBED POLYGON
CENTER OF REGULAR POLYGON
CIRCUMRADIUS OF REGULAR POLYGON
INRADIUS OF REGULAR POLYGON
CENTRAL ANGLE OF REGULAR POLYGON
CENTRAL TRIANGLE OF REGULAR POLYGON
CIRCUMFERENCE OF CIRCLE
THE NUMBER π
CIRCULAR REGION
AREA OF CIRCLE
LENGTHS OF ARC
SECTOR
ARC OF A SECTOR
RADIUS OF A SECTOR

There were 18 theorems in this chapter, most of which consisted of formulas. In developing many of these formulas, we used the idea of a LIMIT of a sequence of numbers. A complete treatment of limits is too difficult for a first course in geometry. Our goal was to give an intuitive feeling for the concept of a limit and to make the formulas seem plausible. Be sure that you know the following list of formulas and that you know how to apply them.

Sum S of the measures of the interior angles of a convex polygon of n sides:

$$S = (n - 2)180.$$

Measure m of each angle of a regular polygon of n sides:

$$m = \frac{(n - 2)180}{n}.$$

Sum S of the measures of the exterior angles, one at each vertex, of a convex polygon of n sides:

$$S = 360.$$

Measure m of each exterior angle of a regular polygon of n sides:

$$m = \frac{360}{n}.$$

Perimeter of a regular polygon of n sides:

$$p = ns,$$

where p is the perimeter and s is the length of a side of the polygon.

Area of a regular polygon of n sides:

$$S = \tfrac{1}{2}ap,$$

where S is the area, a is the inradius, and p is the perimeter of the polygon.

Circumference of a circle:

$$C = \pi d \text{ or } C = 2\pi r,$$

where C is the circumference, r is the radius, and d is the diameter of the circle.

Area of a circle:

$$S = \pi r^2,$$

where S is the area and r is the radius of the circle.

Length of an arc of a circle:

$$L = \left(\frac{M}{180}\right)\pi r,$$

where L is the length of the arc, M is the degree measure of the arc, and r is the radius of the circle.

Area of a sector:

$$S = \tfrac{1}{2}rL,$$

where S is the area, r is the radius, and L is the length of the arc of the sector.

REVIEW EXERCISES

1. Find the measure of each interior angle of a regular pentagon; of a regular 7-gon.
2. Find the measure of each exterior angle of a regular 15-gon.
3. If the measure of each interior angle of a regular n-gon is 150, find n.
4. If the measure of each exterior angle of a regular n-gon is 40, find n.
5. If the measure of each of the n central angles of a regular n-gon is 20, find n.
6. Using only a pair of compasses for drawing circles and a straightedge for drawing lines, explain how you would construct a regular hexagon; an equilateral triangle; a square.
7. Find the perimeter and the area of a regular hexagon each of whose sides is 20 cm. long.

In Exercises 8–13, the radius of a circle is given. In each exercise, find the area and the circumference of the circle. Use $\pi = 3.14$ and express each answer to the nearest tenth.

8. $r = 13.5$ cm.
9. $r = 1$
10. $r = 6.04$ in.
11. $r = 3.14$ ft.
12. $r = 12$
13. $r = 62.8$

In Exercises 14–19, the circumference C of a circle or the area S of a circle is given. In each exercise, find the radius of the circle. Give an exact answer in each case.

14. $S = 64\pi$
15. $C = 64\pi$
16. $S = 216\pi$ sq. cm.
17. $S = 225$ sq. in.
18. $C = 143\pi$ ft.
19. $C = 72$ cm.

In Exercises 20–24, r is the radius of a circle, L is the length of an arc of the circle, and M is the degree measure of the arc.

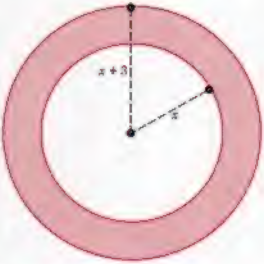
20. If $r = 16$ and $M = 80$, find L.
21. If $r = 4$ and $L = 3\pi$, find M.

22. If $L = \frac{3\pi}{2}$ and $M = 240$, find r.

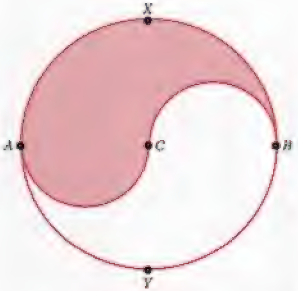
23. If $r = 10$ and $L = \frac{14\pi}{3}$, find M.

24. If $r = 4.5$ and $M = 220$, find L.

25. An annulus (the shaded region shown in the figure) has an inner radius of x and an outer radius of $x + 3$. If its area is 48π, find its inner and outer radii.

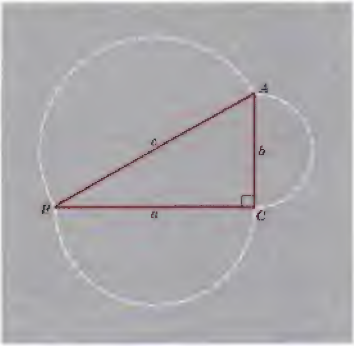


26. In the figure, $\widehat{AC}$, $\widehat{BC}$, $\widehat{AXB}$, and $\widehat{AYB}$ are semicircles, with A, B, C collinear. If $AC = BC = 7$, find the area of the shaded region. $\left(\text{Use } \pi = \frac{22}{7}.\right)$

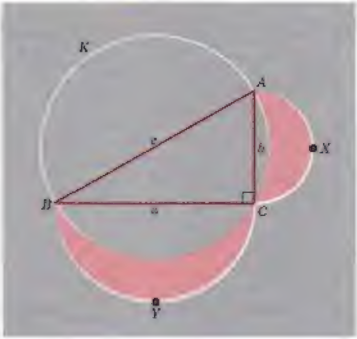


27. Find the area of a sector if the radius is 14 and if the length of the arc of the sector is 11π.

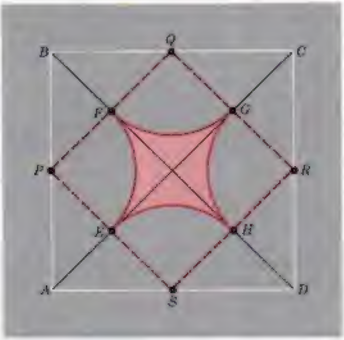
28. In the figure, $\triangle ABC$ is a right triangle with the right angle at C. $\widehat{AB}$, $\widehat{BC}$, and $\widehat{AC}$ are semicircles with diameters $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively. If a, b, c are the lengths of the sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively, show that the area of the semicircle with diameter c is equal to the sum of the areas of the two semicircles with diameters a and b.



29. In the figure, $\triangle ABC$ is a right triangle with the right angle at C. $\overline{AB}$ is a diameter of circle K, and $\widehat{AXC}$ and $\widehat{BYC}$ are semicircles with diameters $\overline{AC}$ and $\overline{BC}$, respectively. Show that the sum of the areas of the two shaded regions is equal to the area of the triangle.



30. CHALLENGE PROBLEM. In the figure, $ABCD$ is a square each of whose sides is 10 cm. long. P, Q, R, S are the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Arcs $\widehat{EF}$, $\widehat{FG}$, $\widehat{GH}$, $\widehat{HE}$ are arcs of circles with centers P, Q, R, S, respectively, and are tangent to the diagonals of square $ABCD$ at points E, F, G, H.



(a) Prove that $PQRS$ is a square.
(b) Find the area of the shaded region bounded by the four arcs $\widehat{EF}$, $\widehat{FG}$, $\widehat{GH}$, and $\widehat{HE}$.

Chapter 15

Van Bucher/Photo Researchers

Areas and Volumes of Solids

15.1 INTRODUCTION

In your study of informal geometry you may have learned formulas for finding surface areas and volumes of some of the familiar solids. In this chapter we review and extend this phase of geometry. We continue our somewhat informal development of geometry from Chapters 13 and 14. A formal development of these formulas belongs properly in a calculus course. The development of this chapter is designed to make the formulas plausible. Emphasis is placed on understanding the formulas and on using them.

In Chapter 13 we studied circles and spheres. A sphere may be thought of as a solid or as a surface. In this book a sphere is a surface, and the union of a sphere and its interior is a spherical region. Informally speaking, the *area* of a sphere is a number that expresses the measure of the sphere, and the *volume* of a sphere is a number that expresses the measure of the associated spherical region.

Although prisms, pyramids, cylinders, and cones may be thought of as either solids or surfaces, in this book we consider them as solids. The

area of a cylinder (more properly, the surface area of a cylinder), for example, is a number that expresses the measure of the surface of the cylinder, and the volume of a cylinder is a number that expresses the measure of the cylinder itself.

15.2 PRISMS

Figure 15-1 shows some diagrams of prisms. We might think of a prism as the solid swept out by a polygonal region moving parallel to itself from one position to another. Each point P in the region moves along a segment $\overline{PP'}$ as suggested in Figure 15-2, and all of these segments are parallel to each other. The *prism* is the union of all such segments. We make these ideas formal with the following definition.

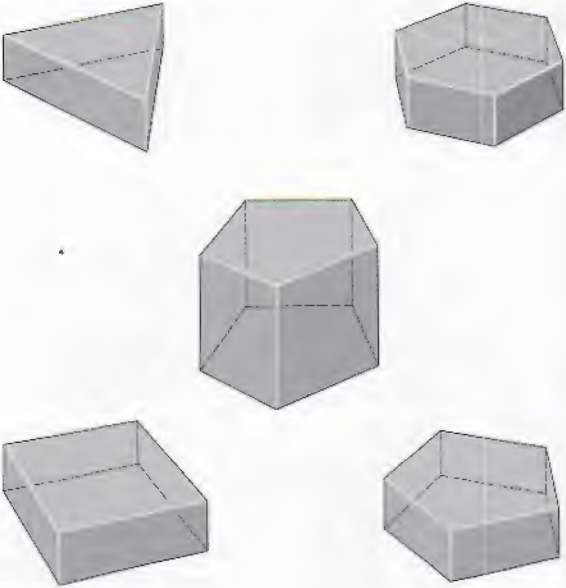


Figure 15-1

Definition 15.1 (See Figure 15-2.) Let α and β be distinct parallel planes. Let Q and Q' be points in α and β, respectively. Let R be a polygonal region in α. For each point P in R let P' be the point in β such that $\overline{PP'} \parallel \overline{QQ'}$. The union of all such segments $\overline{PP'}$ is a prism. If $\overline{QQ'}$ is perpendicular to α and β, the prism is a right prism.

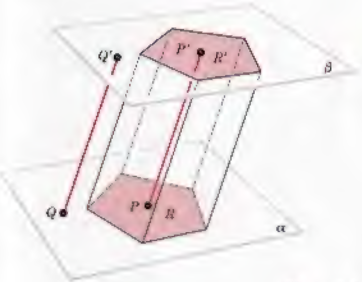


Figure 15-2

In this chapter we shall limit our discussion of prisms to those prisms whose bases are *convex* polygonal regions, that is, regions whose boundaries are convex polygons.

Definition 15.2 (See Figure 15-2.) Let R' be the polygonal region consisting of all the points P' in β. The polygonal regions R and R' are called the bases of the prism. Depending upon the orientation of the prism it is sometimes convenient to call one of the bases the lower base and the other base the upper base. Sometimes we call the lower base simply the base. A segment that is perpendicular to both α and β and with its endpoints in these planes is an altitude of the prism. Sometimes the length of an altitude is called the altitude of the prism.

Prisms are often classified according to their bases. Thus a triangular prism is one whose base is a triangular region; a rectangular prism is one whose base is a rectangular region, and so on.

Figure 15-3 shows a triangular prism. The bases are the triangular regions ABC and $A'B'C'$. The triangular region DEF in the figure, which lies in a plane parallel to the plane of the base, is called a *cross section* of the prism.

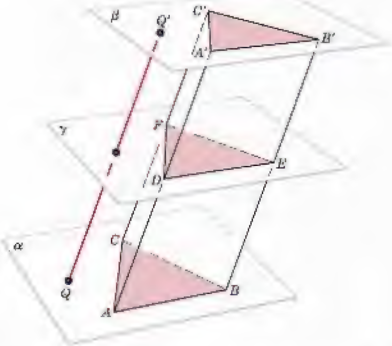


Figure 15-3

> ***Definition 15.3*** If a plane parallel to the plane of the base of a prism intersects the prism, the intersection is called a cross section of the prism.

Is the triangular region ABC a cross section of the prism shown in Figure 15-3? Is the region $A'B'C'$ a cross section of the prism?

We say that a point P of the lower base *corresponds* to a point R of a cross section (other than the lower base) if $\overline{PR} \parallel \overline{QQ'}$. We call $\triangle DEF$ in Figure 15-3 the boundary of the triangular region DEF. Similarly, $\triangle ABC$ is the boundary of the lower base of the prism, $\triangle A'B'C'$ is the boundary of the upper base, and so on.

THEOREM 15.1 The boundary of each cross section of a triangular prism is congruent to the boundary of the base of the prism.

Proof: Let the triangular region ABC in plane α be the lower base of the prism and the triangular region $A'B'C'$ in plane β be the upper base as shown in Figure 15-3. Let γ be a plane parallel to α and intersecting $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ in points D, E, F, respectively. Then, by definition, the region DEF is a cross section of the prism. That the region DEF is a

triangular region can be proved using separation properties. We omit the details here. We shall prove $\triangle DEF \cong \triangle ABC$.

If $\gamma = \alpha$, then $D = A$, $E = B$, $F = C$, and $\triangle DEF = \triangle ABC$. Therefore $\triangle DEF \cong \triangle ABC$. Why? Suppose, then, that $\gamma \neq \alpha$ as suggested in Figure 15-3. By the definition of a prism, there are points Q in α and Q' in β such that $\overline{AA'} \parallel \overline{QQ'}$ and $\overline{BB'} \parallel \overline{QQ'}$. Then $\overline{AA'} \parallel \overline{BB'}$ and $\overline{AD} \parallel \overline{BE}$. Also, by Theorem 8.12, $\overline{AB} \parallel \overline{DE}$. Therefore $ABED$ is a parallelogram and $\overline{DE} \cong \overline{AB}$. In the same way we can prove $\overline{DF} \cong \overline{AC}$ and $\overline{FE} \cong \overline{CB}$. Therefore $\triangle DEF \cong \triangle ABC$ by the S. S. S. Postulate. Since γ is an arbitrary plane parallel to α and intersecting the prism, we have proved that the boundary of each cross section of a triangular prism is congruent to the boundary of its base, and the proof is complete.

Since the upper base of a prism is a cross section of the prism, we have the following corollary.

COROLLARY 15.1.1 The boundaries of the upper and lower bases of a triangular prism are congruent.

THEOREM 15.2 (The Prism Cross Section Theorem) All cross sections of a prism have the same area.

Proof: Let the prism as shown in Figure 15-4 be given. (We have shown a prism whose base is a polygonal region consisting of five sides. The following argument can be modified to apply to a prism whose base is a polygonal region consisting of n sides, where $n \geq 3$.) Let R be the base and let R' be a cross section of the prism. Then R can be divided into nonoverlapping triangular regions t_1, t_2, t_3 as shown in Figure 15-4. Let t_1', t_2', t_3' be the corresponding triangular regions in R'. Then the boundary of t_1 is congruent to the boundary of t_1', the boundary of t_2 is congruent to the boundary of t_2', and the boundary of t_3 is congruent to the boundary of t_3'. Why? The areas of t_1', t_2', t_3' are equal, respectively, to the areas of t_1, t_2, t_3. Why? The area of R' is the sum of the areas of t_1', t_2', t_3', and the area of R is the sum of the areas of t_1, t_2, t_3. Why? Since these two sums are equal, it follows that the area of R' is the same as the area of R. Since R' is an arbitrary cross section of the prism, it follows that all cross sections have the same area and the proof is complete.

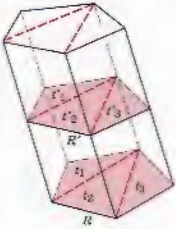


Figure 15-4

COROLLARY 15.2.1. The two bases of a prism have the same area.

Proof: Assigned as an exercise.

Definition 15.4 (See Figure 15-5.) A **lateral edge** of a prism is a segment $\overline{AA'}$, where A is a vertex of the base and A' is the corresponding vertex of the upper base. Given any side of one base of a prism, the **lateral face** of the prism corresponding to that side is the union of all segments $\overline{PP'}$ parallel to a lateral edge and with P on the given side of the base. The **lateral surface** of a prism is the union of its lateral faces. The **total surface** of a prism is the union of its lateral surface and its bases.

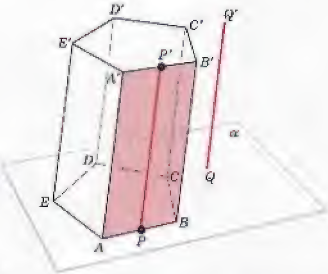


Figure 15-5

For the prism shown in Figure 15-5, $\overline{AA'}$ is a lateral edge and $ABB'A'$ is a lateral face. Name four other lateral edges and four other lateral faces in the figure.

Note that the lateral edges of a prism are parallel to the segment $\overline{QQ'}$ in the definition of a prism. The segment $\overline{AB}$ in Figure 15-5 is not a lateral edge of the prism although $\overline{AB}$ is an edge of the prism. Indeed, it is an edge, or side, of a base of the prism.

THEOREM 15.3 The lateral faces of a prism are parallelogram regions and the lateral faces of a right prism are rectangular regions.

Proof: (See Figure 15-5.) A complete proof involves a discussion of separation properties which we omit here. Suppose that we are given a

prism with points labeled as in Figure 15-5. We shall content ourselves with proving that $ABB'A'$ is a parallelogram and that $ABB'A'$ is a rectangle if the prism is a right prism. Similar arguments could be given for each of the lateral faces of the prism.

By the definition of a prism, there is a segment $\overline{QQ'}$ such that $\overline{AA'} \parallel \overline{QQ'}$ and $\overline{BB'} \parallel \overline{QQ'}$. Therefore $\overline{AA'} \parallel \overline{BB'}$. Why? By the definition of a prism, the planes containing the bases are parallel. Therefore it follows from Theorem 8.12 that $\overline{A'B'} \parallel \overline{AB}$. This proves that $ABB'A'$ is a parallelogram.

If the prism is a right prism, then $\overline{QQ'}$ is perpendicular to plane α. Why? Therefore $\overline{AA'} \perp \alpha$ (Why?) and $\overline{AA'} \perp \overline{AB}$. Why? It follows that $ABB'A'$ is a rectangle.

Definition 15.5 A **parallelepiped** is a prism whose base is a parallelogram region. A **rectangular parallelepiped** is a right prism whose base is a rectangular region. A **cube** is a rectangular parallelepiped all of whose edges are congruent. A **diagonal** of a parallelepiped is a segment joining any two of its vertices which are not contained in the same lateral face or base of the parallelepiped.

Figure 15-6 shows pictures of a parallelepiped, a rectangular parallelepiped, and a cube. In each picture, the segment $\overline{RS}$ is a diagonal of the parallelepiped. How many diagonals does a parallelepiped have?

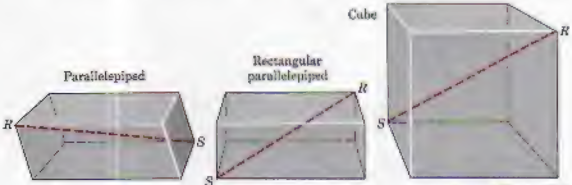


Figure 15-6

Definition 15.6 The **lateral surface area** of a prism is the sum of the areas of its lateral faces. The **total surface area** of a prism is the sum of the lateral surface area and the areas of the two bases.

EXERCISES 15.2

1. Copy and complete:

   All the faces of a parallelepiped (lateral, upper base, and lower base) are [?] regions.

2. Copy and complete:

   All the faces of a [?] parallelepiped are rectangular regions.

3. Copy and complete:

   All the faces of a [?] are square regions.

Exercises 4–16 refer to the prism shown in Figure 15-7. In this figure, P is a point in α and P' is a point in β such that $\overline{PP'} \perp \alpha$. In each exercise, complete the statement.

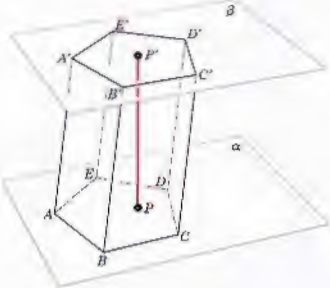


Figure 15-7

4. The region $ABCDE$ is called a [?] of the prism.
5. The region $A'B'C'D'E'$ is called a [?] of the prism.
6. $\overline{BB'}$ is called a (an) [?] of the prism.
7. There are [?] lateral edges in all.
8. Counting the lateral edges and the edges (sides) of the two bases, there are [?] edges in all.
9. The parallelogram region $BCC'B'$ is called a (an) [?] of the prism.
10. There are [?] lateral faces in all.

11. Counting the lateral faces and the two bases, there are $\boxed{?}$ faces in all.
12. A is called a (an) $\boxed{?}$ of the prism.
13. There are $\boxed{?}$ vertices in all.
14. If V is the number of vertices, E is the number of edges, and F is the number of faces, then $V - E + F = \boxed{?}$.
15. $\overline{PP'}$ is called an $\boxed{?}$ of the prism.
16. If $\overline{BB'} \perp \alpha$, then the prism is called a $\boxed{?}$.

17. Prove that the total surface area of a cube is $6e^2$, where e is the length of one of its edges.
18. Find the total surface area of a cube whose edge is 7 cm. in length.
19. Prove that the total surface area of a rectangular parallelepiped is

$$2ab + 2bc + 2ac,$$

where a and b are the dimensions of the base and c is the altitude of the prism. Draw an appropriate figure.

20. Find the total surface area of a rectangular parallelepiped if the dimensions of the base are 4 cm. by 6 cm. and if the altitude of the prism is 8 cm.
21. Given that the pentagonal prism of Figure 15-7 is a right prism, that the lengths of the edges of the base are 3, 7, 4, $9\frac{1}{2}$, and 6, and that the altitude is 8, find the lateral surface area of the prism.
22. If S is the lateral surface area, a is the altitude, and p is the perimeter of the base of a right prism, prove that

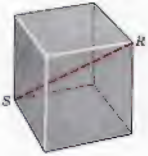
$$S = ap.$$

23. Use the formula in Exercise 22 to find the lateral surface area of the prism in Exercise 21. Does your answer agree with the one obtained in Exercise 21?
24. Find the altitude of a right prism if the lateral surface area is 336 and the perimeter of the base is 28.
25. Find the perimeter of the base of a right prism if the lateral surface area is 351 and the altitude is $13\frac{1}{2}$.
26. If the base of the right prism in Exercise 25 is a square region, find the length of each of its edges.
27. Find the total surface area of a right triangular prism if the boundary of each base is an equilateral triangle whose sides have length 10 and if the altitude of the prism is 12.
28. The area of a cross section of a prism is 32. The lateral surface area is 128. Find the total surface area of the prism.
29. If $\overline{AB}$ and $\overline{PQ}$ are two lateral edges of a prism, prove that $\overline{AB}$ and $\overline{PQ}$ are coplanar.

30. Prove Corollary 15.2.1.

31. Given the rectangular parallelepiped shown in the figure with $AB = 12$, $BC = 6$, and $CD = 8$, find the length of the diagonal $\overline{AD}$. (*Hint:* Draw $\overline{AC}$. What kind of triangle is $\triangle ABC$? What kind of triangle is $\triangle ACD$?)

32. In the figure $\overline{RS}$ is a diagonal of a cube and the length of each edge of the cube is 8. Prove that $RS = 8\sqrt{3}$.



33. Prove that the length of every diagonal of a cube is $e\sqrt{3}$, where e is the length of one of its edges.

34. CHALLENGE PROBLEM. Use the Distance Formula for a three-dimensional coordinate system to prove that the diagonals of a rectangular parallelepiped have equal lengths.

35. CHALLENGE PROBLEM. If h is the altitude of a prism, prove that $h \leq r$, where r is the length of any one of its lateral edges.

15.3 PYRAMIDS

Figure 15-8 on page 673 shows some pictures of pyramids. Compare Figure 15-8 with Figure 15-1. In what respect does a pyramid differ from a prism? How are they similar?

Since a pyramid is similar in many respects to a prism, some terms for parts of a prism are also used for parts of a pyramid. We shall use these terms without giving formal definitions.

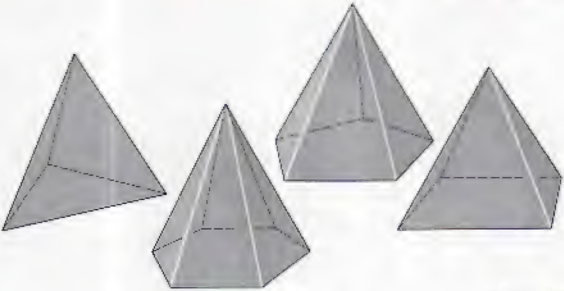


Figure 15-8

Definition 15.7 (See Figure 15-9.) Let R be a polygonal region in a plane α and V a point not in α. For each point P of R there is a segment $\overline{PV}$. The union of all such segments is called a pyramid. The polygonal region R is called the base and V is called the vertex of the pyramid. The distance VT from V to α is the altitude of the pyramid.

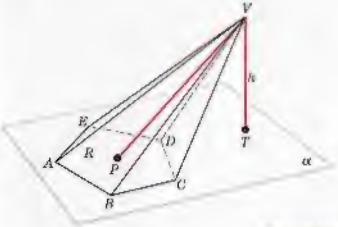


Figure 15-9

For the pyramid in Figure 15-9, $\overline{AV}$ is a lateral edge and the triangular region ABV is a lateral face. Name four other lateral edges and four other lateral faces of the pyramid shown. How many edges (lateral and base) does this pyramid have in all? How many faces in all? How many vertices in all?

A cross section of a pyramid is the intersection of the pyramid with a plane parallel to the base provided the intersection contains more than one point.

THEOREM 15.4 The boundary of each cross section of a triangular pyramid is a triangle similar to the boundary of the base, and the areas of any two cross sections are proportional to the squares of the distances of their planes from the vertex of the pyramid.

Proof: Let the triangular region ABC in plane α be the base of the pyramid as shown in Figure 15-10. Let β be a plane parallel to α and intersecting $\overline{AV}$, $\overline{BV}$, and $\overline{CV}$ in distinct points A', B', and C', respectively. Then the triangular region $A'B'C'$ is a cross section of the pyramid. Let S be the area of $\triangle ABC$, let S' be the area of $\triangle A'B'C'$, let k be the distance from the vertex to the cross section plane, and let h be the altitude of the prism. In Figure 15-10, $k = VP'$ and $h = VP$.

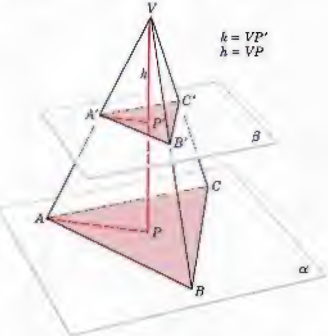


Figure 15-10

To complete the proof of the theorem we shall prove statements 1 and 2.

1. $\triangle A'B'C' \sim \triangle ABC$
2. $(S', S) \underset{p}{=} (k^2, h^2)$

If $\beta = \alpha$, then $A' = A$, $B' = B$, $C' = C$, $P' = P$, and $k = h$. Therefore $\triangle A'B'C' = \triangle ABC$ and hence

$$\triangle A'B'C' \cong \triangle ABC.$$

It follows that $\triangle A'B'C' \sim \triangle ABC$ and that

$$S' = S, \qquad k = h, \qquad \text{and} \qquad (S', S) \underset{p}{=} (k^2, h^2).$$

Suppose, then, that $\beta \neq \alpha$. We have V, A, P, A', P' coplanar (Why?) with $A\text{–}A'\text{–}V$ and $P\text{–}P'\text{–}V$. $\overline{AP} \perp \overline{VP}$ and $\overline{A'P'} \perp \overline{VP'}$. Why? Therefore

$\triangle A'P'V \sim \triangle APV$ by the A.A. Similarity Theorem. ($\angle A'VP' \cong \angle AVP$ and $\angle A'P'V \cong \angle APV$.) Hence

$$(VA', VA) \underset{p}{=} (k, h).$$

In the same way, we can show that $\triangle B'P'V \sim \triangle BPV$ and hence

$$(VB', VB) \underset{p}{=} (k, h).$$

Therefore

$$(VA', VA) \underset{p}{=} (VB', VB)$$

and

$$\triangle A'VB' \sim \triangle AVB$$

by the S. A. S. Similarity Theorem. Therefore

$$(A'B', AB) \underset{p}{=} (VA', VA)$$

and

$$(A'B', AB) \underset{p}{=} (k, h)$$

and

$$A'B' = \frac{k}{h} \cdot AB.$$

In the same way it can be shown that

$$B'C' = \frac{k}{h} \cdot BC$$

and

$$C'A' = \frac{k}{h} \cdot CA.$$

Then

$$(A'B', B'C', C'A') \underset{p}{=} (AB, BC, CA)$$

and it follows that $\triangle A'B'C' \sim \triangle ABC$ by the S. S. S. Similarity Theorem. Recall that in Chapter 10 it was proved (Theorem 10.15) that if two triangles are similar, then their areas are proportional to the square of the lengths of any two corresponding sides. Therefore

$$(S', S) \underset{p}{=} ((A'B')^2, (AB)^2).$$

Since

$$(A'B', AB) \underset{p}{=} (k, h),$$

it follows that

$$((A'B')^2, (AB)^2) \underset{p}{=} (k^2, h^2).$$

Therefore

$$(S', S) \underset{p}{=} (k^2, h^2),$$

and the proof is complete.

Compare our next theorem for pyramids with Theorem 15.2 for prisms.

> *THEOREM 15.5* In any pyramid the areas of any two cross sections are proportional to the squares of the distances of their planes from the vertex of the pyramid.

Proof: Let a pyramid be given as shown in Figure 15-11. (We have shown a pyramid whose base is a polygonal region consisting of five sides. The following argument can be modified to apply to a pyramid whose base is a polygonal region consisting of n sides, where $n \geq 3$.)

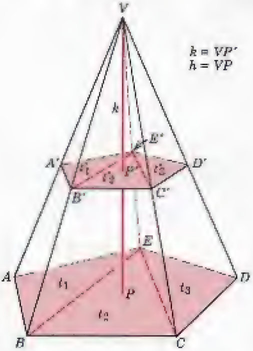


Figure 15-11

Let the polygonal region $ABCDE$ be the base and let $A'B'C'D'E'$ be any cross section of the pyramid. Let S be the area of the region $ABCDE$ and let S' be the area of the region $A'B'C'D'E'$. Let k be the distance from the vertex to the cross section plane and let h be the altitude of the pyramid. The region $ABCDE$ can be divided into nonoverlapping triangular regions t_1, t_2, t_3 as shown in Figure 15-11. Let t'_1, t'_2, t'_3 be the corresponding triangular regions in $A'B'C'D'E'$. Let T_1, T_2, T_3 be the areas of t_1, t_2, t_3, respectively, and let T'_1, T'_2, T'_3 be the areas of t'_1, t'_2, t'_3, respectively.

Then

$$S' = T'_1 + T'_2 + T'_3$$

and

$$S = T_1 + T_2 + T_3.$$

By Theorem 15.4,

$$(T_1', T_1) \underset{p}{=} (k^2, h^2).$$

From the product property of a proportion it follows that

$$T_1'h^2 = T_1k^2.$$

Similarly,

$$T_2'h^2 = T_2k^2$$

and

$$T_3'h^2 = T_3k^2.$$

Then adding and using the Distributive Property, we get

$$(T_1' + T_2' + T_3')h^2 = (T_1 + T_2 + T_3)k^2$$

$$S'h^2 = Sk^2$$

$$(S', S) \underset{p}{=} (k^2, h^2)$$

and this completes the proof.

We now use Theorem 15.5 to prove our next theorem.

THEOREM 15.6 ***(The Pyramid Cross Section Theorem)*** If two pyramids have equal altitudes and if their bases have equal areas, then cross sections equidistant from the vertices have equal areas.

Proof: Let two pyramids be given as shown in Figure 15-12. (We have shown pyramids whose bases are triangular regions. The theorem, however, is not restricted to this case and our proof applies equally as well to pyramids whose bases are polygonal regions with more than 3 sides.)

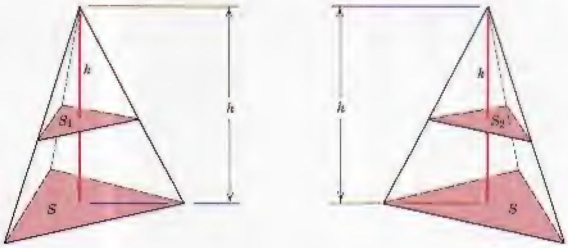


Figure 15-12

Let the base area of each pyramid be S, let h be the altitude of each, and let k be the distance from the vertex to the plane of the cross section of each. Let the areas of the cross sections be S_1 and S_2. We must prove that $S_1 = S_2$. By Theorem 15.5,

$$(S_1, S) \underset{p}{=} (k^2, h^2) \underset{p}{=} (S_2, S).$$

Therefore

$$S_1S = SS_2,$$

$$S_1 = S_2,$$

and the proof is complete.

EXERCISES 15.3

In Exercises 1–13, copy and complete each statement. Exercises 3–13 refer to the pyramid shown in Figure 15-13 in which P is a point in α, the plane of the base, such that $\overline{VP} \perp \alpha$.

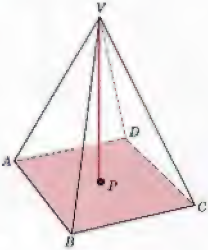


Figure 15-13

1. All the lateral faces of a pyramid are [?] regions.
2. If the boundary of the base of a pyramid is an equilateral triangle, then the boundary of each cross section is [?].
3. The region $ABCD$ is called the [?] of the pyramid.
4. V is called the [?] of the pyramid.
5. $\overline{AV}$ is called a [?] of the pyramid.
6. There are [?] lateral edges in all.
7. Counting the lateral edges and the edges of the bases, there are [?] edges in all.
8. The triangular region ABV is called a [?] of the pyramid.
9. There are [?] lateral faces in all.

10. Counting the base, the total number of faces is $\boxed{?}$.
11. There are $\boxed{?}$ vertices in all.
12. If V is the number of vertices, E the number of edges, and F the number of faces, then $V - E + F = \boxed{?}$.
13. VP is the $\boxed{?}$ of the pyramid.

14. Compare your answer to Exercise 12 with that of Exercise 14 in Exercises 15.2. Are they the same?
15. Compute $V - E + F$ (see Exercise 12) for the pyramid shown in Figure 15-11 and for the prism shown in Figure 15-5. Are your answers the same for both solids? Do you think $V - E + F = 2$ for every prism or pyramid? Try some more examples using figures from this chapter.
16. Recall that the center of a regular polygon is the center of the circumscribed circle. The center of a regular polygonal region is the center of the polygon bounding the region. A pyramid is a **regular pyramid** if and only if its base is a regular polygonal region and the foot of the perpendicular from its vertex to its base is the center of the base. Figure 15-13 shows a regular pyramid whose base is a square. Prove that the boundary of the lateral face AVB is an isosceles triangle. (*Hint:* Draw $\overline{PA}$ and $\overline{PB}$ and prove that $\triangle APV \cong \triangle BPV$ by the S. A. S. Postulate.)
17. Given that the pyramid shown in Figure 15-13 is a regular pyramid whose base is a square region (see Exercise 16), prove that $\triangle AVB \cong \triangle BVC$.
18. Draw a picture of a regular pentagonal pyramid, that is, a regular pyramid whose base is a regular pentagonal region (see Exercise 16). Label the vertex V and the vertices of the base A, B, C, D, E. Pick any two lateral faces and prove that they are congruent isosceles triangles. (*Hint:* Draw the perpendicular from V to the center of the base as in Figure 15-13.)
19. Recall that corresponding altitudes of congruent triangles are congruent. Prove that the lateral surface area of the regular pyramid shown in Figure 15-13 is given by $S = \frac{1}{2}ap$, where p is the perimeter of the base and a is the length of the segment whose endpoints are the vertex of the pyramid and the foot of the perpendicular from the vertex to an edge of the base. (See Exercises 16, 17, and 18.)
20. Let two pyramids, one triangular and one hexagonal, with equal base areas be given. The altitude of each pyramid is 9 in. The cross section of the triangular pyramid that is 3 in. from the base has an area of 40 sq. in. What is the area of the cross section of the hexagonal pyramid that is 3 in. from its base?
21. The area of the base of a pentagonal pyramid is 1024. The distance from the vertex to the plane of a cross section is 3 and the altitude of the pyramid is 8. Find the area of the cross section. (*Hint:* Use Theorem 15.5.)

22. The boundary of the base of a pyramid is an equilateral triangle, and the boundary of each lateral face is an equilateral triangle with sides of length 8. Find the total surface area of the pyramid.
23. The boundary of the base of a pyramid is a square whose sides are 10 cm. in length, and the boundary of each lateral face is an equilateral triangle. Find the total surface area of the pyramid.
24. Find the altitude of the pyramid of Exercise 23.
25. Find the altitude of the pyramid of Exercise 22.
26. The area of a cross section of a pyramid is 125. The area of the base is 405. The altitude of the pyramid is 9. Find the distance k from the vertex to the plane of the cross section.
27. Given $(a, b) \underset{p}{=} (c, d)$, prove that $(a^2, b^2) \underset{p}{=} (c^2, d^2)$. If

$$(a, b) \underset{p}{=} (c, d)$$

with constant of proportionality t, then what is the constant of proportionality for

$$(a^2, b^2) \underset{p}{=} (c^2, d^2)?$$

Refer to the proof of Theorem 15.4. In which sentence of this proof is this property of a proportionality used?

15.4 AREAS AND VOLUMES OF PRISMS AND CYLINDERS

As stated in the introduction to this chapter, we shall not treat areas and volumes of solids as rigorously as we did areas of polygonal regions. The concepts of surface area and volume are natural extensions of the area concept developed in Chapter 9. We accept without proof the fact that each solid discussed in this chapter has a surface area and a volume.

For volumes, we accept two postulates and use them to prove the volume formulas for prisms, cylinders, pyramids, and cones. Later in the chapter we develop a formula for the volume of a sphere.

For surface areas of cylinders, cones, and spheres, we develop the formulas informally and then state them formally as theorems whose proofs are beyond the scope of this book.

Our first postulate is similar to Postulate 28, the Rectangle Area Postulate of Chapter 9.

POSTULATE 31 *(Rectangular Parallelepiped Volume Postulate)* The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

Figure 15-14 is a picture of a rectangular parallelepiped. By definition, its base is a rectangular region and each of its faces is a rectangular region. Thus any one of its faces could be called the base, and the length of any one of its edges that is perpendicular to that face could be called the altitude.

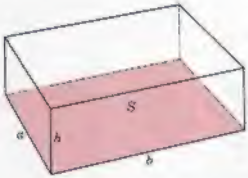


Figure 15-14

By Postulate 31, the volume V of the rectangular parallelepiped shown in Figure 15-14 is given by the formula

$$V = Sh,$$

where S is the area of the base and h is the altitude. By Postulate 28, $S = ab$. Therefore we have

$$V = abh$$

as a formula for the volume of a rectangular parallelepiped.

Suppose that an ordinary deck of playing cards is arranged so that its lateral faces are vertical as suggested in Figure 15-15a. It seems reasonable that if the shape of the deck is changed as in Figure 15-15b, the volume of the deck remains the same. It also seems reasonable that the fifteenth card from the bottom in Figure 15-15a has the same volume as the corresponding card in Figure 15-15b.

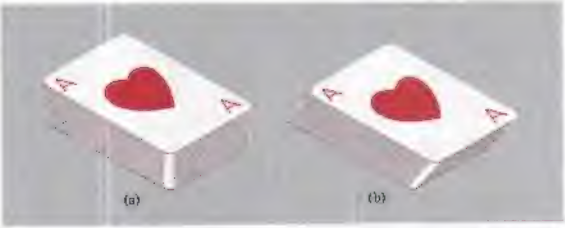


Figure 15-15

As a second example, suppose that we make an approximate model of a rectangular pyramid by forming a stack of thin cards as suggested in Figure 15-16. Of course, the thinner we make the cards, the more cards there will be and the closer the approximation will be. Suppose that the cards in the model are kept at the same level but are allowed to change position by sliding along each other.

Figure 15-16

Then the shape of the model changes, but its volume does not change. For different positions of the cards, the base area and the thickness of any two cards at the same level are the same, hence their volume is the same. The total volume of the model is the total volume of the cards, and the total volume does not change when the cards slide along each other.

More generally, imagine two solids with equal altitudes and with bases in the same horizontal plane. Suppose that every two cross sections of these solids at equal distances from the bases have equal areas. Then it seems reasonable that the two solids should have equal volumes. The reason is that if we imagine the solids being cut into thin slices by planes parallel to the bases, the volumes of slices at the same distance from the bases will be approximately equal. Therefore the volumes of the two solids should be equal.

The principle that we have tried to make plausible here is called Cavalieri's Principle after Professor Bonaventura Cavalieri (1598–1647) of the University of Bologna. He used this principle in obtaining some results that we now find in the calculus. We state this principle formally as the second postulate of this section.

POSTULATE 32 (***Cavalieri's Principle***) If two solids have equal altitudes, and if cross sections of these solids at equal distances from the bases have equal areas, then the solids have equal volumes.

Cavalieri's Principle is the key to calculating volumes other than rectangular parallelepipeds. We use the principle in the proof of our next theorem.

THEOREM 15.7 The volume V of any prism is the product of its altitude h and the area S of its base, that is, $V = Sh$.

Proof: Let a prism with altitude h and base area S be given as shown in Figure 15-17a. Let the rectangular parallelepiped shown in Figure 15-17b have the same altitude h and the same base area S, and let its base and the base of the given prism be in the same plane.

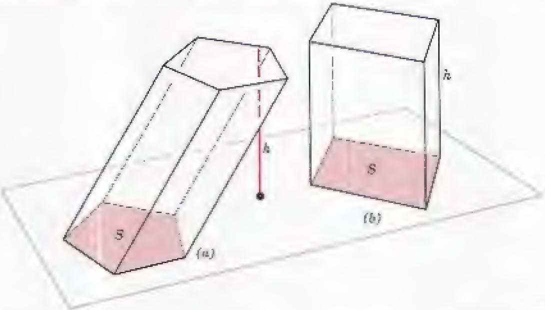


Figure 15-17

By the Prism Cross Section Theorem, all cross sections for both prisms have the same area S. By Cavalieri's Principle, the two prisms have the same volume. It follows from the Rectangular Parallelepiped Volume Postulate that $V = Sh$ for the given prism.

Example 1 The base boundary of a prism is an equilateral triangle 8 in. on a side. Its altitude is 12 in. Find the volume of the prism.

Solution: The volume of the prism is given by $V = Sh$, where S is the area of the triangular base and h is the altitude. We have $S = 16\sqrt{3}$ (show this) and $h = 12$. Therefore

$$V = 16 \cdot \sqrt{3} \cdot 12 = 192\sqrt{3},$$

and the volume is $192\sqrt{3}$ cu. in.

Figure 15-18 shows a diagram of a *circular cylinder*. Many of the terms used in defining a prism apply equally as well to a circular cylinder. The base of a prism is a polygonal region.

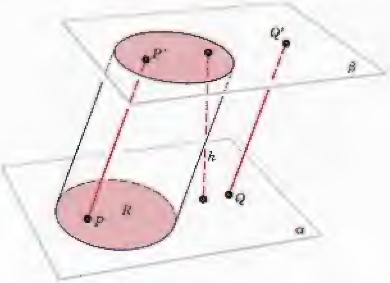


Figure 15-18

The base of a circular cylinder is a **circular region**, that is, the union of a circle and its interior. Make appropriate changes in the wording of the definition of a prism (Definition 15.1), refer to Figure 15-18, and define a circular cylinder.

There are other cylinders in addition to circular cylinders, that is, cylinders whose bases are not circular regions; but we shall consider only circular cylinders in this text. Therefore, when we speak of a cylinder, we mean a circular cylinder.

If $\overline{QQ'}$ in Figure 15-18 is perpendicular to α, then the cylinder is called a **right circular cylinder**. The altitude, bases, and cross sections of a cylinder are defined in the same way as are the corresponding parts of prisms.

The following two theorems are analogous to Theorems 15.1 and 15.2 for prisms and can be proved in a similar way. We omit the details of the proofs.

THEOREM 15.8 The boundary of each cross section of a cylinder is a circle that is congruent to the boundary of the base.

Outline of Proof: (See Figure 15-19 on page 685.) Let R be the base of the given cylinder and let R' be any cross section of the cylinder. Let C be the center of the circle that bounds the base, let P be a point on that circle, and let C' and P' be the corresponding points in R'. Let

$CP = r$ and let $C'P' = r'$. Then $PCC'P'$ is a parallelogram (Why?), and

$$P'C' = r' = PC = r.$$

Since PC has a constant value regardless of the position of P on the base circle, then $P'C'$ has a constant value. Thus all points P' lie on a circle with radius r' and center C'.

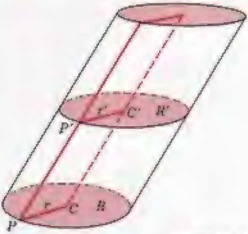


Figure 15-19

Therefore the cross section is a circular region and its boundary is a circle of radius r. Then its boundary is congruent to the boundary of the base.

> *THEOREM 15.9* (***The Cylinder Cross Section Theorem***) The area of a cross section of a cylinder is equal to the area of the base.

Proof: Assigned as an exercise.

Cavalieri's Principle is used in the proof of the following theorem on the volume of a cylinder.

> *THEOREM 15.10* The volume of a cylinder is the product of the altitude and the area of the base.

Proof: The proof is similar to that of Theorem 15.7 and is assigned as an exercise.

Imagine slitting a right circular cylinder and unrolling its lateral surface onto a plane. Which figure is obtained? Figure 15-20 on page 686 suggests that the boundary of the region thus obtained is a rectangle whose altitude is the altitude of the cylinder and whose base is the circumference of the base of the cylinder. Thus, if r is the radius

of the boundary of the base of a right cylinder and h is the altitude, then the lateral surface area of the cylinder is equal to the area of the rectangular region obtained by "unrolling" the cylinder, that is, the *lateral surface area* is $2\pi rh$. Since the area of each of the circular bases is πr^2, the *total surface area* of a right cylinder is $2\pi rh + 2\pi r^2$.

Figure 15-20

We state these results formally as our next theorem.

THEOREM 15.11 The lateral surface area of a circular cylinder of base radius r and altitude h is $2\pi rh$ and its total surface area is $2\pi rh + 2\pi r^2$.

Example 2 Find the total surface area of a cylinder if the radius of the circle that bounds the base is 7 and the altitude of the cylinder is 10.

Solution: The lateral surface area is

$$2\pi rh = 2\pi \cdot 7 \cdot 10 = 140\pi.$$

The area of each base is

$$\pi r^2 = \pi(7)^2 = 49\pi.$$

Therefore the total surface area is

$$140\pi + 2 \cdot 49\pi = 140\pi + 98\pi = 238\pi.$$

EXERCISES 15.4

1. A rectangular tank 6 ft. by 4 ft. is used for watering horses. If the tank is filled with water to a depth of 3 ft., how many cubic feet of water are in the tank?
2. One gallon of water occupies 231 cu. in. of space. To the nearest hundredth, how many gallons of water are contained in a space of 1 cu. ft.?
3. To the nearest gallon, how many gallons of water are in the tank of Exercise 1? (See Exercise 2.)
4. Show that the volume V of a cube of side e is given by $V = e^3$.

5. Find the volume of a cube whose edge is 6 in. Find its total surface area.
6. Write a formula for the volume V of a cylinder if its altitude is h and the radius of its base circle is r.
7. A water tank in the shape of a cylinder is 40 ft. high. The diameter of its base is 28 ft. Find the volume of the tank.
8. To the nearest thousand gallons, how many gallons of water will the tank of Exercise 7 hold? (Use $\pi = \frac{22}{7}$ and see Exercise 2.)
9. The altitude of a cylinder is 8 in. and the diameter of its base circle is 3 in. Find the volume and total surface area of the cylinder.
10. On a shelf in Roy's supermarket there are two cylindrical cans of coffee. The first is $1\frac{1}{2}$ times as tall as the second, but the second has a diameter $1\frac{1}{2}$ times that of the first. How should Roy price the second can in relation to the first if he wants the price per unit of volume to be the same for both cans?
11. How do the volumes of two cylinders compare if their altitudes are the same but the radius of the base circle of the second cylinder is three times that of the first?
12. How do the volumes of two cylinders compare if the radii of their base circles are the same but the altitude of the second cylinder is three times that of the first?
13. Draw a suitable figure and prove Theorem 15.9.
14. Draw a suitable figure and prove Theorem 15.10.
15. A brick chimney in the form of a cylindrical shell and 25 ft. tall is to be built. The inside and outside diameters are 24 in. and 16 in., respectively. If it takes 31 bricks per cubic foot of chimney, find the approximate number of bricks needed. (Use $\pi = 3.14$.)
16. An air conditioning unit is to be installed in a rectangular building. In order to install the correct size unit, it is necessary to know the number of cubic feet of air inside the building. If the dimensions of the building are as shown in Figure 15-21, find the volume of air inside the building.

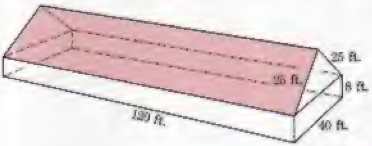


Figure 15-21

17. A block of wood in the shape of a cube has edges 16 in. in length. A circular hole 7 in. in diameter is bored through the block from top to bottom. Find the volume of the part of the block that remains.
18. In Exercise 17, if e is the length of the edge of the cube and r is the radius of the circular hole, write a formula for the volume V of the block that remains after the hole has been bored.

15.5 VOLUMES OF PYRAMIDS AND CONES

Here we develop formulas for calculating volumes of pyramids and cones and for the surface area of a cone. As in Section 15.4, Cavalieri's Principle plays a key role in the proofs of theorems on volume.

THEOREM 15.12 Two pyramids with the same altitude and the same base area have the same volume.

Proof: Let two pyramids be given as suggested in Figure 15-22. By the Pyramid Cross Section Theorem, corresponding cross sections of the two pyramids have the same area. Therefore, by Cavalieri's Principle, their volumes are the same.

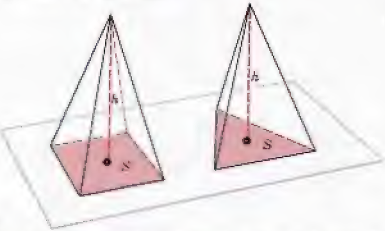


Figure 15-22

Our next theorem provides a formula for calculating the volume of a triangular pyramid. Suppose that we are given a triangular pyramid with base ABC and vertex W. (See Figure 15-23a.) Next, we take a triangular prism with the same base area and altitude as shown in Figure 15-23b. Imagine two planes cutting the prism and dividing it into three triangular pyramids as shown in Figure 15-24. (Name the two cutting planes that divide the prism as shown in Figure 15-24.)

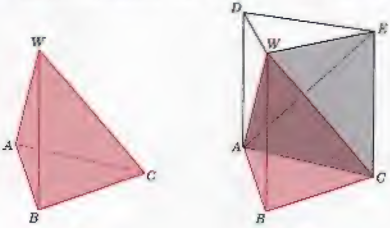


Figure 15-23 (a) (b)

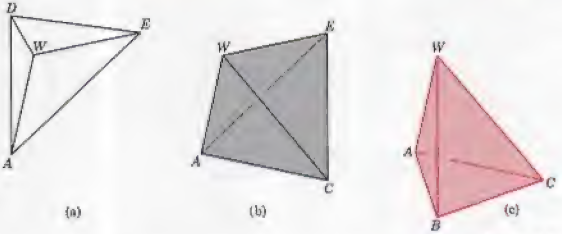


Figure 15-24

Pyramid (c) in Figure 15-24 has the same base area and the same altitude as the given pyramid in Figure 15-23. Therefore, by Theorem 15.12, they have the same volume. In Figure 15-23b, $\overline{DE} \parallel \overline{AC}$, $\overline{AD} \parallel \overline{CE}$, and $ADEC$ is a parallelogram with $\overline{AE}$ one of its diagonals. Therefore A, D, E, C are coplanar and $\triangle ADE \cong \triangle ECA$. Why? Think of pyramids (a) and (b) in Figure 15-24 as having bases ADE and ECA, respectively, and common vertex W. Then pyramids (a) and (b) have the same base area (Why?) and the same altitude (the distance from W to plane $ADEC$). By Theorem 15.12, these two pyramids have the same volume.

Next, consider $WECB$ in Figure 15-23. We have $\overline{WE} \parallel \overline{BC}$, $\overline{WB} \parallel \overline{CE}$, so $WECB$ is a parallelogram with $\overline{WC}$ one of its diagonals. Therefore W, E, C, B are coplanar and $\triangle WEC \cong \triangle CBW$. Think of pyramids (b) and (c) in Figure 15-24 as having bases WEC and CBW, respectively, and common vertex A. Then pyramids (b) and (c) have the same base area and the same altitude and, by Theorem 15.12, these two pyramids have the same volume. Therefore all three pyramids, (a), (b), (c) in Figure 15-24, have the same volume, say V, and the volume of the prism in Figure 15-23 is $3V$.

Now consider ABC as the base of the prism in Figure 15-23. Let the area of $\triangle ABC$ be S and let h be the altitude of the prism. Then

$$3V = Sh \qquad \text{and} \qquad V = \tfrac{1}{3}Sh.$$

But the given pyramid in Figure 15-23 has the same base area S and the same altitude h. Therefore the volume V of a triangular pyramid is given by the formula

$$V = \tfrac{1}{3}Sh,$$

where S is the area of its base and h is its altitude. This result is our next theorem.

THEOREM 15.13 The volume of a triangular pyramid is one-third the product of its base area and its altitude.

The formula $V = \frac{1}{3}Sh$ holds for any pyramid as our next theorem states.

THEOREM 15.14 The volume of a pyramid is one-third the product of its base area and its altitude.

Proof: Let a pyramid with base area S and altitude h be given as shown on the left in Figure 15-25. Consider a triangular pyramid having the same base area, the same altitude, and with its base and the base of the given pyramid in the same plane. It follows from the Pyramid Cross Section Theorem that cross sections of these two pyramids formed by the same plane have the same area. By Cavalieri's Principle, the two pyramids have the same volume. Since the volume of the triangular pyramid is $\frac{1}{3}Sh$, the volume of the given pyramid is also $\frac{1}{3}Sh$ and the proof is complete.

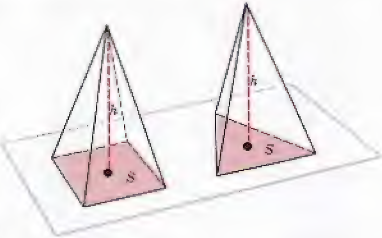


Figure 15-25

Example 1 The dimensions of the base of a rectangular pyramid are 7 cm. by 11 cm. and its altitude is 16 cm. Find its volume.

Solution: The volume V of the pyramid is given by the formula $V = \frac{1}{3}Sh$, where S is the area of the base and h is the altitude. We have

$$S = 77 \qquad \text{and} \qquad V = \tfrac{1}{3} \cdot 77 \cdot 16 = 410\tfrac{2}{3}.$$

Therefore the volume is $410\frac{2}{3}$ cu. cm.

Figure 15-26 shows a picture of a circular cone. Just as the definition of a circular cylinder is analogous to the definition of a prism, the definition of a circular cone is analogous to the definition of a pyramid.

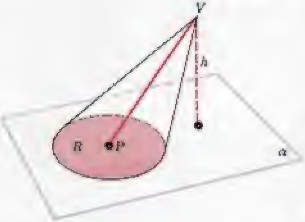


Figure 15-26

Many of the terms used in defining a pyramid apply equally as well to a circular cone. The base of a pyramid is a polygonal region. The base of a circular cone is a circular region. Make appropriate changes in the wording of the definition of a pyramid (Definition 15.7), refer to Figure 15-26, and define a circular cone.

There are other cones in addition to circular cones, that is, cones whose bases are not circular regions; but we consider only circular cones here. Therefore, when we speak of a cone, we mean a circular cone. Also, when we speak of the base circle of a cone, we mean the circle which is the boundary of the base.

Definition 15.8 (See Figure 15-27.) If the center of the base circle of a cone is the foot of the perpendicular from the vertex V to plane α, the cone is called a right circular cone.

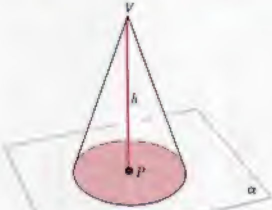


Figure 15-27

The following theorem is analogous to Theorems 15.4 and 15.5 on pyramids and can be proved in a similar way. We give only an outline of the proof and omit the details.

THEOREM 15.15 (*The Cone Cross Section Theorem*) A cross section of a cone of altitude h, made by a plane at a distance k from the vertex, is a circular region whose area and the area of the base are proportional to k^2 and h^2.

Outline of Proof: (See Figure 15-28.) Let R be the base of the given cone and let R' be any cross section of the cone. Let C be the center of the base circle, let P be any point on that circle, and let C' and P' be the corresponding points in R'. Let $VA = h$, $VB = k$, $CP = r$, and $C'P' = r'$.

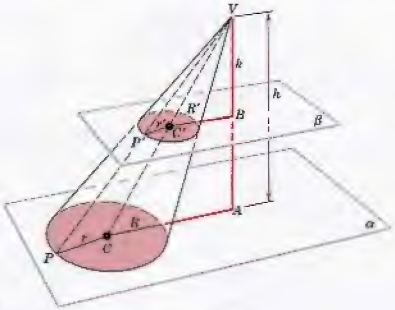


Figure 15-28

1. $\triangle VC'B \sim \triangle VCA$ Why?
   $(VC', k) \underset{p}{=} (VC, h)$ Why?
2. $\triangle VC'P' \sim \triangle VCP$ Why?
   $(VC', r') \underset{p}{=} (VC, r)$ Why?
   $(VC', k, r') \underset{p}{=} (VC, h, r)$ Why?

Then

$$(k, r') \underset{p}{=} (h, r), \qquad (r', r) \underset{p}{=} (k, h), \qquad \text{and} \qquad r' = \frac{kr}{h}.$$

3. Since PC has a constant value regardless of the position of the point P on the base circle, then $P'C'$ has a constant value. Thus all points P' lie on a circle with radius r' and center C'. The corresponding circular region is the cross section.
4. Let S' be the area of the circular cross section and let S be the area of the base. Then

$$(S', S) \underset{p}{=} [\pi(r')^2, \pi r^2] \underset{p}{=} [(r')^2, r^2] \underset{p}{=} (k^2, h^2) \quad \text{and} \quad (S', S) \underset{p}{=} (k^2, h^2).$$

This completes the outline of the proof.

Cavalieri's Principle is used in the proof of the following theorem which tells us how to find the volume of a cone.

THEOREM 15.16 The volume of a circular cone is one-third the product of the area of the base and the altitude.

Proof: The proof is similar to that of Theorem 15.14 and is left as an exercise.

Figure 15-29 shows a picture of a right circular cone. C is the center of the base circle and P is a point of that circle. We call the distance VP, which is the same for any point P on the base circle, the **slant height** of the cone and denote it by s.

Figure 15-29

If you imagine slitting the right cone of Figure 15-29 along $\overline{VP}$ and unrolling its lateral surface onto a plane, you get the sector of a circle shown on the right in Figure 15-29. You learned in Chapter 14 that the area of a sector of a circle is one-half the product of the radius of the sector and the length of the intercepted arc. In the case of a sector obtained by "unrolling" a cone, the radius of the sector is the slant height of the cone, and the length of the intercepted arc is the circumference of the base circle of the cone. Therefore a formula for the *lateral surface area* of a right circular cone is $\frac{1}{2}sC$, where s is the slant height of the cone and C is the circumference of the base circle. If r is the base radius, then

$$\tfrac{1}{2}sC = \tfrac{1}{2}s(2\pi r) = \pi rs.$$

If S is the total surface area of a right circular cone, s its slant height, and r the radius of its base circle, then a formula for the *total surface area* is

$$S = \tfrac{1}{2} \cdot s \cdot 2\pi r + \pi r^2 \qquad \text{or} \qquad S = \pi r(s + r).$$

This brings us to our next theorem.

THEOREM 15.17 The lateral surface area of a right circular cone of slant height s, base radius r, and base circumference C is $\frac{1}{2}sC$, or πrs, and its total surface area is $\pi r(s + r)$.

Example 2 The slant height of a right circular cone is 12 and the radius of its base circle is 5. Find the lateral surface area and the total surface area of the cone.

Solution: The lateral surface area is $\frac{1}{2}sC$, where

$$s = 12 \qquad \text{and} \qquad C = 2\pi r = 2 \cdot \pi \cdot 5 = 10\pi.$$

Therefore the lateral surface area is

$$\tfrac{1}{2} \cdot 12 \cdot 10\pi = 60\pi.$$

The total surface area S is given by the formula

$$S = \pi r(s + r).$$

Therefore

$$S = \pi \cdot 5(12 + 5) = 85\pi.$$

EXERCISES 15.5

1. The length of one edge of the base of a regular triangular pyramid is 12 in. and the altitude of the pyramid is 18 in. Find the lateral surface area of the pyramid. Find the total surface area. Find the volume. (Recall that a regular pyramid is one whose base boundary is a regular polygon and that the center of the polygon is the foot of the perpendicular from the vertex of the pyramid.)

Exercises 2–9 refer to the regular hexagonal pyramid in Figure 15-30, with $VC = 18$ in., $AB = 12$ in., and P the projection of V on $\overleftrightarrow{AB}$.

2. Find CP.
3. Find VP.
4. Find the area of the lateral face AVB.
5. Find the lateral surface area of the pyramid.
6. Find the area of $\triangle ABC$.
7. Find the area of the base of the pyramid.
8. Find the total surface area of the pyramid.
9. Find the volume of the pyramid.

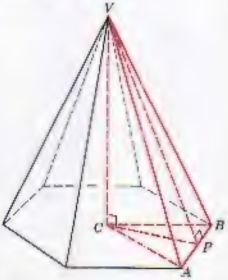
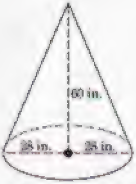


Figure 15-30

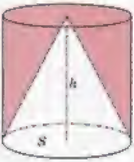
10. A plane bisects the altitude of a pyramid and is parallel to the base of the pyramid. Find the ratio of the volume of the pyramid above the bisecting plane to the volume of the given pyramid. (*Hint:* To what numbers are the area of the cross section in the bisecting plane and the area of the base proportional?)

11. Find the total surface area and the volume of the right circular cone shown in the figure.



12. Find the capacity in gallons of a right conical tank if it is 60 in. deep and if the radius of its circular top is 28 in. (To "see" the tank, invert the cone of Exercise 11. Use $\pi = \frac{22}{7}$.)

13. In the figure a right circular cone stands inside a right circular cylinder of same base and altitude. Write a formula for the volume of that portion of the cylinder not occupied by the cone.



14. Figure 15-31 shows two right cylinders having the same base area and the same altitude. Figure 15-31b shows two cones with a common vertex V inside the cylinder. If V is midway between the bases of the cylinder, show that the sum of the volumes of the two cones shown in Figure 15-31b is equal to the volume of the cone shown in (a).

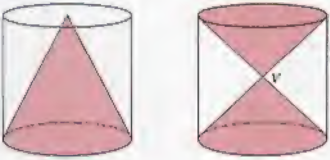
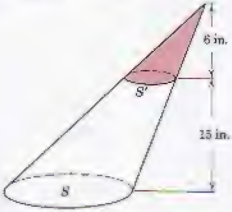


Figure 15-31

15. The volume of the cone shown in the figure is 250 cu. in. and its altitude is 15 in. A second cone is cut from the first by a plane parallel to the base and 6 in. from the vertex. Find the volume of the smaller cone. (*Hint:* If S and S' are the areas of the bases of the larger and smaller cones, respectively, to what numbers are S' and S proportional?)



16. CHALLENGE PROBLEM. If a plane parallel to the base of a cone (or pyramid) cuts off another cone (or pyramid), then the solid between the cutting plane and the base is called a frustum. Figure 15-32b shows a frustum of a right cone. If the radius of the base circle is 9 in., the radius of the top circle 6 in., and the height of the frustum 12 in., find the volume of the frustum. (*Hint:* First find x, where x is the altitude of the smaller cone in Figure 15-32a.)

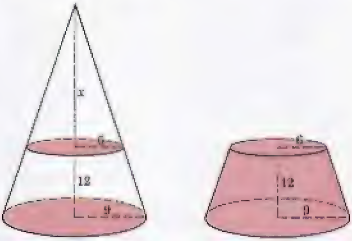
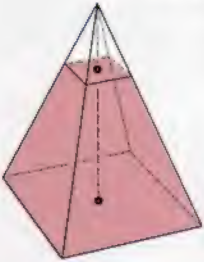


Figure 15-32

17. CHALLENGE PROBLEM. The figure (on page 697) shows a picture of a regular square pyramid. (See Exercise 1.) A plane parallel to the base intersects the altitude of the pyramid at a point whose distance from the vertex is one-third the distance from the vertex to the base. If the altitude of the pyramid is 24 in. and the length of an edge of the base is 18 in., find the lateral surface area and the volume of the frustum. (See Exercise 16.)



15.6 SURFACE AREAS OF SPHERES AND VOLUMES OF SPHERICAL REGIONS

The surface area of a sphere may be found very roughly by winding a string around a hemisphere and by covering with string a circular disk having the same radius as the sphere as suggested in Figure 15-33.

Figure 15-33

A comparison of the lengths of the two strings suggests that the surface area of the hemisphere is twice the area of the circle. Since the area of a circle is πr^2, this suggests that the surface area of a hemisphere is $2\pi r^2$ and that the surface area of a sphere is $4\pi r^2$.

We now proceed to a more sophisticated approach for finding the surface area of a sphere, but first we need a definition.

> *Definition 15.9* A spherical region is the union of a sphere and its interior.

Think of slicing a spherical region into n thin slices of thickness t where nt is the diameter $2r$ of the sphere. (See Figure 15-34.) This partitions the surface into n zones.

Figure 15-34

Figure 15-35 suggests a vertical cross section C of the sphere made by a plane through the center E of the sphere. The sphere has been sliced by equally spaced horizontal planes into n zones.

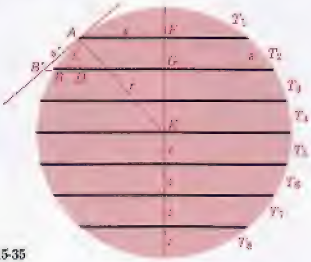


Figure 15-35

One of these zones, T_2, for example, is the surface generated when the arc $\widehat{AB}$ rotates about the line $\overleftrightarrow{FE}$. Let B' be the point in which the tangent line to C at A intersects $\overleftrightarrow{GB}$. Then AB' is approximately equal to AB and the area of the surface generated when $\overline{AB'}$ rotates about $\overleftrightarrow{FE}$ is approximately equal to the area of the surface generated when $\widehat{AB}$ rotates about $\overleftrightarrow{FE}$.

We have $\overline{AB'} \perp \overline{AE}$. Why? Let D be the point on $\overleftrightarrow{GB'}$ such that $\overline{DA} \perp \overline{FA}$. Then $\overline{AD} \perp \overline{B'D}$. Why? Therefore $\triangle AB'D \sim \triangle AEF$. (Show this by finding two pairs of congruent corresponding angles in the two triangles.) Let $s' = AB'$, $t = AD$, $r = AE$, and $x = AF$. Then

$$(s', t) \underset{p}{=} (r, x)$$

and

$$s'x = rt.$$

Think of the zone T_2 as a narrow ribbon of width s; s is the length of the arc $\overset{\frown}{AB}$ and is approximately equal to s'. Then the area of the zone is approximately equal to the length of the ribbon, about $2\pi x$, times the width of the ribbon s. Therefore the area of the zone is approximately $2\pi xs$. Now, $2\pi xs$ is approximately equal to $2\pi xs'$, and

$$2\pi xs' = 2\pi rt$$

since $s'x = rt$. If we combine the areas of the n zones, we find the area S of the sphere is given approximately by

$$\begin{aligned} S &= n(2\pi rt) \\ &= (2\pi r)(nt) \\ &= (2\pi r)(2r) \\ &= 4\pi r.^2 \end{aligned}$$

The total error introduced by using areas of ribbons to approximate areas of zones can be made as small as desired if the thickness of the slices is made small enough. The formula

$$S = 4\pi r^2$$

for the surface area of a sphere is an exact formula. Our approach has involved approximations, but our result is the correct one. In higher mathematics, the area of a sphere is carefully defined and the assertion that $S = 4\pi r^2$ is proved. We state it as a theorem.

THEOREM 15.18 The surface area of a sphere is 4π times the square of the radius of the sphere; that is, $S = 4\pi r^2$.

In our discussion concerning Figure 15-35, we showed that the area Z of a zone T of a sphere is approximately equal to $2\pi rt$, where r is the radius of the sphere and t is the thickness of the zone. Actually,

$$Z = 2\pi rt$$

is also an exact formula.

The formula for the volume of a sphere can now be obtained without difficulty. Note that when we say "volume of a sphere," we mean "volume of a spherical region."

Suppose that the surface of the sphere is divided into n pieces of equal areas by "latitude" circles and "longitude" circles, regular spaced as suggested in Figure 15-36. Suppose that the pieces are denoted by

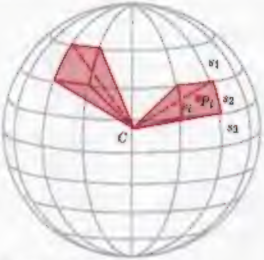


Figure 15-36

$s_1, s_2, \ldots, s_n$. For $i = 1, 2, \ldots, n$, let P_i be a point of s_i. That is, P_1 is a point of s_1, P_2 is a point of s_2, etc. Join each of the boundary points of s_i to the center C of the sphere with a segment. The union of these segments and the piece s_i encloses a portion of the spherical region which is approximately a pyramid whose altitude is the radius of the sphere and whose base is the piece s_i. (Note that we could "flatten" the base by using a portion of the plane that is tangent to the sphere at P_i. The error introduced by this flattening of the base when accumulated for all n pieces s_i, can be made as small as desired by making n sufficiently large.)

Recall that the volume of a pyramid is $\frac{1}{3}Sh$, where S is the area of the base and h is the altitude. Therefore the volume of one of the pyramidal portions of the spherical region is approximately $\frac{1}{3}rS$ where r is the radius of the sphere and S is the area of each s_i. By summing the volumes of the portions of the sphere corresponding to all of the pieces s_i, we find that the volume V of the sphere is given approximately by

$$\begin{aligned} V &= n(\tfrac{1}{3}rS) \\ &= (\tfrac{1}{3}r)(nS) \\ &= (\tfrac{1}{3}r)(4\pi r^2) \\ &= \tfrac{4}{3}\pi r^3. \end{aligned}$$

Although we used approximations to obtain the formula for the volume of a sphere, the formula

$$V = \frac{4}{3}\pi r^3$$

is an exact formula. We write it formally as our last theorem.

THEOREM 15.19 If r is the radius of a sphere, the volume of the sphere is $\frac{4}{3}\pi r^3$.

EXERCISES 15.6

In working the exercises in this set, do not use a replacement for π unless instructed to do so.

1. Find the surface area and the volume of a sphere whose diameter is 12.
2. A sphere having a diameter of 14 in. is placed in a cubical box. If the length of each edge of the box is 14 in., how many cubic inches of the volume of the box are not occupied by the sphere? (Use $\pi = \frac{22}{7}$.)
3. The radii of two spheres are 3 in. and 6 in., respectively. Find the ratio of their surface areas. Find the ratio of their volumes.
4. One sphere has a diameter that is three times that of a second sphere. Find the ratio of their surface areas. Find the ratio of their volumes.
5. In Exercise 4, if the larger sphere has a volume of 38,808, what is the volume of the smaller sphere? Use $\pi = \frac{22}{7}$ and find the radius of each of the two spheres.
6. Given a hemisphere, a cylinder, and a cone such that the radii of the base circles of the cylinder and the cone are equal to the radius of the hemisphere. If the cylinder and the cone have altitudes equal to the radius of the hemisphere, prove that the volume of the cylinder is equal to the sum of the volumes of the hemisphere and the cone.
7. If the altitude of a circular cylinder is equal to the diameter of a sphere, and if the radius of the base circle of the cylinder is equal to the radius of the sphere, prove that the volume of the sphere is two-thirds the volume of the cylinder.
8. A sphere with radius $3\frac{1}{2}$ in. is divided into 14 equal sections by planes containing the same diameter of the sphere. Find the volume of each section. (Use $\pi = \frac{22}{7}$.)
9. What is the largest radius a sphere can have if the numerical value of the surface area of the sphere is to be greater than or equal to the numerical value of the volume of the sphere?
10. The volumes of a sphere and a circular cone are equal, and the radius of the sphere equals the radius of the base circle of the cone. Find the altitude of the cone in terms of the radius of the sphere.

11. The radius of a sphere is $5\frac{1}{2}$. The volume of the sphere is equal to the volume of a circular cone. The radius of the base circle of the cone is $5\frac{1}{2}$. Find the altitude of the cone.
12. The surfaces of Earth and its moon are approximately spheres with the diameter of the moon about one-fourth that of Earth. Find the surface area (to the nearest 10,000 square miles) of the moon. (Use $\pi = \frac{22}{7}$ and 4000 miles as the radius of Earth.)
13. A water reservoir in the shape of a hemisphere has a diameter of 42 ft. If 1 cu. ft. of water weighs approximately 62.5 lb., how many tons (to the nearest ton) of water does the reservoir hold?
14. CHALLENGE PROBLEM A sphere and a right circular cylinder have equal volumes. The radius of the sphere is equal to the radius of the base circle of the cylinder. Which has the larger surface area, the sphere or the cylinder? Find the ratio of the larger to the smaller surface area.

CHAPTER SUMMARY

In this chapter we defined the following geometric solids. Review these definitions.

PRISM
RIGHT PRISM
PYRAMID
CIRCULAR CYLINDER
RIGHT CIRCULAR CYLINDER
CIRCULAR CONE
RIGHT CIRCULAR CONE
SPHERICAL REGION
PARALLELEPIPED
RECTANGULAR PARALLELEPIPED
CUBE

A pyramid is a REGULAR PYRAMID if its base boundary is a regular polygon and if the foot of the perpendicular from its vertex to its base is the center of the polygon. If a plane parallel to the plane of the base of a solid intersects the solid, we defined the intersection to be a CROSS SECTION of the solid provided the intersection consists of more than a single point.

We proved that all cross sections of a prism (cylinder) have the same area and that the area of a cross section of a pyramid (cone) and the area of the base are proportional to k^2 and h^2, where k is the distance from the vertex to the cross section plane and h is the altitude of the pyramid (cone). We proved that if two pyramids (cones) have equal altitudes and equal base areas, then cross sections equidistant from the vertices have equal areas.

A large part of this chapter is devoted to the development of surface area and volume formulas. The surface area formulas for solids with polygonal faces are based on the idea that the total surface area is the sum of the areas of the faces. The RECTANGULAR PARALLELEPIPED VOLUME

POSTULATE is a natural extension of the Rectangle Area Postulate. CAVALIERI'S PRINCIPLE, which we accepted as a postulate, plays a key role in the development of the volume formulas of this chapter. Several of the volume and area formulas were made plausible using ideas of limits and then were written formally as theorems. Following are the main formulas of this chapter.

AREA FORMULAS

Prism: The lateral surface area of a *cube* is $4e^2$ and the total surface area is $6e^2$, where e is the length of an edge of the cube. The lateral surface area of a *rectangular parallelepiped* is $2ah + 2bh$ and the total surface area is $2ah + 2bh + 2ab$, where a and b are the dimensions of the base and h is the altitude. The lateral surface area of a *right prism* is hp, where h is the altitude and p is the perimeter of the base of the prism.

Pyramid: The lateral surface area of a *regular pyramid* is $\frac{1}{2}ap$, where p is the perimeter of the base and a is the altitude (to an edge of the base) of the triangle that bounds a lateral face.

Cylinder: The lateral surface area of a *circular cylinder* is $2\pi rh$ and the total surface area is $2\pi rh + 2\pi r^2$, where h is the altitude and r is the radius of the base circle of the cylinder.

Cone: The lateral surface area of a *right circular cone* is πrs and the total surface area is $\pi r(s + r)$, where s is the slant height of the cone and r is the radius of the base circle.

Sphere: The total surface area of a *sphere* is $4\pi r^2$, where r is the radius of the sphere.

VOLUME FORMULAS

Prism: The volume of a *prism* is Sh, where S is the area of the base and h is the altitude of the prism.

Pyramid: The volume of a *pyramid* is $\frac{1}{3}Sh$, where S is the area of the base and h is the altitude of the pyramid.

Cylinder: The volume of a *circular cylinder* is πr^2h, where r is the radius of the base circle and h is the altitude of the cylinder.

Cone: The volume of a *circular cone* is $\frac{1}{3}\pi r^2h$, where r is the radius of the base circle and h is the altitude of the cone.

Sphere: The volume of a *sphere* is $\frac{4}{3}\pi r^3$, where r is the radius of the sphere.

REVIEW EXERCISES

In Exercises 1–15, complete each statement, given the prism in Figure 15-37 with P and P' in planes α and β such that $\overline{PP'} \perp \alpha$.

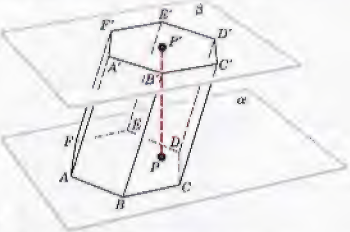


Figure 15-37

1. The region $ABCDEF$ is called the ? of the prism.
2. The region $A'B'C'D'E'F'$ is called the ? of the prism.
3. $\overline{DD'}$ is called a ? of the prism.
4. There are ? lateral edges in all.
5. Counting the lateral edges and the edges of the two bases, there are ? edges in all.
6. The parallelogram region $BCC'B'$ is called a ? of the prism.
7. There are ? lateral faces in all.
8. Counting the lateral faces and the two bases, there are ? faces in all.
9. E' is called a ? of the prism.
10. There are ? vertices in all.
11. If V is the number of vertices, E is the number of edges, and F is the number of faces, then $V - E + F =$?.
12. $\overline{PP'}$ is called an ? of the prism.
13. If $\overline{CC'} \perp \alpha$, then the prism is called a ? prism.
14. If $\overline{BB'} \perp \alpha$, $BB' = 16$, and the perimeter of the base is $42\frac{1}{2}$, then the lateral surface area of the prism is ?.
15. If $PP' = 24\frac{2}{3}$ in. and the area of the base is 326 sq. in., then the volume of the prism is ?.

In Exercises 16–30, complete the statement, given the pyramid of Figure 15-38 with point P in α, the plane of the base, such that $\overline{VP} \perp \alpha$.

16. The region $ABCDE$ is called the ? of the pyramid.
17. V is called the ? of the pyramid.
18. $\overline{BV}$ is called a ? of the pyramid.
19. There are ? lateral edges in all.

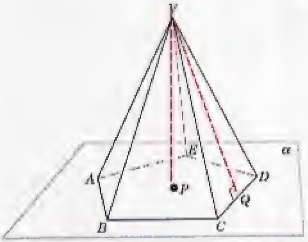


Figure 15-38

20. Counting the lateral edges and the edges of the bases there are $\boxed{?}$ edges in all.
21. The triangular region CVD is called a $\boxed{?}$ of the pyramid.
22. There are $\boxed{?}$ lateral faces in all.
23. Counting the base, there are $\boxed{?}$ faces in all.
24. There are $\boxed{?}$ vertices in all.
25. If V is the number of vertices, E is the number of edges, and F is the number of faces, then $V - E + F = \boxed{?}$.
26. VP is the $\boxed{?}$ of the pyramid.
27. If the boundary of the base is a regular pentagon and if P is the center of the pentagon, then the pyramid is called a $\boxed{?}$ pyramid.
28. Given the hypothesis of Exercise 27, the boundary of each of the lateral faces of the pyramid is a(n) $\boxed{?}$ triangle, and all these triangles are $\boxed{?}$ to each other.
29. Given the hypothesis of Exercise 27, if $CD = 12$, $\overline{VQ} \perp \overline{CD}$, and $VQ = 18$, then the lateral surface area of the pyramid is $\boxed{?}$.
30. If the area of the base is 262 and $VP = 16\frac{1}{4}$, then the volume of the pyramid is $\boxed{?}$.

31. Suppose that two pyramids, one triangular and one rectangular, with equal base areas are given. The altitude of each pyramid is 12 in. A cross section of the triangular pyramid is 4 in. from the base and has an area of 60 sq. in. What is the area of a cross section 4 in. from the base of the rectangular pyramid?
32. The area of the base of a hexagonal pyramid is 729. The distance from the vertex to the plane of a cross section is 4 and the altitude of the pyramid is 9. Find the area of the cross section. Find the volume of the pyramid.
33. The boundary of the base of a pyramid is a square whose sides are 12 cm. in length. If the boundary of each of the lateral faces is an equilateral triangle, find the total surface area and the volume of the pyramid.

In Exercises 34–37, complete the statement, given the solid shown in Figure 15-39 with $\alpha \parallel \beta$, circular regions R and R' contained in α and β, and points P and P' in α and β such that $\overline{PP'} \perp \alpha$.

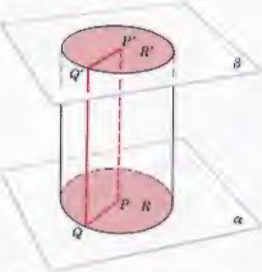


Figure 15-39

34. The solid shown in the figure is called a ?.
35. $\overline{PP'}$ is called an ? of the cylinder.
36. If P and P' are the centers of the two base circles, then the cylinder is called a ? cylinder.
37. If P and P' are the centers of the base circles, if Q and Q' are points on the two base circles of the cylinder, and if $PQ = 9$ and $QQ' = 16$, the total surface area of the cylinder is ? and the volume is ?.

38. Find the volume of a cylinder if the altitude is 24.6 in. and the radius of the base circle is 8 in. (Use $\pi = 3.14$.)
39. Find the lateral surface area of the cylinder of Exercise 38. (Use $\pi = 3.14$.)

In Exercises 40–45, complete each statement, given the solid shown in Figure 15-40 with circular region R, the base of the solid, contained in plane α and with points P and Q in α such that $\overline{VP} \perp \alpha$ and Q is on the base circle of the solid.

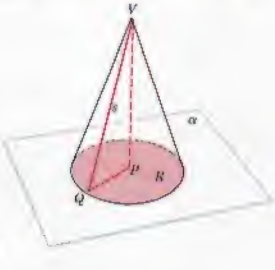
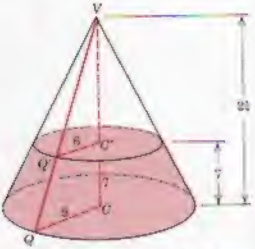


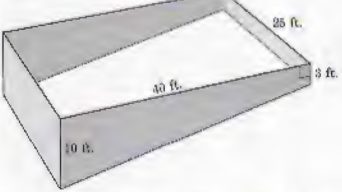
Figure 15-40

40. The solid shown in the figure is called a $\boxed{?}$.
41. If P is the center of the base circle, then the cone is called a $\boxed{?}$.
42. V is called the $\boxed{?}$ of the cone.
43. VP is the $\boxed{?}$ of the cone.
44. If $VQ = s$, then s is called the $\boxed{?}$ of the cone.
45. If the radius r of the base circle is 10 and $s = 27$, then the total surface area of the cone is $\boxed{?}$ and the volume of the cone is $\boxed{?}$.

46. Find the volume of the frustum of the right circular cone shown in the figure if the radius of the boundary of the cross section is 6, the radius of the base circle is 9, the altitude VC of the given cone is 21, and the distance between the plane of the base and the plane of the cross section is 7.



47. Find the surface area and the volume of a sphere whose radius is 8.
48. Find the volume of a sphere whose diameter is 21. (Use $\pi = \frac{22}{7}$.)
49. The swimming pool with dimensions as shown in the figure is a right prism, the bases of which are regions whose boundaries are trapezoids. The altitude of the trapezoidal base is 40 ft. and the altitude of the prism is 25 ft. Find the volume of the pool.



50. To the nearest hundred gallons, how many gallons of water are needed to fill the swimming pool of Exercise 49 to within 1 ft. of the top? (Recall that one gallon of water occupies 231 cu. in. of space.)

Appendix

LIST OF SYMBOLS

SYMBOL		PAGE
$\{\ \}$	set braces	14
$\in$	is an element of or is a member of	14
$\cap$	intersection	15
$\varnothing$	null set or empty set	15
$\cup$	union	17
$\subset$	is a subset of	18
$\overleftrightarrow{AB}$	line	24
α	alpha	24
β	beta	24
γ	gamma	24
A–B–C	point B is between points A and C	42
$\overline{AB}$	segment	47
$\overrightarrow{AB}$	ray	48
opp $\overrightarrow{AB}$	the ray opposite $\overrightarrow{AB}$	49
$\angle$	angle	52
$\overset{\circ\!\!\longrightarrow}{AB}$	halfline	62
$\triangle$	triangle	73
PQ	the distance between points P and Q	100
$\lvert a\rvert$	absolute value	105
$\cong$	is congruent to	111
$=$	equals or is equal to	111

SYMBOL		PAGE
$\overrightarrow{AB}$	directed segment from A to B	130
$m\angle ABC$	the measure of $\angle ABC$	141
cd $\overrightarrow{VX}$	ray-coordinate of $\overrightarrow{VX}$	152
$\perp$	perpendicular or is perpendicular to	172
$\llcorner$	right angle	173
$\angle A\text{-}BC\text{-}D$	dihedral angle $A\text{-}BC\text{-}D$	181
$\longleftrightarrow$	one to-one correspondence	190
$\perp$ bis	is the perpendicular bisector of	236
$<$	is less than	248
$>$	is greater than	248
$\leq$	is less than or equal to	248
$\geq$	is greater than or equal to	250
$\parallel$	parallel or is parallel to	282
$\lvert\triangle ABC\rvert$	the area of triangle ABC	369
$\overset{=}{p}$	are proportional to	393
$\sim$	is similar to	402
δ	delta	529
$\widehat{AB}$	arc	577
π	pi (the ratio of the circumference of a circle to its diameter)	639

POSTULATES

1. *The Three-Point Postulate.* Space contains at least three noncollinear points.
2. *The Line-Point Postulate.* Every line is a set of points and contains at least two distinct points.
3. *The Point-Line Postulate.* For every two distinct points, there is one and only one line that contains both points.
4. *The Four-Point Postulate.* Space contains at least four noncoplanar points.
5. *The Plane-Point Postulate.* Every plane is a set of points and contains at least three noncollinear points.
6. *The Point-Plane Postulate.* For every set of three noncollinear points, there is one and only one plane that contains them.
7. *The Flat-Plane Postulate.* If two distinct points of a line belong to a plane, then every point of the line belongs to that plane.
8. *The Plane-Intersection Postulate.* If two distinct planes intersect, then their intersection is a line.

9. *The A–B–C Betweenness Postulate.* If point B is between points A and C, then point B is also between C and A, and all three points are distinct and collinear.

10. *The Three-Point Betweenness Postulate.* If three distinct points are collinear, then one and only one is between the other two.

11. *The Line-Building Postulate.* If A and B are any two distinct points, then there is a point X_1 such that X_1 is between points A and B, a point Y_1 such that B is between A and Y_1, and a point Z_1 such that A is between Z_1 and B.

12. *The Line Separation Postulate.* Each point A on a line separates the line. The points of the line other than the point A form two distinct sets such that
    1. each of the two sets is convex;
    2. if two points are in the same set, then A is not between them;
    3. if two points are in different sets, then A is between them.

13. *The Plane Separation Postulate.* Each line l in a plane separates the plane. The points of the plane other than the points on line l form two distinct sets such that
    1. each of the two sets is convex;
    2. if two points are in the same set, then no point of line l is between them;
    3. if two points are in different sets, then there is a point of line l between them.

14. *The Space Separation Postulate.* Each plane α in space separates space. The points in space other than the points in plane α form two distinct sets such that
    1. each of the two sets is convex;
    2. if two points are in the same set, then no point of plane α is between them;
    3. if two points are in different sets, then there is a point of plane α between them.

15. *Distance Existence Postulate.* If $\overline{AB}$ is any segment, there is a correspondence which matches with every segment $\overline{CD}$ in space a unique positive number, the number matched with $\overline{AB}$ being 1.

16. *Distance Betweenness Postulate.* If A, B, C are collinear points such that A–B–C, then for any distance function we have $AB + BC = AC$.

17. *Triangle Inequality Postulate.* If A, B, C are noncollinear points, then for distances in any system we have $AB + BC > AC$.

18. *Distance Ratio Postulate.* If $\overline{PQ}$ and $\overline{RS}$ are unit segments and A, B, C, D are points such that $A \neq B$, $C \neq D$, then

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{CD \text{ (in } \overline{PQ} \text{ units)}} = \frac{AB \text{ (in } \overline{RS} \text{ units)}}{CD \text{ (in } \overline{RS} \text{ units)}}$$

or, equivalently,

$$\frac{AB \text{ (in } \overline{PQ} \text{ units)}}{AB \text{ (in } \overline{RS} \text{ units)}} = \frac{CD \text{ (in } \overline{PQ} \text{ units)}}{CD \text{ (in } \overline{RS} \text{ units)}}.$$

19. *Ruler Postulate.* If $\overline{AB}$ is a unit segment and if P and Q are distinct points on a line l, then there is a unique coordinate system on l relative to $\overline{AB}$ such that the origin is P and the coordinate q of Q is a positive number.
20. *Angle Measure Existence Postulate.* There exists a correspondence which associates with every angle in space a unique real number between 0 and 180.
21. *Angle Measure Addition Postulate.* If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are distinct coplanar rays, then $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if $m\angle AVC = m\angle AVB + m\angle BVC$.
22. *Protractor Postulate.* If α is any plane and $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are noncollinear rays in α, then
    1. there is a unique ray-coordinate system $\mathcal{S}$ in α relative to V in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VB} = m\angle AVB$, and
    2. if X is any point on the B-side of $\overleftrightarrow{VA}$, then $cd\ \overrightarrow{VX}$ (in $\mathcal{S}$) $= m\angle AVX$.
23. *The S.A.S. Postulate.* Let a one-to-one correspondence between the vertices of two triangles (not necessarily distinct) be given. If two sides and the included angle of the first triangle are congruent, respectively, to the corresponding parts of the second triangle, then the correspondence is a congruence.
24. *The A.S.A. Postulate.* Let a one-to-one correspondence between the vertices of two triangles (not necessarily distinct) be given. If two angles and the included side of the first triangle are congruent, respectively, to the corresponding parts of the second triangle, then the correspondence is a congruence.
25. *The S.S.S. Postulate.* Let a one-to-one correspondence between the vertices of two triangles (not necessarily distinct) be given. If

the three sides of the first triangle are congruent, respectively, to the corresponding sides of the second triangle, then the correspondence is a congruence.

26. *Parallel Postulate.* There is at most one line parallel to a given line and containing a given point not on the given line.
27. *Area Existence Postulate.* Every polygon has an area, and that area is a (unique) positive number.
28. *Rectangle Area Postulate.* If two adjacent sides of a rectangle are of length a and b, then the area S of the rectangle is given by the formula $S = ab$.
29. *Area Congruence Postulate.* Congruent polygons have equal areas.
30. *Area Addition Postulate.* If a polygonal region is partitioned into a finite number of polygonal subregions by a finite number of segments (called boundary segments) such that no two subregions have points in common except for points on the boundary segments, then the area of the region is the sum of the areas of the subregions.
31. *Rectangular Parallelepiped Volume Postulate.* The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.
32. *Cavalieri's Principle Postulate.* If two solids have equal altitudes, and if cross sections of these solids at equal distances from the bases have equal areas, then the solids have equal volumes.

DEFINITIONS

1.1. *Space* is the set of all points.

1.2. The points of a set are *collinear* if and only if there is a line which contains all of them. The points of a set are *noncollinear* if and only if there is no line which contains all of them.

1.3. The points of a set are *coplanar* if and only if there is a plane which contains all of them. The points of a set are *noncoplanar* if and only if there is no plane which contains all of them.

2.1. If A and B are any two distinct points, *segment* $\overline{AB}$ is the set consisting of points A, B, and all points between A and B. The points A and B are called *endpoints* of $\overline{AB}$.

2.2. If A and B are any two distinct points, *ray* $\overrightarrow{AB}$ is the union of segment $\overline{AB}$ and all points X such that A–B–X. The point A is called the *endpoint* of $\overrightarrow{AB}$.

2.3. If A is between B and C, then rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called *opposite* rays.

2.4. The *interior of a segment* is the set of all points of the segment except its endpoints. The *interior of a ray*, also called a *half-line*, is the set of all points of the ray except its endpoint.

2.5. An *angle* is the union of two noncollinear rays with the same endpoint. Each of the two rays is called a *side* of the angle. The common endpoint of the two rays is called the *vertex* of the angle.

2.6. A set of points is called *convex* if for every two points P and Q in the set, the entire segment $\overline{PQ}$ is in the set. The null set and every set that contains only one point are also called *convex sets*.

2.7. Let a line l and a point A on l be given.
1. The two convex sets described in Postulate 12 are called *halflines* or *sides* of point A on line l; A is the *endpoint* of each of them.
2. If C and D are two points in one of these sets, we say that C and D are on the *same side* of A, or that C is on the *D-side* of A, or that D is on the *C-side* of A.
3. If B is a point in one of these sets and C is a point in the other set, we say that B and C are on *opposite sides* of A on line l, or that B and C are in the *opposite halflines* of l determined by the point A.

2.8. Let a plane α and a line l in α be given.
1. The two convex sets described in Postulate 13 are called *halfplanes* or *sides* of l in plane α; l is the *edge* of each of them.
2. If C and D are two points in one of these sets, then we say that C and D are on the *same side* of l in plane α, or that C is on the *D-side* of l, or that D is on the *C-side* of l, or that C and D are in the *same halfplane*.
3. If B is a point in one of these sets and C is a point in the other set, we say that B and C are on *opposite sides* of l in plane α, or that B and C are in the *opposite halfplanes* of α determined by the line l.

2.9. Let a plane α be given.
1. The two convex sets described in Postulate 14 are called *halfspaces* or *sides* of plane α, and plane α is called the *face* of each of them.
2. If C and D are any two points in one of these sets, then we

say that C and D are on the *same side* of α, or that C is on the D-*side* of α, or that D is on the C-*side* of α, or that C and D are in the *same halfspace*.

3. If B is a point in one of these sets and C is a point in the other set, then we say that B and C are on *opposite sides* of α, or that B and C are in *opposite halfspaces*.

2.10. The *interior* of an angle, say $\angle ABC$, is the intersection of two halfplanes, the C-side of $\overleftrightarrow{AB}$ and the A-side of $\overleftrightarrow{BC}$.

2.11. The *exterior* of an angle is the set of all points in the plane of the angle except those points on the sides of the angle and in its interior.

2.12. If A, B, C are three noncollinear points, then the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is a *triangle*.

2.13. Let $\triangle ABC$ be given.

1. Each of the points A, B, C is a *vertex* of $\triangle ABC$.
2. Each of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is a *side* of $\triangle ABC$.
3. Each of the angles $\angle ABC$, $\angle BCA$, $\angle CAB$ is an *angle* of $\triangle ABC$.
4. A side and a vertex not on that side are *opposite* to each other.
5. A side and an angle are *opposite* to each other if that side and the vertex of that angle are opposite to each other.

2.14. The intersection of the interiors of the three angles of a triangle is the *interior* of the triangle.

2.15. The *exterior* of a triangle is the set of all points in the plane of the triangle that are neither points of the triangle nor points of the interior of the triangle.

2.16. Let A, B, C, D be four coplanar points such that no three of them are collinear and such that none of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersects any other at a point which is not one of its endpoints. Then the union of the four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ is a *quadrilateral*. Each of the four segments is a *side* of the quadrilateral and each of the points A, B, C, D is a *vertex* of the quadrilateral.

3.1. The *distance between any point and itself* is 0.

3.2. 1. The correspondence that matches a unique positive number with each pair of distinct points C and D, as in Postulate 15, and the number 0 with the points C and D if $C = D$, as in Definition 3.1, is called the *distance function determined by* $\overline{AB}$ or the *distance function based on* $\overline{AB}$.

2. The segment $\overline{AB}$ that determines a distance function is the *unit segment* for that distance function.
3. The number matched with C and D, in Postulate 15, is the *distance from C to D*, or the *distance between C and D*, or the *length of* $\overline{CD}$.

3.3. Let $\overline{PQ}$ be a unit segment and l a line. A *coordinate system* on l relative to $\overline{PQ}$ is a one-to-one correspondence between the set of all points of l and the set of all real numbers such that if points A, B, C are matched with the real numbers a, b, c, respectively, then
1. B is between A and C if and only if b is between a and c and
2. AB (in $\overline{PQ}$ units) $= |a - b|$.

3.4. 1. The *origin* of a coordinate system on a line is the point matched with 0.
2. The *unit point* is the point matched with 1.
3. The number matched with a point is its *coordinate*.

3.5. Two segments (distinct or not) are *congruent* if and only if they have the same length. If two segments are congruent, we say that each of them is congruent to the other one and we refer to them as congruent segments.

3.6. The *midpoint* of a segment $\overline{AB}$ is the point P on $\overline{AB}$ such that $AP = PB = \frac{1}{2}AB$. The midpoint of a segment is said to *bisect* the segment or to *divide* it into two congruent parts.

3.7. The *trisection points* of a segment $\overline{AB}$ are the two points P and Q on $\overline{AB}$ such that $AP = PQ = QB = \frac{1}{3}AB$. The trisection points of a segment are said to *divide* the segment into three congruent parts. Similarly, points C, D, and E on $\overline{AB}$ such that $AC = CD = DE = EB = \frac{1}{4}AB$ are said to *divide* $\overline{AB}$ into four congruent parts. This idea may be extended to any number of congruent parts

3.8. The *directed segment* from A to B, denoted by $\overrightarrow{AB}$, is the set $\{\overline{AB}, A\}$.

3.9. Let a directed segment $\overrightarrow{AB}$ and two points P and Q on $\overleftrightarrow{AB}$ be given. If $P \in \overline{AB}$, $Q \notin \overline{AB}$, and $\dfrac{AP}{PB} = \dfrac{AQ}{QB}$, then P and Q are said to divide $\overrightarrow{AB}$ in the same ratio, P dividing it *internally* and called an *internal point of division*, Q dividing it *externally* and called an *external point of division*. The ratio $\dfrac{AP}{PB}$ is the *ratio of division*.

4.1. The number which corresponds to an angle as in the Angle Measure Existence Postulate is called the *measure of the angle*.

4.2. Two angles (whether distinct or not) are *congruent angles*, and each is said to be *congruent* to the other, if they have the same measure.

4.3. If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are rays, then $\overrightarrow{VB}$ is *between* $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if

1. A and B are in the same halfplane with edge $\overleftrightarrow{VC}$.
2. B and C are in the same halfplane with edge $\overleftrightarrow{VA}$.
3. A and C are in opposite halfplanes with edge $\overleftrightarrow{VB}$.

4.4. Let V be a point in a plane α. A *ray-coordinate system* in α relative to V is a one-to-one correspondence between the set of all rays in α with endpoint V and the set of all real numbers x such that $0 \leq x < 360$ with the following property: If numbers r and s correspond to rays $\overrightarrow{VR}$ and $\overrightarrow{VS}$ in α, respectively, and if $r > s$, then

$m\angle RVS = r - s$ if $r - s < 180$,
$m\angle RVS = 360 - (r - s)$ if $r - s > 180$,
$\overrightarrow{VR}$ and $\overrightarrow{VS}$ are opposite rays if $r - s = 180$.

4.5. The number that corresponds to a ray in a given ray-coordinate system is called the *ray-coordinate* of that ray. The ray whose ray-coordinate is zero is called the *zero-ray* of that system.

4.6. Two angles are called *vertical angles* if and only if their sides form two pairs of opposite rays.

4.7. Two angles are called a *linear pair of angles* if and only if they have one side in common and the other sides are opposite rays.

4.8. Two angles (distinct or not) are *complementary*, and each is called a *complement* of the other if the sum of their measures is 90. Two angles (distinct or not) are *supplementary*, and each is called a *supplement* of the other if the sum of their measures is 180.

4.9. A ray is a *midray* of an angle if it is between the sides of the angle and forms with them two congruent angles. A midray of an angle is said to *bisect* the angle; it is sometimes called the *bisector of the angle* or, briefly, the *angle bisector*.

4.10. Two coplanar angles are *adjacent angles* if they have one side in common and the intersection of their interiors is empty.

4.11. An angle whose measure is 90 is a *right angle*. An angle whose measure is less than 90 is an *acute angle*. An angle whose measure is greater than 90 is an *obtuse angle*.

4.12. If the union of two intersecting lines contains a right angle, then the lines are *perpendicular*.

4.13. Two sets, each of which is a segment, a ray, or a line, and which determine two perpendicular lines are called *perpendicular sets*, and each is said to be perpendicular to the other.

4.14. Let n be any integer greater than or equal to 3. Let P_1, $P_2, \ldots, P_{n-1}, P_n$ be n distinct coplanar points such that the n segments $\overline{P_1P_2}$, $\overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ have the following properties:

1. No two of these segments intersect except at their endpoints.
2. No two of these segments with a common endpoint are collinear.

Then the union of these n segments is a *polygon*. Each of the n given points is a *vertex* of the polygon. Each of the n segments is a *side* of the polygon.

4.15. Two vertices of a polygon that are endpoints of the same side are called *consecutive* vertices. Two sides of a polygon that have a common endpoint are called *consecutive* sides. A *diagonal* of a polygon is a segment whose endpoints are vertices, but not consecutive vertices, of the polygon.

4.16. A polygon is a *convex polygon* if and only if each of its sides lies on the edge of a halfplane which contains all of the polygon except that one side.

4.17. The *interior of a convex polygon* is the intersection of all of the halfplanes, each of which has a side of the polygon on its edge and each of which contains all of the polygon except that side.

4.18. An angle determined by two consecutive sides of a convex polygon is called an *angle of the polygon*. Two angles of a polygon are called *consecutive angles* of the polygon if their vertices are consecutive vertices of the polygon.

4.19. If two sides (or vertices, or angles) of a quadrilateral are not consecutive sides (or vertices, or angles), then they are *opposite sides* (or vertices, or angles) and each is said to be *opposite* the other.

4.20. If two noncoplanar halfplanes have the same edge, then the

union of these halfplanes and the line which is their common edge is a *dihedral angle*. The union of this common edge and either one of these two halfplanes is a *face* of the dihedral angle.

4.21. Two dihedral angles which have a common edge and whose union is the union of the two intersecting planes are *vertical dihedral angles*.

5.1. Two triangles (not necessarily distinct) are *congruent* if and only if there exists a one-to-one correspondence between their vertices in which the corresponding parts are congruent. Such a one-to-one correspondence between the vertices of two congruent triangles is called a *congruence*.

5.2. An angle of a triangle is said to be *included* by two sides of that triangle if the angle contains these sides. A side of a triangle is said to be *included* by two angles of that triangle if the endpoints of the side are the vertices of those angles.

5.3. An *isosceles* triangle is a triangle with (at least) two congruent sides. If two sides are congruent, then the remaining side is called the *base*. The angle opposite the base is called the *vertex angle*. The two angles that are opposite the congruent sides are called the *base angles*.

5.4. A triangle with three congruent sides is called an *equilateral* triangle. A triangle with three congruent angles is called an *equiangular* triangle.

5.5. A *median of a triangle* is a segment whose endpoints are a vertex of the triangle and the midpoint of the side opposite that vertex.

5.6. The *perpendicular bisector of a segment* in a given plane is the line in that plane which is perpendicular to the segment at its midpoint.

6.1. $\overline{AB} > \overline{CD}$ if and only if $AB > CD$; $\overline{AB} < \overline{CD}$ if and only if $AB < CD$.

6.2. $\angle ABC > \angle DEF$ if and only if $m\angle ABC > m\angle DEF$; $\angle ABC < \angle DEF$ if and only if $m\angle ABC < m\angle DEF$.

6.3. If a and b are numbers, then $a < b$ if and only if there is a *positive* number p such that $b = a + p$. Also $a > b$ if and only if there is a *positive* number p such that $a = b + p$.

6.4. If a and b are numbers, then $a \leq b$ if and only if $a < b$ or $a = b$.

6.5. Each angle of a triangle is called an *interior* angle of the triangle. An angle which forms a linear pair with an interior angle of a triangle is called an *exterior* angle of the triangle. Each exterior angle is said to be *adjacent* to the interior angle with which it forms a linear pair and *nonadjacent* to the other two interior angles of the triangle.

6.6. A *right triangle* is a triangle with one right angle. The *hypotenuse* of a right triangle is the side opposite the right angle. The other two sides of a right triangle are called *legs*.

6.7. An *obtuse triangle* is a triangle with one obtuse angle.

6.8. An *acute triangle* is a triangle with three acute angles.

6.9. The *distance* between a point and a line not containing the point is the length of the perpendicular segment joining the point to the line. The distance between a line and a point on the line is defined to be zero.

6.10. 1. Any side of a triangle is a *base* of that triangle. Given a base of a triangle, the segment joining the opposite vertex to a point of the line containing its base, and perpendicular to the line containing the base, is the *altitude* corresponding to that base.

2. The length of any side of a triangle is a *base* of that triangle. The distance between the opposite vertex and the line containing that side is the corresponding *altitude*.

7.1. Two distinct lines which are coplanar and nonintersecting are *parallel* lines, and each is said to be parallel to the other. Also, a line is parallel to itself. The lines in a set of lines are said to be parallel lines if each two in the set are parallel.

7.2. Two lines which do not lie in the same plane are called *skew* lines.

7.3. A *transversal* of two distinct coplanar lines is a line which intersects their union in exactly two distinct points.

7.4. Two coplanar angles are *alternate interior angles* if their intersection is a segment and if their interiors do not intersect.

7.5. Two coplanar angles are *consecutive interior angles* if their intersection is a segment, or a segment and a point, and if their interiors intersect.

7.6. Two coplanar angles are *corresponding angles* if their intersection is a ray, or a ray and a point, and if their interiors intersect.

7.7. If the lines which contain two segments are parallel, then the segments are said to be *parallel segments*, and each is said to be *parallel* to the other. The segments in a set of segments are parallel if every two of them are parallel.

7.8. A *parallelogram* is a quadrilateral each of whose sides is parallel to the side opposite it.

7.9. A *trapezoid* is a convex quadrilateral with at least two parallel sides.

7.10. A *rhombus* is a parallelogram with two adjacent sides congruent.

7.11. A *rectangle* is a parallelogram with a right angle.

7.12. A *square* is a rectangle with two adjacent sides congruent.

7.13. The *distance between two distinct parallel lines* is the length of a segment which is perpendicular to both lines and whose endpoints lie on these lines, one endpoint on one line and the other endpoint on the other line. *The distance between a line and itself* is zero.

7.14. Two noncollinear rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are *parallel* if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines and if B and D lie on the same side of $\overleftrightarrow{AC}$. Two collinear rays are *parallel* if one of them is a subset of the other.

7.15. Two noncollinear rays $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are *antiparallel* if $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are parallel lines and if F and H lie on opposite sides of $\overleftrightarrow{EG}$. Two collinear rays are *antiparallel* if neither is a subset of the other.

8.1. A line and a plane are *perpendicular* if the line intersects the plane and is perpendicular to every line in the plane through the point of intersection.

8.2. If A and B are distinct points, the unique plane that is perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$ is called the *perpendicular bisecting plane* of $\overline{AB}$.

8.3. Two planes are *parallel* if their intersection is not a line.

8.4. A line and a plane are *parallel* if their intersection is not a point.

8.5. The intersection of a dihedral angle and a plane perpendicular to its edge is a *plane angle of the dihedral angle.*

8.6. The *measure of a dihedral angle* is the measure of any one of its plane angles.

8.7. A *right dihedral angle* is a dihedral angle whose measure is 90.

8.8. Two planes are *perpendicular* if their union is the union of four right dihedral angles.

8.9. A segment, or ray, is *perpendicular* to a plane if the line which contains it is perpendicular to the plane. If a segment is perpendicular to a plane and one endpoint lies in the plane, then that segment is a *perpendicular* to the plane, and its endpoint in the plane is called the *foot of the perpendicular.*

8.10. If α is a plane and S is a set of points, then the *projection* of S on α is the set of all points Q, each of which is the foot of the perpendicular from some point of S.

8.11. The *distance between a point and a plane* not containing it is the length of the perpendicular segment joining the given point to the given plane.

8.12. The *distance between two distinct parallel planes* is the length of a segment that joins a point of one of the planes to a point of the other plane and is perpendicular to both of them.

9.1. A *polygonal region* is a triangular region, or it is the union of a finite number (two or more) of triangular regions such that the intersection of every two of them is the null set, or a vertex of each of them, or a side of each of them.

9.2. 1. Any side of a parallelogram is a *base* of that parallelogram. Given a base of a parallelogram, the segment whose endpoints are on the line containing the base and the line containing the side that is opposite that base, and perpendicular to these lines, is the *altitude* corresponding to that base.

2. The length of any side of a parallelogram is a *base* of that parallelogram. The distance between the parallel lines containing that side and the side that is opposite to it is the corresponding *altitude*, or *height*.

10.1. Let a one-to-one correspondence between the real numbers $a, b, c, \ldots$ and the real numbers $a', b', c', \ldots$ in which a is matched with a', b is matched with b', c is matched with c', and so on, be given. Then the numbers $a, b, c, \ldots$ are said to be *proportional* to the numbers $a', b', c', \ldots$ if there is a nonzero number k such that $a = ka'$, $b = kb'$, $c = kc'$, $\ldots$. The number k is called the *constant of proportionality*.

10.2. If a, b, c, d are numbers such that $(a, b) \underset{p}{=} (c, d)$ is a proportionality, then that proportionality is a *proportion*.

10.3. 1. A one-to-one correspondence $ABC\ldots \longleftrightarrow A'B'C'\ldots$ between the vertices of polygon $ABC\ldots$ and polygon $A'B'C'\ldots$ is a *similarity* between the polygons if and only if corresponding angles are congruent and lengths of corresponding sides are proportional.

2. If $ABC\ldots \longleftrightarrow A'B'C'\ldots$ is a similarity, then polygon $ABC\ldots$ and polygon $A'B'C'\ldots$ are *similar polygons* and each is *similar* to the other.

3. If $ABC\ldots \longleftrightarrow A'B'C'\ldots$ is a similarity with $AB = kA'B'$, $BC = kB'C'$, and so on, then k is the *constant of proportionality*, or the *proportionality constant*, for that similarity.

10.4. If P is a point and l is a line, the *projection of P on l* is (1) the point P if P is on l and (2) the foot of the perpendicular from P to l if P is not on l.

10.5. The *projection of a set S on a line l* is the set of all points Q on l such that each Q is the projection on l of some P in S.

10.6. If a and b are positive numbers such that $(a, x) \underset{p}{=} (x, b)$ or that $(x, a) \underset{p}{=} (b, x)$, then x is called a *geometric mean* of a and b.

11.1. The one-to-one correspondence between the set of all points in an xy-plane and the set of all ordered pairs of real numbers in which each point P in the plane corresponds to the ordered pair (a, b), in which a is the x-coordinate of P and b is the y-coordinate of P, is an *xy-coordinate system*.

11.2. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points and if $x_1 \neq x_2$, then the *slope* of $\overline{AB}$ is $(y_2 - y_1)/(x_2 - x_1)$.

11.3. *The slope of a nonvertical line* is equal to the slope of any of its segments. *The slope of a nonvertical ray* is equal to the slope of the line that contains the ray.

12.1. Given an x-axis, a y-axis, and a z-axis, the one-to-one correspondence between all the points in space and all the ordered triples of real numbers in which each point P corresponds to the ordered triple (a, b, c) where a, b, c are the x-, y-, z-coordinates, respectively, of P, is the *xyz-coordinate system*.

13.1. Let r be a positive number and let O be a point in a given plane. The set of all points P in the given plane such that $OP = r$ is called a *circle*. The given point O is called the

center of the circle, the given number r is called the *radius* of the circle, and the number $2r$ is called the *diameter* of the circle.

13.2. Let r be a positive number and let O be a point in space. The set of all points P in space such that $OP = r$ is called a *sphere*. The given point O is called the *center* of the sphere, the given number r is called the *radius* of the sphere, and the number $2r$ is called the *diameter* of the sphere.

13.3. Two or more coplanar circles, or two or more spheres, with the same center are said to be *concentric*.

13.4. A *chord* of a circle or a sphere is a segment whose endpoints are points of the circle or sphere. A *secant* of a circle or sphere is a line containing a chord of the circle or sphere. A *diameter* of a circle or sphere is a chord containing the center of the circle or sphere. A *radius* of a circle or sphere is a segment with one endpoint at the center and the other endpoint on the circle or sphere.

13.5. Two circles (distinct or not) are *congruent* if their radii are equal. Two spheres (distinct or not) are *congruent* if their radii are equal.

13.6. Let a circle with center O and radius r in plane α be given. The *interior* of the circle is the set of all points P in plane α such that $OP < r$. The *exterior* of the circle is the set of all points P in plane α such that $OP > r$.

13.7. If a line in the plane of a circle intersects the circle in exactly one point, the line is called a *tangent* to the circle and the point is called the *point of tangency*, or the *point of contact*. We say that the line and the circle are *tangent* at this point. If a segment or a ray intersects a circle and if the line that contains that segment or ray is tangent to the circle, then the segment or ray is said to be *tangent* to the circle.

13.8. Two circles are *tangent* if and only if they are coplanar and tangent to the same line at the same point. If the centers of the tangent circles are on the same side of the tangent line, the circles are said to be *internally tangent*. If their centers are on opposite sides of the tangent line, the circles are said to be *externally tangent*.

13.9. Let a sphere with center O and radius r be given. The *interior* of the sphere is the set of all points P in space such that $OP < r$. The *exterior* of the sphere is the set of all points P in space such that $OP > r$.

13.10. If a plane intersects a sphere in exactly one point, the plane is called a *tangent plane* to the sphere. The point is called the *point of tangency*, or the *point of contact*, and we say that the plane and the sphere are tangent at this point.

13.11. A circle that is the intersection of a sphere with a plane through the center of the sphere is called a *great circle* of the sphere.

13.12. An angle which is coplanar with a circle and has its vertex at the center of the circle is called a *central angle*.

13.13. If A and B are distinct points on a circle with center P and if A and B are not the endpoints of a diameter of the circle, then the union of A, B, and all points of the circle in the interior of $\angle APB$ is called a *minor arc* of the circle. The union of A, B, and all points of the circle in the exterior of $\angle APB$ is called a *major arc* of the circle. If A and B are the endpoints of a diameter of the circle, then the union of A, B, and all points of the circle in one of the two halfplanes, with edge $\overleftrightarrow{AB}$, lying in the plane of the circle is called a *semicircle*.

13.14. If $\widehat{AXB}$ is any arc of a circle with center P, then its *degree measure* (denoted by $m\widehat{AXB}$) is given as follows:

1. If $\widehat{AXB}$ is a minor arc, then $m\widehat{AXB}$ is the measure of the associated central angle; that is,

$$m\widehat{AXB} = m\angle APB.$$

2. If $\widehat{AXB}$ is a semicircle, then

$$m\widehat{AXB} = 180.$$

3. If $\widehat{AXB}$ is a major arc and $\widehat{AYB}$ is the corresponding minor arc, then

$$m\widehat{AXB} = 360 - m\widehat{AYB}.$$

13.15. An angle is said to be *inscribed* in an arc of a circle and is called an *inscribed angle* if and only if both of the following conditions are satisfied:

1. Each side of the angle contains an endpoint of the arc.
2. The vertex of the angle is a point, but not an endpoint, of the arc.

13.16. An angle is said to *intercept an arc* of a circle and the arc is called an *intercepted arc* of the angle if and only if all three of the following conditions are satisfied:

1. The endpoints of the arc lie on the angle.
2. Each side of the angle contains at least one endpoint of the arc.
3. Each point of the arc, except its endpoints, lies in the interior of the angle.

13.17. Two arcs (not necessarily distinct) are *congruent* if and only if they have the same measure and are arcs of congruent circles.

13.18. If V and T are distinct points and if the line $\overleftrightarrow{VT}$ is tangent to a circle at T, then the segment $\overline{VT}$ is called a *tangent-segment* from V to the circle. If secant $\overleftrightarrow{PA}$ intersects a circle in points A and B such that A is between P and B, then the segment $\overline{PB}$ is called a *secant-segment* from P to the circle and the segment $\overline{PA}$ is called an *external secant-segment* from P to the circle.

13.19. Given a circle S with center O and radius r, and a point P in the same plane with S, the *power* of P with respect to S is $(OP)^2 - r^2$.

14.1. A *regular polygon* is a convex polygon all of whose sides are congruent and all of whose angles are congruent.

14.2. Each angle of a convex polygon is called an *interior* angle of the polygon. An angle which forms a linear pair with an interior angle of a convex polygon is called an *exterior* angle of the polygon. Each exterior angle is said to be *adjacent* to the interior angle with which it forms a linear pair.

14.3. The circle which contains the three vertices of a given triangle is called the *circumscribed circle,* or *circumcircle,* of the triangle and we say that it *circumscribes* the triangle. The triangle is said to be *inscribed* in the circle and is called an *inscribed triangle* of the circle.

14.4. The *circumcenter* of a regular polygon is the center of its circumscribed circle. A *circumradius* of a regular polygon is a segment (or its length) joining the center of the polygon to one of the vertices of the polygon. An *inradius* of a regular polygon is a segment (or its length) whose endpoints are the center of the polygon and the foot of the perpendicular from the center of the polygon to a side of the polygon. A *central angle* of a regular polygon is an angle whose vertex is at the center of the polygon and whose sides contain adjacent vertices of the polygon.

14.5. A circle is said to be *inscribed* in a polygon and is called an *inscribed circle* or *incircle,* of the polygon if each of the sides of the polygon is tangent to the circle. We also say that the polygon *circumscribes* the circle. The center of an incircle of a polygon is an *incenter* of the polygon.

14.6. The *circumference* of a circle is the limit of the sequence of perimeters p_n of the inscribed regular polygons (that is, $C = \lim p_n$).

14.7. If C is the circumference of a circle and d is its diameter, then the number C/d, which is the same for all circles, is denoted by the Greek letter π.

14.8. A *circular region* is the union of a circle and its interior.

14.9. The *area of a circle* is the limit of the sequence of areas of the inscribed regular polygons.

14.10. The *length of arc* $\widehat{AB}$ (denoted by $l\widehat{AB}$) is the limit of $\{A_n\}$ where

$$A_n = AP_1 + P_1P_2 + \cdots + P_{n-1}B$$

and where $P_1, P_2, \ldots, P_{n-1}$ are $n - 1$ distinct points of $\widehat{AB}$ subtending congruent angles at the center V of the circle containing $\widehat{AB}$.

14.11. In the same circle or in congruent circles, two arcs are congruent if and only if they have the same length.

14.12. Given a circle of radius r with center P, and an arc $\widehat{AB}$ of this circle, the union of all segments $\overline{PQ}$ such that Q is a point on arc $\widehat{AB}$ is called a *sector.* We call $\widehat{AB}$ the *arc of the sector* and we call r the *radius of the sector.*

15.1. Let α and β be distinct parallel planes. Let Q and Q' be points in α and β, respectively. Let R be a polygonal region in α. For each point P in R let P' be the point in β such that $\overline{PP'} \| \overline{QQ'}$. The union of all such segments $\overline{PP'}$ is a *prism.* If $\overline{QQ'}$ is perpendicular to α and β, the prism is a *right prism.*

15.2. Let R' be the polygonal region consisting of all the points P' in β. The polygonal regions R and R' are called the *bases* of the prism. Depending upon the orientation of the prism it is sometimes convenient to call one of the bases the *lower base* and the other base the *upper base.* Sometimes we call the lower base simply *the base.* A segment that is perpendicular to both α and β and with its endpoints in these planes is *an*

altitude of the prism. Sometimes the length of an altitude is called *the altitude* of the prism.

15.3. If a plane parallel to the plane of the base of a prism intersects the prism, the intersection is called a *cross section* of the prism.

15.4. A *lateral edge* of a prism is a segment $\overline{AA'}$ where A is a vertex of the base and A' is the corresponding vertex of the upper base. A *lateral face* is the union of all segments $\overline{PP'}$ of which P is a point in an edge of one base and P' is the corresponding point in the edge of the other base. The *lateral surface* of a prism is the union of its lateral faces. The *total surface* of a prism is the union of its lateral surface and its bases.

15.5. A *parallelepiped* is a prism whose base is a parallelogram region. A *rectangular parallelepiped* is a right prism whose base is a rectangular region. A *cube* is a rectangular parallelepiped all of whose edges are congruent. A *diagonal* of a parallelepiped is a segment joining any two of its vertices which are not contained in the same lateral face or base of the parallelepiped.

15.6. The *lateral surface area* of a prism is the sum of the areas of its lateral faces. The *total surface area* of a prism is the sum of the lateral surface area and the areas of the two bases.

15.7. Let R be a polygonal region in a plane α and V a point not in α. For each point P of R there is a segment $\overline{PV}$. The union of all such segments is called a *pyramid*. The polygonal region R is called the *base* and V is called the *vertex* of the pyramid. The distance VT from V to α is the *altitude* of the pyramid.

15.8. If the center of the base circle of a cone is the foot of the perpendicular from the vertex V to plane α, the cone is called a *right circular cone*.

15.9. A *spherical region* is the union of a sphere and its interior.

THEOREMS

1.1. There are at least three distinct lines in space.

1.2. If two distinct lines intersect, then they intersect in exactly one point.

1.3. Space contains at least two distinct planes.

1.4. If a line intersects a plane which does not contain the line, then the intersection is a single point.

1.5. If m is a line and P is a point not on m, then there is exactly one plane which contains m and P.

1.6. If two distinct lines intersect, then there is exactly one plane that contains them.

2.1. The intersection of two convex sets is a convex set.

2.2. If a segment has only one endpoint on a given line, then the entire segment, except for that endpoint, lies in one halfplane whose edge is the given line.

2.3. If the intersection of a line and a ray is the endpoint of the ray, then the interior of the ray is contained in one halfplane whose edge is the given line.

2.4. If P is any point in the interior of $\angle ABC$, then the interior points of ray $\overrightarrow{BP}$ are points of the interior of $\angle ABC$.

2.5. The intersection of the interiors of two angles of a triangle is the interior of the triangle.

2.6. If a line and a triangle are coplanar, if the line does not contain a vertex of the triangle, and if the line intersects one side of the triangle, then it also intersects just one of the other two sides.

3.1. If $\overline{PQ}$ and $\overline{RS}$ are segments such that the length of $\overline{RS}$ in $\overline{PQ}$ units is 1, then for all points A and B it is true that AB (in $\overline{RS}$ units) $= AB$ (in $\overline{PQ}$ units).

3.2. *The Origin and Unit Point Theorem.* If P and Q are any two distinct points, then there is a unique coordinate system on $\overleftrightarrow{PQ}$ with P as origin and Q as unit point.

3.3. Congruence for segments is reflexive, symmetric, and transitive.

3.4. *The Length-Addition Theorem for Segments.* If distinct points B and C are between points A and D and if $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

COROLLARY 3.4.1. If distinct points B and C are between points A and D and if $\overline{AC} \cong \overline{BD}$, then $\overline{AB} \cong \overline{CD}$.

COROLLARY 3.4.2. If A, B, C, D, E, F are points such that $A\text{–}B\text{–}C$, $D\text{–}E\text{–}F$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

COROLLARY 3.4.3. If A, B, C, D, E, F are points such that $A\text{–}B\text{–}C$, $D\text{–}E\text{–}F$, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, then $\overline{BC} \cong \overline{EF}$.

3.5. *Segment Construction Theorem.* Given a segment $\overline{CD}$ and a ray $\overrightarrow{AB}$, there is exactly one point P on $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.

LEMMA 3.6.1. Let x_1 and x_2 be the coordinates of distinct points X_1 and X_2, respectively, on a line l. If x is the coordinate of a point X on l, then

$$\frac{XX_1}{X_2X_1} = \frac{x - x_1}{x_2 - x_1} \text{ if } X \in \overrightarrow{X_1X_2},$$

and

$$\frac{XX_1}{X_2X_1} = -\frac{x - x_1}{x_2 - x_1} \text{ if } X \in \text{opp } \overrightarrow{X_1X_2}.$$

3.6. *The Two Coordinate Systems Theorem.* If X_1 and X_2 are two distinct points of a line l, if the coordinates of X_1 and X_2 are x_1 and x_2, respectively, in a coordinate system $\mathcal{S}$, and x_1' and x_2', respectively, in a coordinate system $\mathcal{S}'$, then for every point X on l, it is true that

$$\frac{x - x_1}{x_2 - x_1} = \frac{x' - x_1'}{x_2' - x_1'}$$

where x and x' are the coordinates of X in $\mathcal{S}$ and in $\mathcal{S}'$, respectively.

COROLLARY 3.6.1. Let X_1 and X_2 be the origin and unit point, respectively, in a coordinate system $\mathcal{S}_k$ on a line l. Let x_1 and x_2 be the coordinates of X_1 and X_2, respectively, in a coordinate system $\mathcal{S}_x$ on l. Let k and x be the coordinates of a point X on l in the systems $\mathcal{S}_k$ and $\mathcal{S}_x$, respectively. Then

1. $\dfrac{XX_1}{X_2X_1} = |k|$

and

2. $\dfrac{x - x_1}{x_2 - x_1} = k$, that is, $x = x_1 + k(x_2 - x_1)$.

4.1. If $\angle AVB$ is any angle in a plane α and if $\mathcal{S}$ is the ray-coordinate system in α relative to V in which $cd\ \overrightarrow{VA} = 0$ and $cd\ \overrightarrow{VB} = m\angle AVB$, then the ray-coordinate of $\overrightarrow{VX}$ is:

1. 0 if $\overrightarrow{VX} = \overrightarrow{VA}$.
2. 180 if $\overrightarrow{VX} = opp\ \overrightarrow{VA}$.
3. between 0 and 180 if X is on the B-side of $\overleftrightarrow{VA}$.
4. between 180 and 360 if X is on the not-B-side of $\overleftrightarrow{VA}$.

4.2. *Angle Construction Theorem.* If $\angle DEF$ is any angle, if $\overrightarrow{VA}$ is

any ray, if $\mathcal{H}$ is any halfplane with edge $\overleftrightarrow{VA}$, then there is one and only one halfline $\overset{\circ\!\!\rightarrow}{VB}$ in $\mathcal{H}$ such that $\angle AVB \cong \angle DEF$.

4.3. If a ray-coordinate system in which $cd\ \overrightarrow{VA} = 0$, $cd\ \overrightarrow{VB} = b$, $cd\ \overrightarrow{VC} = c$ with $c < 180$ is given, then $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if b is between 0 and c.

4.4. *Angle Measure Addition Theorem.* If distinct rays $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are between rays $\overrightarrow{VA}$ and $\overrightarrow{VD}$ and if $\angle AVB \cong \angle CVD$, then $\angle AVC \cong \angle BVD$.

COROLLARY 4.4.1. If distinct rays $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are between rays $\overrightarrow{VA}$ and $\overrightarrow{VD}$ and if $\angle AVC \cong \angle BVD$, then $\angle AVB \cong \angle CVD$.

COROLLARY 4.4.2. If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$ are distinct coplanar rays such that $A\text{–}V\text{–}D$, B and C are on the same side of $\overleftrightarrow{AD}$ and $\angle AVB \cong \angle CVD$, then $\angle AVC \cong \angle BVD$.

4.5. Vertical angles are congruent.

4.6. If two angles form a linear pair of angles, then they are supplementary angles.

4.7. Complements of congruent angles are congruent.

4.8. Supplements of congruent angles are congruent.

4.9. Every angle has a unique midray.

LEMMA 4.10.1. If a point is in the interior of an angle, then it is an interior point of a ray between the sides of that angle.

4.10. The interior of an angle is the union of the interiors of all rays between the sides of the angle.

4.11. If $\overline{AB}$ is a segment joining an interior point of one side of an angle to an interior point of the other side, then the interior of $\overline{AB}$ is contained in the interior of the angle.

4.12. If the two angles in a linear pair are congruent, they are right angles.

4.13. Any two right angles are congruent.

4.14. For each point on a line in a plane, there is one and only one line which lies in the given plane, contains the given point, and is perpendicular to the given line.

5.1. *The Isosceles Triangle Theorem.* The base angles of an isosceles triangle are congruent.

COROLLARY 5.1.1. If a triangle is equilateral, then it is equiangular.

5.2. *Converse of the Isosceles Triangle Theorem.* If a triangle has two congruent angles, then the sides opposite these angles are congruent and the triangle is isosceles.

COROLLARY 5.2.1. If a triangle is equiangular, then it is equilateral.

5.3. The median to the base of an isosceles triangle bisects the vertex angle and is perpendicular to the base.

5.4. The midray of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

5.5. *The Perpendicular Bisector Theorem.* If, in a given plane α, P is a point on the perpendicular bisector of $\overline{AB}$, then P is equidistant from the endpoints of $\overline{AB}$.

5.6. Converse of the Perpendicular Bisector Theorem. If, in a given plane α, P is equidistant from the endpoints of $\overline{AB}$, then P lies on the perpendicular bisector $\overline{AB}$.

6.1. If x and y are numbers, then $x < y$ if and only if $y > x$.

6.2. $\overline{AB} > \overline{CD}$ if and only if $\overline{CD} < \overline{AB}$.

6.3. $\angle ABC > \angle DEF$ if and only if $\angle DEF < \angle ABC$.

6.4. Let three distinct collinear points A, B, C be given. Then A–C–B if and only if $AB > AC$ and $AB > BC$.

6.5. If point D is the interior of $\angle ABC$, then $m\angle ABC > m\angle ABD$ and $m\angle ABC > m\angle DBC$.

6.6. *The Exterior Angle Theorem.* Each exterior angle of a triangle is greater than either of its nonadjacent interior angles.

COROLLARY 6.6.1. If one of the angles of a triangle is a right angle, then the other two angles of the triangle are acute angles.

COROLLARY 6.6.2. If one of the angles of a triangle is an obtuse angle, then the other two angles of the triangle are acute angles.

6.7. Given a line and a point not on the line, there is one and only one line which contains the given point and which is perpendicular to the given line.

6.8. *Angle-Comparison Theorem.* If two sides of a triangle are not congruent, then the angles opposite them are not congruent and the greater angle lies opposite the greater side.

6.9. *Side-Comparison Theorem.* If two angles of a triangle are not congruent, then the sides opposite them are not congruent and the greater side lies opposite the greater angle.

COROLLARY 6.9.1. The hypotenuse of a right triangle is the longest side of the triangle.

COROLLARY 6.9.2. The shortest segment from a point to a line not containing the point is the segment perpendicular to the line.

6.10. *Triangle Inequality Theorem.* The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

6.11. *Side-Comparison Theorem for Two Triangles.* If two sides of one triangle are congruent, respectively, to two sides of a second triangle, and if the angle included by the sides of the first triangle is greater than the angle included by the sides of the second triangle, then the third side of the first triangle is greater than the third side of the second triangle.

6.12. *Angle-Comparison Theorem for Two Triangles.* If two sides of one triangle are congruent, respectively, to two sides of a second triangle, and if the third side of the first triangle is greater than the third side of the second triangle, then the angle included by the two sides of the first triangle is greater than the angle included by the two sides of the second triangle.

7.1. *Existence of Parallel Lines Theorem.* If l is a line and P is a point, then there is at least one line through P and parallel to l. If P is on l, there is exactly one line through P and parallel to l.

7.2. Let two distinct coplanar lines and a transversal be given. If the transversal is perpendicular to both lines, then the lines are parallel.

7.3. *Alternate Interior Angle Theorem.* If two alternate interior angles determined by two distinct coplanar lines and a transversal are congruent, then the lines are parallel.

7.4. *Corresponding Angle Theorem.* If two corresponding angles determined by two distinct coplanar lines and a transversal are congruent, then the lines are parallel.

7.5. *Consecutive Interior Angle Theorem.* If two consecutive interior angles determined by two distinct coplanar lines and a transversal are supplementary, then the lines are parallel.

7.6. *Converse of Alternate Interior Angle Theorem.* If two distinct

lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent.

7.7. *Converse of Corresponding Angle Theorem.* If two distinct lines are parallel, then any two corresponding angles determined by a transversal of the lines are congruent.

7.8. *Converse of Consecutive Interior Angle Theorem.* If two distinct lines are parallel, then any two consecutive interior angles determined by a transversal of the lines are supplementary.

7.9. Let a and b be two distinct coplanar lines, and let t be a transversal of them that is perpendicular to a. If t is perpendicular to b, the lines a and b are parallel.

7.10. Let a and b be two distinct coplanar lines, and let t be a transversal of them that is perpendicular to a. If a and b are parallel, then t is perpendicular to b.

7.11. Two coplanar lines parallel to the same line are parallel to each other.

7.12. Let three distinct coplanar lines with two of them parallel be given. If the third line intersects one of the two parallel lines, then it intersects the other also.

7.13. Let two sets $\mathcal{S}$ and $\mathcal{T}$ of parallel lines in a plane α be given. (This means that every two lines in $\mathcal{S}$ are parallel and that every two lines in $\mathcal{T}$ are parallel.) If one line in $\mathcal{S}$ is perpendicular to one line in T, then every line in S is perpendicular to every line in T.

7.14. If a convex quadrilateral is a parallelogram, then its opposite sides are congruent.

7.15. If a convex quadrilateral is a parallelogram, then its opposite angles are congruent.

7.16. If a convex quadrilateral is a parallelogram, then its diagonals bisect each other.

7.17. If two sides of a convex quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

7.18. If the diagonals of a convex quadrilateral bisect each other, then the quadrilateral is a parallelogram.

7.19. If each two opposite sides of a convex quadrilateral are congruent, then the quadrilateral is a parallelogram.

7.20. A rhombus is an equilateral parallelogram.

7.21. A rectangle is a parallelogram with four congruent angles.

7.22. A square is an equilateral rectangle.

7.23. A square is an equiangular rhombus.

7.24. The diagonals of a rhombus are perpendicular.

7.25. For every two distinct parallel lines there is a number that is the common length of all segments perpendicular to both of the given lines and with one endpoint on one of the given lines and one endpoint on the other one.

7.26. If $\angle ABC$ and $\angle DEF$ are coplanar angles with $\overrightarrow{BA}$ and $\overrightarrow{ED}$ parallel and with $\overrightarrow{BC}$ and $\overrightarrow{EF}$ parallel, then $\angle ABC \cong \angle DEF$.

7.27. If $\angle ABC$ and $\angle DEF$ are coplanar angles with $\overrightarrow{BA}$ and $\overrightarrow{ED}$ parallel and with $\overrightarrow{BC}$ and $\overrightarrow{EF}$ antiparallel, then $\angle ABC$ and $\angle DEF$ are supplementary.

7.28. If $\angle ABC$ and $\angle DEF$ are coplanar angles with $\overrightarrow{BA}$ and $\overrightarrow{ED}$ antiparallel and with $\overrightarrow{BC}$ and $\overrightarrow{EF}$ antiparallel, then $\angle ABC \cong \angle DEF$.

7.29. The sum of the measures of the angles of a triangle is 180.

7.30. The measure of an exterior angle of a triangle is equal to the sum of the measures of its nonadjacent interior angles.

7.31. Let a one-to-one correspondence between the vertices of two triangles be given. If two angles of one triangle are congruent, respectively, to the corresponding angles of the other triangle, then the third angles of the two triangles are also congruent.

7.32. *The S.A.A. Theorem.* Let a one-to-one correspondence between the vertices of two triangles be given. If two angles and a side opposite one of them in one triangle are congruent, respectively, to the corresponding parts of the second triangle, then the correspondence is a congruence.

7.33. The sum of the measures of the angles of a convex quadrilateral is 360.

7.34. The acute angles of a right triangle are complementary.

7.35. *The Hypotenuse-Leg Theorem.* Let there be a one-to-one correspondence between the vertices of two right triangles in which the vertices of the right angles correspond. If the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

8.1. If A, A', B, C, D are distinct points with B and C each equidistant from A and A' and with D on $\overleftrightarrow{BC}$, then D is equidistant from A and A'.

8.2. If a line is perpendicular to each of two distinct intersecting lines at their point of intersection, then it is perpendicular to the plane that contains them.

8.3. If a line and a plane are perpendicular, then the plane contains every line perpendicular to the given line at the point of intersection of the given line and the given plane.

8.4. Given a line and a point, there is a unique plane perpendicular to the line and containing the point.

8.5. The perpendicular bisecting plane of a segment is the set of all points equidistant from the endpoints of the segment.

8.6. Given two perpendicular lines, there is a unique line that is perpendicular to each of the given lines at their point of intersection.

8.7. If two lines are perpendicular to the same plane, they are parallel.

8.8. If one of two distinct parallel lines is perpendicular to a plane, then the other line is also perpendicular to that plane.

8.9. Given a plane and a point, there is a unique line containing the given point and perpendicular to the given plane.

8.10. If a plane intersects one of two distinct parallel lines but does not contain it, then it intersects the other line and does not contain it.

8.11. If a plane is parallel to one of two parallel lines, it is parallel to the other line also.

8.12. If a plane intersects two distinct parallel planes, the intersections are two distinct parallel lines.

8.13. If α, β, γ are three distinct planes such that β is parallel to γ and such that α intersects β, then α intersects γ.

8.14. If a line intersects one of two distinct parallel planes in a single point, then it intersects the other plane in a single point.

8.15. If a line is parallel to one of two distinct parallel planes, it is parallel to the other plane.

8.16. There is a unique plane that contains a given point and is parallel to a given plane.

8.17. Given a point and a plane, then every line containing the given point and parallel to the given plane lies in the plane containing the given point and parallel to the given plane.

8.18. Any two plane angles of a dihedral angle are congruent.

8.19. If a line is perpendicular to a plane, then any plane containing the given line is perpendicular to the given plane.

8.20. If a line is perpendicular to one of two parallel planes, then it is perpendicular to the other plane also.

8.21. If two planes are perpendicular, then any line in one of the planes and perpendicular to their line intersection is perpendicular to the other plane.

8.22. If two distinct intersecting planes are perpendicular to a third plane, then their line of intersection is perpendicular to the third plane.

8.23. The projection of a line on a plane is either a line or a point.
COROLLARY 8.23.1. The projection of a segment on a plane is either a point or a segment.

8.24. Given a plane α and two distinct points A and B such that $\overleftrightarrow{AB}$ is parallel to α, if $\overline{A'B'}$ is the projection of $\overline{AB}$ on α, then $\overline{A'B'} \cong \overline{AB}$.

8.25. Given parallel planes α and β and $\triangle ABC$ in α, if A', B', C' are the projections of A, B, C, respectively, on β, then $\triangle ABC \cong \triangle A'B'C'$.

8.26. The shortest segment joining a given point not in a given plane to a point in the given plane is the perpendicular that joins the given point to its projection in the given plane.

8.27. All segments that are perpendicular to each of two distinct parallel planes and have their endpoints in these planes have the same length.

9.1. If b is a base of a parallelogram and if h is the corresponding height, then the area S of the parallelogram is given by the formula $S = bh$.

9.2. If b is a base of a triangle and if h is the corresponding altitude, then the area S of the triangle is given by the formula $S = \frac{1}{2}bh$.

9.3. If b_1 and b_2 are the lengths of the parallel sides of a trapezoid and if h is the distance between the lines that contain these parallel sides, then the area S of the trapezoid is given by the formula $S = \frac{1}{2}(b_1 + b_2)h$.

9.4. *The Pythagorean Theorem.* If a, b, c are the lengths of the sides of a right triangle ABC, with $c = AB$ the length of the hypotenuse, then $a^2 + b^2 = c^2$.

9.5. *Converse of The Pythagorean Theorem.* If a, b, c are the lengths of the sides of a triangle and if $a^2 + b^2 = c^2$, then the triangle is a right triangle and the right angle is opposite the side of length c.

10.1. The proportionality relation is an equivalence relation.

10.2. If $(a, b, c) \underset{p}{=} (d, e, f)$, then
1. $(a, b, c, a + b + c) \underset{p}{=} (d, e, f, d + e + f)$,
and, if $h \neq 0$,
2. $(ha, b, c) \underset{p}{=} (hd, e, f)$.

10.3. Proportions involving nonzero numbers a, b, c, d have the following properties:
1. *Alternation Property:* If $(a, b) \underset{p}{=} (c, d)$, then $(a, c) \underset{p}{=} (b, d)$ and $(d, b) \underset{p}{=} (c, a)$.
2. *Inversion Property:* If $(a, b) \underset{p}{=} (c, d)$, then $(b, a) \underset{p}{=} (d, c)$.
3. *Product Property:* $(a, b) \underset{p}{=} (c, d)$ if and only if $ad = bc$.
4. *Ratio Property:* $(a, b) \underset{p}{=} (c, d)$ if and only if $a/b = c/d$.

10.4. The relation of similarity between polygons is reflexive, symmetric, and transitive.

10.5. The perimeters of two similar polygons are proportional to the lengths of any two corresponding sides.

10.6. *Triangle Proportionality Theorem.* If a line parallel to one side of a triangle intersects a second side in an interior point, then it intersects the third side in an interior point, and the lengths of the segments formed on the second side are proportional to the lengths of the segments formed on the third side.

10.7. *Converse of the Triangle Proportionality Theorem.* Let $\triangle ABC$ with points D and E such that A–D–B and A–E–C be given. If $(AD, AB) \underset{p}{=} (AE, AC)$, then $\overline{DE} \parallel \overline{BC}$.

10.8. If two distinct transversals cut three or more distinct lines that are coplanar and parallel, then the lengths of the segments formed on one transversal are proportional to the lengths of the segments formed on the other transversal.

COROLLARY 10.8.1. If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side.

10.9. If $\triangle ABC$ is any triangle and k is any positive number, then there is a triangle $\triangle A'B'C'$ such that $\triangle A'B'C' \sim \triangle ABC$ with constant of proportionality k.

10.10. S.S.S. *Similarity Theorem.* Given $\triangle ABC$ and $\triangle DEF$, if $(AB, BC, CA) \underset{p}{=} (DE, EF, FD)$, then $\triangle ABC \sim \triangle DEF$.

10.10. S.S.S. *Similarity Theorem—Alternate Form.* If the lengths of the sides of one triangle are proportional to the lengths of the corresponding sides of the other triangle, then the triangles are similar.

10.11. S.A.S. *Similarity Theorem.* Given $\triangle ABC$ and $\triangle DEF$, if $\angle A \cong \angle D$ and $(AB, AC) \underset{p}{=} (DE, DF)$, then $\triangle ABC \sim \triangle DEF$.

10.11. S.A.S. *Similarity Theorem—Alternate Form.* If an angle of one triangle is congruent to an angle of another triangle and if the lengths of the including sides are proportional to the lengths of the corresponding sides in the other triangle, then the triangles are similar.

COROLLARY 10.11.1. A segment which joins the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.

10.12. A.A. *Similarity Theorem.* Given $\triangle ABC$ and $\triangle DEF$, if $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$.

10.12. A.A. *Similarity Theorem—Alternate Form.* If two angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles are similar.

10.13. If triangles $\triangle ABC$ and $\triangle DEF$ are such that $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CA} \parallel \overline{FD}$, then $\triangle ABC \sim \triangle DEF$.

10.14. If two triangles are similar, then the lengths of any two corresponding altitudes are proportional to the lengths of any two corresponding sides.

10.15. If two triangles are similar, then their areas are proportional to the squares of the lengths of any two corresponding sides.

10.16. In any right triangle the altitude to the hypotenuse separates the right triangle into two triangles each similar to the original triangle, and hence also to each other.

COROLLARY 10.16.1. The square of the length of the altitude to the hypotenuse of a right triangle is equal to the product of the lengths of the projections of the legs on the hypotenuse.

COROLLARY 10.16.2. The square of the length of a leg of a right triangle is equal to the product of the lengths of the hypotenuse and the projection of that leg on the hypotenuse.

COROLLARY 10.16.1. Alternate Form. The length of the altitude to the hypotenuse of a right triangle is the geometric mean of the lengths of the projections of the legs on the hypotenuse.

COROLLARY 10.16.2. Alternate Form. The length of a leg of a right triangle is the geometric mean of the lengths of the hypotenuse and the projection of that leg on the hypotenuse.

10.17. *The Pythagorean Theorem.* In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

10.18. *Converse of the Pythagorean Theorem.* If $a^2 + b^2 = c^2$, where a, b, c are the lengths of the sides of a triangle, then the triangle is a right triangle with c the length of the hypotenuse.

10.19. The median to the hypotenuse of a right triangle is one-half as long as the hypotenuse.

10.20. *The 3, 4, 5 Theorem.* If x is any positive number, then every triangle with side lengths $3x$, $4x$, $5x$ is a right triangle.

10.21. *The 5, 12, 13 Theorem.* If x is a positive number and if the lengths of the sides of a triangle are $5x$, $12x$, and $13x$, then the triangle is a right triangle.

10.22. *The 1, 1, $\sqrt{2}$ Theorem.* If the lengths of the sides of a triangle are proportional to 1, 1, $\sqrt{2}$, then the triangle is an isosceles right triangle.

10.23. *The 1, $\sqrt{3}$, 2 Theorem.* If the lengths of the sides of a triangle are proportional to 1, $\sqrt{3}$, 2, then it is a right triangle with its shortest side half as long as its hypotenuse.

10.24. A triangle is a 30, 60, 90 triangle if and only if it is a 1, $\sqrt{3}$, 2 triangle with the shortest side opposite the 30 degree angle.

11.1. The correspondence which matches each point in the xy-plane with its xy-coordinates is a one-to-one correspondence between the set of all ordered pairs of real numbers and the set of all points in the xy-plane.

11.2. If $P(x_1, y_1)$ and $Q(x_2, y_1)$ are points on the same horizontal line in an xy-plane, then

$$PQ = |x_1 - x_2|.$$

11.3. If $P(x_1, y_1)$ and $Q(x_1, y_2)$ are points on the same vertical line in an xy-plane, then

$$PQ = |y_1 - y_2|.$$

11.4. If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are any two points in an xy-plane, then

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

11.5. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points in an xy-plane, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

11.6. If l is any nonvertical (nonhorizontal) line in an xy-plane, then the one-to-one correspondence between the points of l

and their x-coordinates (y-coordinates) is a coordinate system on l.

11.7. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two distinct points, then $\overleftrightarrow{AB} = \{(x, y) : x = x_1 + k(x_2 - x_1), \quad y = y_1 + k(y_2 - y_1),$ k is real$\}$.

11.8. If a, b, c, d are real numbers, if b and d are not both zero, and if $S = \{(x, y) : x = a + bk, y = c + dk, k \text{ is real}\}$, then S is a line.

11.9. The slopes of all segments of a nonvertical line are equal.

11.10. Given a point A and a real number m, there is one and only one line which contains A and has slope m.

11.11. The line l given by
1. $l = \{(x, y) : x = x_1 + k, y = y_1 + mk, k \text{ is real}\}$ or by
2. $l = \{(x, y) : x = x_1 + sk, y = y_1 + rk, k \text{ is real}\}$

is the line through (x_1, y_1) with slope $m = \dfrac{r}{s}$.

11.12. If l is the horizontal line through (x_1, y_1), then

$$l = \{(x, y) : y = y_1\}.$$

11.13. If l is the vertical line through (x_1, y_1), then

$$l = \{(x, y) : x = x_1\}.$$

11.14. *The Two-Point Form.* If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are distinct points, and if $\overleftrightarrow{AB}$ is an oblique line, then

$$\overleftrightarrow{AB} = \left\{(x, y) : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}\right\}.$$

11.15. *The Point Slope Form.* If l is the line through $A = (x_1, y_1)$ with slope m, then

$$l = \{(x, y) : y - y_1 = m(x - x_1)\}.$$

11.16. Two nonvertical lines are parallel if and only if their slopes are equal.

11.17. Two oblique lines are perpendicular if and only if the product of their slopes is -1.

11.18. A segment which joins the midpoints of two sides of a triangle is parallel to the third side and has half the length of the third side.

11.19. The medians of a triangle are concurrent in a point (centroid) which is two-thirds of the distance from each vertex to the midpoint of the opposite side.

11.20. Let quadrilateral $ABCD$ with $A = (0, 0)$, $B = (a, 0)$, $D = (b, c)$ be given. $ABCD$ is a parallelogram if and only if $C = (a + b, c)$.

11.21. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

11.22. The diagonals of a parallelogram bisect each other.

11.23. If the vertices of a parallelogram are $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, $D = (b, c)$, then the parallelogram is a rectangle if and only if $b = 0$.

11.24. The diagonals of a rectangle are congruent.

11.25. If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

11.26. A rectangle is a square if and only if its diagonals are perpendicular.

11.27. If the vertices of a parallelogram are $A = (0, 0)$, $B = (a, 0)$, $C = (a + b, c)$, and $D = (b, c)$, then the parallelogram is a rhombus if and only if $a^2 = b^2 + c^2$.

11.28. If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

11.29. The median of a trapezoid is parallel to each of the bases and its length is one-half the sum of the lengths of the two bases.

11.30. A trapezoid is isosceles if its diagonals are congruent.

11.31. If a line bisects one side of a triangle and is parallel to a second side, then the line bisects the third side of the triangle.

11.32. The midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.

11.33. The lines which contain the altitudes of a triangle are concurrent. (Their common point is called the *orthocenter* of the triangle.)

12.1. *Distance Formula Theorem.* The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

12.2. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are any two distinct points, then

$$\overleftrightarrow{PQ} = \left\{ (x, y, z) : \begin{array}{l} x = x_1 + k(x_2 - x_1), \\ y = y_1 + k(y_2 - y_1), \text{ and } k \text{ is real} \\ z = z_1 + k(z_2 - z_1), \end{array} \right\}$$

If $R = (a, b, c)$, where $a = x_1 + k(x_2 - x_1)$,
$b = y_1 + k(y_2 - y_1)$,
$c = z_1 + k(z_2 - z_1)$,

is a point of $\overleftrightarrow{PQ}$, then

R is the point P if $k = 0$,
$R \in \overrightarrow{PQ}$ and $PR = k \cdot PQ$ if $k \geq 0$,
$R \in opp\ \overrightarrow{PQ}$ and $PR = -k \cdot PQ$ if $k \leq 0$.

12.3. Given an xyz-coordinate system, every plane has a linear equation.

12.4. Given an xyz-coordinate system, the graph of every linear equation

$$ax + by + cz + d = 0,$$

in which a, b, c, d are real numbers and a, b, c are not all zero, is a plane.

12.5. If a, b, c, d are real numbers with a, b, c not all zero, then the plane

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}$$

is perpendicular to the line through $O(0, 0, 0)$ and $P(a, b, c)$.

12.6. Consider the plane

$$\alpha = \{(x, y, z) : ax + by + cz + d = 0\}.$$

If $a = 0$, α is parallel to the x-axis. If $b = 0$, α is parallel to the y-axis. If $c = 0$, α is parallel to the z-axis.

12.7. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two distinct points with $x_1 \neq x_2$, $y_1 \neq y_2$, $z_1 \neq z_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}\right\}.$$

12.8. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_1)$ are distinct points with $x_1 \neq x_2$ and $y_1 \neq y_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, z = z_1\right\}.$$

12.9. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_1, z_2)$ are distinct points with $x_1 \neq x_2$ and $z_1 \neq z_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{x - x_1}{x_2 - x_1} = \frac{z - z_1}{z_2 - z_1}, y = y_1\right\}.$$

12.10. If $P(x_1, y_1, z_1)$ and $Q(x_1, y_2, z_2)$ are two distinct points with $y_1 \neq y_2$ and $z_1 \neq z_2$, then

$$\overleftrightarrow{PQ} = \left\{(x, y, z) : \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, x = x_1\right\}.$$

12.11. If $P(x_1, y_1, z_1)$ and $Q(x_1, y_1, z_2)$ are two distinct points, then

$$\overleftrightarrow{PQ} = \{(x, y, z) : x = x_1 \text{ and } y = y_1\}.$$

12.12. If $P(x_1, y_1, z_1)$ and $Q(x_1, y_2, z_1)$ are two distinct points, then

$$\overleftrightarrow{PQ} = \{(x, y, z) : x = x_1 \text{ and } z = z_1\}.$$

12.13. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_1, z_1)$ are two distinct points, then

$$\overleftrightarrow{PQ} = \{(x, y, z) : y = y_1 \text{ and } z = z_1\}.$$

13.1. Let an xy-plane be given and let O be the origin and let r be a positive number. Let C be the circle in the xy-plane with center O and radius r. Then

$$C = \{(x, y) : x^2 + y^2 = r^2\}.$$

13.2. Given a line l and a circle C in the same plane, let O be the center of the circle and let P be the foot of the perpendicular from O to line l.

1. Every point of l is outside C if and only if P is outside C.
2. l is tangent to C if and only if P is on C.
3. l is a secant of C if and only if P is inside C.

13.3. Given a circle and a line in the same plane, if the line is tangent to the circle, then it is perpendicular to the radius whose outer end is the point of tangency.

13.4. Given a circle and a line in the same plane, if the line is perpendicular to a radius at its outer end, then the line is a tangent to the circle.

13.5. A diameter of a circle bisects a chord of the circle other than a diameter if and only if it is perpendicular to the chord.

13.6. In the plane of a circle, the perpendicular bisector of a chord contains the center of the circle.

13.7. Let a circle C and a line l in the plane of the circle be given. If l intersects the interior of C, then l intersects C in exactly two distinct points.

13.8. Chords of congruent circles are congruent if and only if they are equidistant from the centers of the circles.

13.9. Let O be a point, r a positive number, and S the sphere with center O and radius r. Given an xyz-coordinate system with origin O,

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}.$$

13.10. Given a sphere S with center O and a plane α which does not contain O, let P be the foot of the perpendicular from O to α.
1. Every point of α is in the exterior of S if and only if P is in the exterior of S.
2. α is tangent to S if and only if P is on S.
3. α intersects S in a circle with center P if and only if P is in the interior of S.

13.11. Let a sphere S with center O and radius r and a plane α be given. If the intersection of S with α contains the center O of the sphere, then the intersection is a circle whose center and radius are the same as those of the sphere.

13.12. The perpendicular from the center of a sphere to a chord of the sphere bisects the chord.

13.13. The segment joining the center of a sphere to the midpoint of a chord of the sphere is perpendicular to the chord.

13.14. A plane is tangent to a sphere if and only if it is perpendicular to a radius of the sphere at its outer end.

13.15. *Arc Measure Addition Theorem.* If A, B, C are distinct points on a circle, then

$$\mathrm{m}\widehat{ABC} = \mathrm{m}\widehat{AB} + \mathrm{m}\widehat{BC}.$$

COROLLARY 13.15.1. If $A_1, A_2, \ldots, A_n$ are n distinct points on a circle that partition the circle into n arcs $\widehat{A_1A_2}$, $\widehat{A_2A_3}, \ldots, \widehat{A_{n-1}A_n}, \widehat{A_nA_1}$, that intersect only at their endpoints, then $\mathrm{m}\widehat{A_1A_2} + \mathrm{m}\widehat{A_2A_3} + \cdots + \mathrm{m}\widehat{A_{n-1}A_n} + \mathrm{m}\widehat{A_nA_1} = 360$.

13.16. The measure of an inscribed angle is one-half the measure of its intercepted arc.

COROLLARY 13.16.1. Angles inscribed in the same arc are congruent.

COROLLARY 13.16.2. An angle inscribed in a semicircle is a right angle.

COROLLARY 13.16.3. Congruent angles inscribed in the same circle or in congruent circles intercept congruent arcs.

13.17. If two distinct parallel lines in the plane of a circle intersect that circle, they intercept congruent arcs.

13.18. In the same circle, or in congruent circles, two chords that are not diameters are congruent if and only if their associated minor arcs are congruent.

13.19. The measure of a tangent-chord angle is one-half the measure of its intercepted arc.

13.20. The measure of a secant-secant angle is one-half the difference of the measures of the intercepted arcs.

13.21. If an angle has its vertex in the exterior of a circle and if its sides consist of two secant-rays, or a secant-ray and a tangent-ray, or two tangent-rays to the circle, then the measure of the angle is $\boxed{?}$.

13.22. The measure of an angle whose vertex is in the interior of a circle and whose sides are contained in two secants is one-half the sum of the measures of the intercepted arcs.

13.23. If two chords of a circle intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other.

13.24. The two distinct tangent-segments to a circle with center O from an external point P are congruent and the angle whose vertex is P and whose sides contain the two tangent-segments is bisected by the ray $\overrightarrow{PO}$.

13.25. The product of the length of a secant-segment from a given exterior point of a circle and the length of its external secant-segment is the same for any secant to a given circle from a given exterior point.

13.26. Given a tangent-segment $\overline{PC}$ from P to a circle at C and a secant through P intersecting the given circle in points A and B, then

$$PA \cdot PB = (PC)^2.$$

13.27. Let a circle S with center O and radius r be given. Let P be a point in the plane of S and let p be the power of P with respect to S. If a line through P intersects S in points A and B, then

1. $PA \cdot PB = -p$ if P is inside S, and
2. $PA \cdot PB = p$ if P is on S or outside S.

14.1. The sum of the measures of the angles of a convex polygon of n sides is $(n - 2)180$.

COROLLARY 14.1.1. The measure of each angle of a regular polygon of n sides is

$$\frac{(n-2)180}{n}.$$

14.2. The sum of the measures of the exterior angles, one at each vertex, of a convex polygon of n sides is 360.

COROLLARY 14.2.1. The measure of each exterior angle of a regular polygon of n sides is $360/n$.

14.3. A given triangle has exactly one circumcircle.

14.4. A given regular polygon has exactly one circumcircle.

14.5. Let a regular polygon of n sides be given. Then all of the central triangles of the given polygon are congruent and all of the central angles of the given polygon are congruent.

14.6. All the inradii of a given regular polygon are congruent.

14.7. There is exactly one circle that is inscribed in a given regular polygon.

14.8. Two regular polygons are similar if they have the same number of sides.

14.9. The perimeters of two regular polygons, with the same number of sides, are proportional to the lengths of their sides, or their circumradii, or their inradii.

14.10. The area of a regular polygon is equal to one-half the product of its inradius and perimeter, that is, $S = \frac{1}{2}ap$.

14.11. The areas of two regular polygons with the same number of sides are proportional to the squares of their circumradii (or the squares of their inradii, or the squares of their side lengths, or the squares of their perimeters).

14.12. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers with nth terms x_n and y_n, respectively.

1. If the limit of x_n is L_1 and the limit of y_n is L_2, then the sequence whose nth term is $x_n y_n$ has a limit, and $\lim x_n y_n = L_1 \cdot L_2$.
2. If the limit of x_n is L_1 and the limit of y_n is $L_2 \neq 0$, then the sequence whose nth term is x_n/y_n has a limit and $\lim x_n/y_n = L_1/L_2$.
3. If $x_n = y_n$ for every positive integer $n \geq 1$ and if $\{x_n\}$ and $\{y_n\}$ each has a limit, then $\lim x_n = \lim y_n$.
4. If k is a real number and if $x_n = k$, for every $n \geq 1$, then $\lim x_n = k$.

14.13. If C and C' are the circumferences of any two circles with diameters d and d', respectively, then

$$(C, d) \underset{p}{=} (C', d').$$

COROLLARY 14.13.1. If C and d are the circumference and diameter, respectively, of a circle, then the number $\frac{C}{d}$ is the same for all circles.

14.14. The area S of a circle with radius r is πr^2, that is,

$$S = \pi r^2.$$

COROLLARY 14.14.1. The areas of two circles are proportional to the squares of their radii.

14.15. The lengths of arcs of congruent circles are proportional to their degree measures.

14.16. The length L of an arc of degree measure M contained in a circle with radius r is $(M/180)\pi r$, that is,

$$L = \left(\frac{M}{180}\right)\pi r.$$

14.17. The area S of a sector is one-half the product of its radius r and the length L of its arc, that is,

$$S = \tfrac{1}{2}rL.$$

14.18. If M is the degree measure of the arc of a sector with radius r, then the area S of the sector is $(M/360)\pi r^2$, that is,

$$S = \left(\frac{M}{360}\right)\pi r^2.$$

15.1. The boundary of each cross section of a triangular prism is congruent to the boundary of the base of the prism.

COROLLORY 15.1.1. The boundaries of the upper and lower bases of a triangular prism are congruent.

15.2. *The Prism Cross Section Theorem.* All cross sections of a prism have the same area.

COROLLARY 15.2.1. The two bases of a prism have the same area.

15.3. The lateral faces of a prism are parallelogram regions and the lateral faces of a right prism are rectangular regions.

15.4. The boundary of each cross section of a triangular pyramid is

a triangle similar to the boundary of the base, and the areas of any two cross sections are proportional to the squares of the distances of their planes from the vertex of the pyramid.

15.5. In any pyramid, the areas of any two cross sections are proportional to the squares of the distances of their planes from the vertex of the pyramid.

15.6. *The Pyramid Cross Section Theorem.* If two pyramids have equal altitudes and if their bases have equal areas, then cross sections equidistant from the vertices have equal areas.

15.7. The volume V of any prism is the product of its altitude h and the area S of its base, that is, $V = Sh$.

15.8. The boundary of each cross section of a cylinder is a circle that is congruent to the boundary of the base.

15.9. *The Cylinder Cross Section Theorem.* The area of a cross section of a cylinder is equal to the area of the base.

15.10. The volume of a cylinder is the product of the altitude and the area of the base.

15.11. The lateral surface area of a cylinder of base radius r and altitude h is $2\pi rh$, and its total surface area is $2\pi rh + 2\pi r^2$.

15.12. Two pyramids with the same altitude and the same base area have the same volume.

15.13. The volume of a triangular pyramid is one-third the product of its base area and its altitude.

15.14. The volume of a pyramid is one-third the product of its base area and its altitude.

15.15. *The Cone Cross Section Theorem.* A cross section of a cone of altitude h, made by a plane at a distance k from the vertex, is a circular region whose area and the area of the base are proportional to k^2 and h^2.

15.16. The volume of a circular cone is one-third the product of the area of the base and the altitude.

15.17. The lateral surface area of a right circular cone of slant height s, base radius r, and base circumference C is $\frac{1}{2}sC$, or πrs, and its total surface area is $\pi r(s + r)$.

15.18. The surface area of a sphere is 4π times the square of the radius of the sphere; that is, $S = 4\pi r^2$.

15.19. If r is the radius of a sphere, the volume of the sphere is $\frac{4}{3}\pi r^3$.

Squares, Cubes, Square Roots, Cube Roots

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

Glossary

Abscissa. See **Ordered Pair of Real Numbers.**

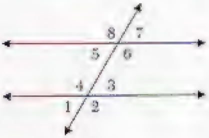
Angle. The union of two noncollinear rays that have a common endpoint. Each of the two rays is called a *side* of the angle. The common endpoint of the two rays is called the *vertex* of the angle.

acute angle. An angle whose measure is less than 90.

angle pairs.

adjacent angles. Two coplanar angles are called a pair of adjacent angles if they have one side in common and the intersection of their interiors is empty.

alternate interior angles. Two coplanar angles whose intersection is a segment and whose interiors do not intersect. In the figure, angles 3 and 5 (or 4 and 6) are a pair of alternate interior angles.



consecutive interior angles. Two coplanar angles whose intersection is a segment, or a segment and a point, and whose interiors intersect. In the figure, angles 3 and 6 (or 4 and 5) are a pair of consecutive interior angles.

corresponding angles. Two coplanar angles whose intersection is a ray, or a ray and a point, and whose interiors intersect. In the figure, angles 3 and 7 (or 4 and 8, or 2 and 6, or 1 and 5) are a pair of corresponding angles.

complementary angles. Two angles (distinct or not) are complementary, and each is called a *complement* of the other if the sum of their measures is 90.

linear pair of angles. Two angles that have one side in common and whose other sides are opposite rays.

supplementary angles. Two angles (distinct or not) are supplementary, and each is called a *supplement* of the other if the sum of their measures is 180.

vertical angles. Two angles whose sides form two pairs of opposite rays.

central angle. An angle that is coplanar with a circle and has its vertex at the center of the circle.

chord-chord angle. An angle determined by the intersection of two distinct chords of a circle. The *measure* of a chord-chord angle is one-half the sum of the measures of the intercepted arcs.

dihedral angle. If two noncoplanar halfplanes have the same edge, then the union of these halfplanes and the line that is their common edge is a dihedral angle. The union of this common edge and either one of these two halfplanes is a *face* of the dihedral angle.

measure of a dihedral angle. The measure of any one of its plane angles.

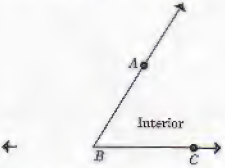
plane angle of a dihedral angle. The intersection of a dihedral angle and a plane perpendicular to its edge.

right dihedral angle. A dihedral angle whose measure is 90.

exterior of an angle. The set of all points in the plane of the angle except those points on the sides of the angle and in its interior.

inscribed angle. An angle is said to be inscribed in an arc of a circle and is called an inscribed angle if and only if each side of the angle contains an endpoint of the arc and the vertex of the angle is a point, but not an endpoint, of the arc. The *measure* of an inscribed angle is one-half the measure of its intercepted arc.

interior of. The interior of an angle, say $\angle ABC$, is the intersection of two half-planes, the C-side of $\overleftrightarrow{AB}$ and the A-side of $\overleftrightarrow{BC}$.



measure of. The measure of an angle in our formal geometry is the unique number (between 0 and 180) associated with the angle as stated in the Angle Measure Existence Postulate. The formal geometry idea of angle measure grows out of the informal geometry idea of degree measure of an angle as the number of degree units between the sides of the angle.

obtuse angle. An angle whose measure is greater than 90.

right angle. An angle whose measure is 90.

secant-secant angle. An angle whose sides are two secant-rays. The *measure* of a secant-secant angle is one-half the difference of the measures of the intercepted arcs.

secant-tangent angle. An angle whose sides are a secant-ray and a tangent-ray. The *measure* of a secant-tangent angle is one-half the difference of the measures of the intercepted arcs.

tangent-chord angle. If $\overline{AB}$ is a chord of a circle and if AC is tangent to the circle at A, then $\angle BAC$ is called a tangent-chord angle. The *measure* of a tangent-chord angle is one-half the measure of its intercepted arc.

tangent-tangent angle. An angle whose sides are two tangent rays. The *measure* of a tangent-tangent angle is one-half the difference of the measures of the intercepted arcs.

Area Formulas, with S = area.

Circle. $S = \pi r^2$, where r is the radius.

Parallelogram. $S = bh$, where b is a base and h is the corresponding height.

Rectangle. $S = ab$, where a and b are the lengths of two adjacent sides.

Right triangle. $S = \frac{1}{2} \cdot ab$, where a and b are the lengths of the legs.

Trapezoid. $S = \frac{1}{2} \cdot h(b_1 + b_2)$, where b_1 and b_2 are the lengths of two parallel sides and h is the distance between the parallel lines containing those sides.

Triangle. $S = \frac{1}{2} \cdot bh$, where b is a base and h is the corresponding height.

Bisect. To separate into two congruent parts.

Circle. The set of all points in a plane at a fixed distance from a fixed point in the plane.

arc of. A subset of a circle bounded by two distinct points of the circle.

area of. See **Area Formulas, Circle.**

center of. The point in the interior of a circle from which all points of the circle are equidistant.

central angle of a circle. An angle in the plane of the circle whose vertex is at the center of the circle.

chord of a circle. A segment whose endpoints are on a circle.

circumcircle. A circle that contains all the vertices of a polygon is called a circumcircle, or **circumscribed circle,** of the polygon.

circumference of. The circumference C of a circle with radius r is given by the formula $C = 2\pi r$.

concentric circles. Circles that lie in the same plane and have the same center.

congruent circles. Circles are called congruent circles if and only if their radii are equal.

diameter of. A chord of a circle that contains the center of the circle.

exterior of. The set of all points in the plane of the circle whose distance from the center of the circle is greater than the radius of the circle.

incircle. A circle that is tangent to all the sides of a polygon is called an incircle, or inscribed circle, of the polygon.

inscribed angle of. See **Angle, inscribed angle.**

interior of. The set of all points in the plane of a circle whose distance from the center of the circle is less than the radius of the circle.

power of a point with respect to a circle. Given a circle S with center O and radius r, and a point P in the same plane with S, the power of P with respect to S is $(OP)^2 - r^2$.

radius of a circle. A segment whose endpoints are the center of a circle and a point of the circle. Also the number that is the length of all radii (plural of radius) of the same circle.

secant of. A line in the plane of a circle that intersects the circle in two distinct points.

sector of. Given a circle of radius r with center P, and an arc $\widehat{AB}$ of this circle, the union of all segments $\overline{PQ}$ such that Q is a point on arc $\widehat{AB}$ is called a sector. We call $\widehat{AB}$ the *arc of the sector* and we call r the *radius of the sector.*

segment of. Let $\overline{AB}$ be a chord of a circle and let $\widehat{AB}$ be the corresponding arc. The union of all segments PQ such that P is a point of $\overline{AB}$ and Q is a point of $\widehat{AB}$ is called a segment of the circle.

tangent of. A line in the plane of a circle that intersects the circle in exactly one point.

Collinear. The points of a set are collinear if and only if there is a line that contains all of them.

Conditional. A statement of the form "if p, then q" is called a conditional. The if-clause (i.e., the p statement) is called the *hypothesis* and the then-clause (i.e., the q statement) is called the *conclusion.*

Conjunction. The conjunction of two statements p, q is the statement *p and q*. A conjunction of two statements is true if and only if both of the statements are true.

Contrapositive. If a theorem has the form "if P, then Q," then the related theorem of the form "if not-Q, then not-P" is called the contrapositive of the given theorem. If a theorem is true, then its contrapositive is true, and conversely.

Converse of a Statement. Each of the statements "If P, then Q" and "If Q, then P" is the converse of the other.

Convex Polygon. See **Polygon, convex.**

Convex Set of Points. A set of points is called convex if for every two points P and Q in the set, the entire segment PQ is in the set. The null set and every set that contains only one point are also called convex sets.

Coordinate System on a Line. Let a unit segment and a line l be given. A coordinate system on l relative to the given unit segment is a one-to-one correspondence between the set of all points of l and the set of all real numbers such that if points A, B, C are matched with the real numbers a, b, c, respectively, then (1) B is between A and C if and only if b is between a and c, and (2) the distance between A and B (relative to the given unit segment) is $|a - b|$. The *origin* of a line coordinate system is the point matched with 0. The *unit point* is the point matched with 1. The number matched with a point is its *coordinate.*

Coplanar. The points of a set are coplanar if and only if there is a plane that contains all of them.

Corollary. A theorem associated with another theorem from which it follows rather easily.

Decagon. A polygon with ten sides.

Deductive Reasoning. Logical arguments by which general statements are obtained from previously accepted statements.

Disjunction. The disjunction of two statements p, q is the statement p *or* q. A disjunction of two statements is true if and only if either (or both) of the statements is true.

Dodecagon. A polygon with twelve sides.

Geometric Mean. If a and b are positive numbers such that $(a, x) \underset{p}{=} (x, b)$ or that $(x, a) \underset{p}{=} (b, x)$, then x is called a geometric mean of a and b. If $(a, x) \underset{p}{=} (x, b)$ or if $(x, a) \underset{p}{=} (b, x)$, then $x^2 = ab$ and $x = \sqrt{ab}$ or $x = -\sqrt{ab}$. We often call $\sqrt{ab}$ the geometric mean of a and b.

Graph. A set of points. To draw a graph or to plot a graph is to draw a picture that suggests which points belong to the graph. The picture of a graph shows the axes, but they are not usually a part of the graph. Of course, a subset of the axes is often a part of the graph.

Halfline. If A is a point on a line l, then all the points on one side of A on line l are called a halfline. The point A is called the *endpoint* of the halfline. (Note that a halfline does not contain its endpoint.)

Halfplane. If α is a plane and l is a line in α, then the set of points on one side of l in α is called a halfplane. The line l is called the *edge* of the halfplane. (Note that a halfplane does not contain its edge.)

Halfspace. If α is a plane, then all points of space on one side of α is called a halfspace. The plane α is called the *face* or *edge* of the halfspace. (Note that a halfspace does not contain its face.)

Heptagon. A polygon with seven sides.

Hexagon. A polygon with six sides.

Hypotenuse. See **Right Triangle.**

Inductive Reasoning. Reasoning based on numerous examples.

Line. An undefined term in our formal geometry. Two distinct points determine a line in the sense that there is exactly one line containing them. Based on the idea of a straight line in informal geometry associated with stretched strings, "lines" of sight, edges of table tops, and rays of light.

parallel lines. Two distinct lines that are coplanar and nonintersecting are parallel lines, and each is said to be parallel to the other. Also, a line is parallel to itself. The lines in a set of lines are said to be parallel lines if each two lines in the set are parallel.

parallel to a plane. A line and a plane are parallel if their intersection is not a point.

perpendicular lines. If the union of two intersecting lines contains a right angle, then the lines are perpendicular, and each is said to be perpendicular to the other.

perpendicular to a plane. A line and a plane are perpendicular if the line intersects the plane and is perpendicular to every line in the plane through the point of intersection.

skew lines. Two lines that do not lie in the same plane.

transversal of two lines. A transversal of two distinct coplanar lines is a line that intersects their union in exactly two distinct points.

Midpoint. The midpoint of a segment is the point between the endpoints of the segment, which divides the segment into two congruent segments.

Midray. A ray is a midray of an angle if it is between the sides of the angle and forms with them two congruent angles. It is sometimes called the *bisector of the angle*, or, briefly, the *angle bisector*.

n-gon. A polygon with n sides, where n is an integer greater than or equal to 3.

Octagon. A polygon with eight sides.

One-to-One Correspondence. A matching between the elements of two sets such that each element of the first set is matched with exactly one element of the second set and each element of the second set is matched with exactly one element of the first set.

Opposite Rays. If A is between B and C, then rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays.

Ordered Pair of Real Numbers. A pair such as (2, 5) in which the order of the two numbers is significant. The ordered pair (2, 5) is different from the ordered pair (5, 2). There is a one-to-one correspondence between the set of all points in an xy-plane and the set of all ordered pairs of real numbers. The first number in an ordered pair of numbers is called the *x-coordinate*, or *abscissa*, of the point associated with it; the second number is called the *y-coordinate*, or *ordinate*, of the point. Taken together, the two numbers in an ordered pair of numbers are called the *coordinates* of the point associated with it.

Ordered Triple of Real Numbers. A triple such as (2, 3, 4) in which the order of the three numbers is significant. The ordered triple (2, 3, 4) is different from the ordered triple (4, 3, 2). There is a one-to-one correspondence between the set of all points in space and the set of all ordered triples of real numbers. The first number in an ordered triple of numbers (x, y, z) is called the *x-coordinate*, the second number is called the *y-coordinate*, and the third number is called the *z-coordinate* of the point associated with it. Taken together, the three numbers in an ordered triple of numbers are called the *coordinates* of the point associated with it.

Ordinate. See **Ordered Pair of Real Numbers.**

Parallelogram. A parallelogram is a quadrilateral each of whose sides is parallel to the side opposite it.

Pentagon. A polygon with five sides.

Pi (π). The ratio of the circumference of a circle to the length of a diameter (approximately 3.1416).

Plane. An undefined term in our formal geometry. A mathematical idea associated with flat surfaces such as chalkboards and table tops. Just as a line contains an unlimited number of points, a plane contains an unlimited number of lines. A plane is "flat" in the sense that if two points of a (straight) line lie in a plane, then the entire line lies in the plane.

parallel planes. Two planes are parallel if their intersection is not a line.

parallel to a line. See **Line, parallel to a plane.**

perpendicular planes. Two planes whose union is the union of four right dihedral angles.

perpendicular to a line. See **Line, perpendicular to a plane.**

Point. An undefined term in our formal geometry. A mathematical idea associated with a "position" in space. When we "mark a point" on paper or on a chalkboard, we are merely drawing a picture to remind us of the idea of a point.

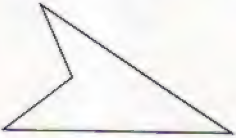
Polygon. Let n be an integer greater than or equal to 3. Let $P_1, P_2, P_3, \ldots, P_{n-1}, P_n$ be n distinct coplanar points such that the n segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}, \overline{P_nP_1}$ have the following properties:

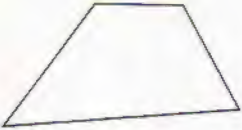
(1) No two of these segments intersect except at their endpoints.

(2) No two of these segments with a common endpoint are collinear.

Then the union of these n segments is a polygon. Each of the n given points is a *vertex* of the polygon. Each of the n segments is a *side* of the polygon. Two vertices of a polygon that are endpoints of the same side are called *consecutive vertices*. Two sides of a polygon that have a common endpoint are called *consecutive sides*. A segment whose endpoints are vertices, but not consecutive vertices, of the polygon is called a *diagonal* of the polygon.

A polygon is a *convex polygon* if and only if each of its sides lies on the edge of a halfplane that contains all of the polygon except that one side.





The *interior* of a convex polygon is the intersection of all of the halfplanes, each of which has a side of the polygon on its edge and each of which contains all of the polygon except for one side.

An angle determined by two consecutive sides of a convex polygon is called an

angle of the polygon. Two angles of a polygon are called *consecutive angles* of the polygon if their vertices are consecutive vertices of the polygon.

regular polygon. A regular polygon is a convex polygon all of whose sides are congruent and all of whose angles are congruent.

center of. The center of a regular polygon is the center of the circumscribed (or inscribed) circle.

central angle of. An angle whose vertex is at the center of the polygon and whose sides contain adjacent vertices of the polygon.

central triangle of. A triangle whose vertices are the center and the endpoints of a side of a regular polygon is called a central triangle of the polygon.

circumcircle of. The circle that contains all the vertices of a regular polygon is called the circumcircle or *circumscribed circle* of the polygon.

circumradius of. The radius of the circumscribed circle.

circumscribed. A regular polygon is said to be circumscribed about a circle and is called a circumscribed polygon if all of its sides are tangent to the circle.

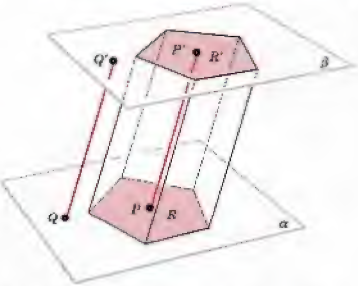
incircle of. If each of the sides of a regular polygon is tangent to a circle, the circle is called an incircle, or *inscribed circle,* of the polygon.

inradius of. The radius of the inscribed circle.

inscribed. A regular polygon is said to be inscribed in a circle and is called an inscribed polygon if all of its vertices are on the circle.

Postulate. A statement in our formal geometry that we accept without proof.

Prism. Let α and β be distinct parallel planes as shown in the figure. Let Q and Q' be points in α and β, respectively. Let R be a polygonal region in α. For each point P in R let P' be the point in β such that $\overline{PP'} \parallel \overline{QQ'}$. The union of all such segments $\overline{PP'}$ is a prism. If $\overline{QQ'}$ is perpendicular to α and β, the prism is a *right prism.*



Let R' be the polygonal region consisting of all points P' in β. The polygonal regions R and R' are called the *bases* of the prism. Depending on the orientation of the prism, it is sometimes convenient to call one of the bases the *lower base* and the other base the *upper base.* Sometimes we call the lower base simply *the base.*

A segment that is perpendicular to both α and β and with its endpoints in these planes is *an altitude* of the prism. Sometimes the length of an altitude is called *the altitude* of the prism.

cross section of. The intersection of a prism and a plane parallel to the base of the prism, provided the intersection is not empty.

lateral edge of. A segment $\overline{AA'}$ where A is a vertex of the base and A' is the corresponding vertex of the upper base.

lateral face of. The union of all segments $\overline{PP'}$ of which P is a point in an edge of one base and P' is the corresponding point in the edge of the other base.

lateral surface of. The union of the lateral faces of a prism.

lateral surface area of. The sum of the areas of the lateral faces.

rectangular prism. A prism whose base is a rectangular region.

total surface of. The union of the lateral surface and the bases of a prism.

total surface area of. The sum of the lateral surface area and the areas of the two bases.

triangular prism. A prism whose base is a triangular region.

volume of. The volume V of any prism is the product of its altitude h and the area S of its base, i.e., $V = Sh$.

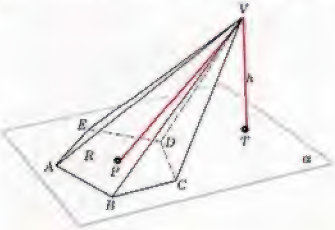
Projection.

on a line. If P is a point and l is a line, the projection of P on l is (1) the point P if P is on l and (2) the foot of the perpendicular from P to l if P is not on l. The projection of a set S on a line l is the set of all points Q on l such that each Q is the projection on l of some P in S.

on a plane. If α is a plane and S is a set of points, then the projection of S on α is the set of all points Q, each of which is the foot of the perpendicular from some point of S.

Proportion. If a, b, c, d are numbers such that $(a, b) \underset{p}{=} (c, d)$ is a proportionality, then that proportionality is a proportion.

Pyramid. Let R be a polygonal region in a plane α and V be a point not in α. For each point P of R there is a segment $\overline{PV}$. The union of all such segments is called a pyramid. The polygonal region R is called the *base* and V is called the *vertex* of the pyramid. The distance VT from V to α is the *altitude* of the pyramid.



cross section of. The intersection of a pyramid with a plane parallel to the base provided the intersection contains more than one point.

volume of. The volume V of any pyramid is one-third the product of its base area S and its altitude h, i.e., $V = \frac{1}{3}Sh$.

Pythagorean Theorem. If a, b, c are the lengths of the sides of a right triangle, with c the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

Quadrilateral. A polygon with four sides.

Ray. If A and B are any two distinct points, ray $\overrightarrow{AB}$ is the union of segment $\overline{AB}$ and all points X such that A–B–X. The point A is called the *endpoint* of $\overrightarrow{AB}$.

antiparallel rays. Two noncollinear rays $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are antiparallel if $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are parallel lines and if F and H lie on opposite sides of $\overleftrightarrow{EG}$. Two collinear rays are antiparallel if neither is a subset of the other.

Rays $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are antiparallel

parallel rays. Two noncollinear rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines and if B and D lie on the same side of $\overleftrightarrow{AC}$. Two collinear rays are parallel if one of them is a subset of the other.

Rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel

perpendicular rays. Two rays are perpendicular if the lines that contain them are perpendicular.

perpendicular to a plane. A ray is perpendicular to a plane if the line that contains the ray is perpendicular to the plane.

secant-ray. A ray whose endpoint is in the exterior of a circle and which intersects the circle in two distinct points.

tangent-ray. A ray whose endpoint is in the exterior of a circle and such that the line containing the ray is tangent to the circle.

Ray-Coordinate System. Let V be a point in plane α. A ray-coordinate system in α relative to V is a one-to-one correspondence between the set of all rays in α with endpoint V and the set of all real numbers x such that $0 \leq x < 360$ with the

following property: If numbers r and s correspond to rays $\overrightarrow{VR}$ and $\overrightarrow{VS}$ in α, respectively, and if $r > s$, then

$$
\begin{array}{ll}
m\angle RVS = r - s & \text{if } r - s < 180, \\
m\angle RVS = 360 - (r - s) & \text{if } r - s > 180, \\
\overrightarrow{VR} \text{ and } \overrightarrow{VS} \text{ are opposite rays} & \text{if } r - s = 180.
\end{array}
$$

The number that corresponds with a ray in a given ray-coordinate system is called the *ray-coordinate* of that ray. The ray whose ray-coordinate is zero is called the *zero-ray* of that system.

Rectangle. A parallelogram with a right angle.

Rhombus. A parallelogram with two adjacent sides congruent.

Segment. A set consisting of two distinct points and all points between them. Thus segment $\overline{AB}$ is the union of points A, B, and all points X such that A–X–B. Points A and B are called the *endpoints* of the segment $\overline{AB}$.

directed segment. The directed segment from A to B, denoted by $\overrightarrow{AB}$, is the set $\{\overline{AB}, A\}$; i.e., it is a segment with one endpoint designated as the starting point. We might think of the directed segment $\overrightarrow{AB}$ as a "path" from A to B if we were to draw a picture of the segment $\overline{AB}$.

parallel segments. If the lines that contain two segments are parallel, then the segments are said to be parallel segments, and each is said to be parallel to the other. The segments in a set of segments are called parallel if every two of them are parallel.

perpendicular bisector of. The perpendicular bisector of a segment in a given plane is the line in that plane that is perpendicular to the segment at its midpoint.

perpendicular bisecting plane of. If A and B are distinct points, the unique plane that is perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$ is called the perpendicular bisecting plane of $\overline{AB}$.

perpendicular to a plane. A segment is perpendicular to a plane if the line that contains it is perpendicular to the plane. If a segment is perpendicular to a plane and one endpoint lies in the plane, then that endpoint is called the *foot of the perpendicular* from the point that is the other endpoint of the segment.

Set. A collection of objects which are called *elements* or *members* of the set.

disjoint sets. Disjoint sets are sets whose intersection is the empty set. For example, the set of odd numbers and the set of even numbers are disjoint sets.

empty set. See **Set, null set.**

equivalent sets. Two sets are equivalent if their elements can be placed in one-to-one correspondence.

intersection of two sets. The intersection of two sets is the set whose members are in both the two given sets. For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e\}$, then the intersection of A and B (denoted by $A \cap B$) is the set $\{c, d\}$.

null set. The set that contains no elements. For example, if $A = \{a, b, c\}$ and

$B = \{d, e, f\}$, then the intersection of A and B is the null or empty set, denoted by $\varnothing$ or $\{ \quad \}$.

perpendicular sets. Two sets, each of which is a segment, a ray, or a line, and which determine two perpendicular lines are called perpendicular sets, and each is said to be perpendicular to the other.

subset. Set A is a subset of set B if every element of set A is also an element of set B. If N is the set of natural numbers and I is the set of integers, then N is a subset of I and we write $N \subset I$ to indicate this.

union of two sets. The union of two sets A and B is the set whose elements are in A, or in B, or in both A and B. For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e\}$, then the union of A and B (denoted by $A \cup B$) is the set $\{a, b, c, d, e\}$.

Set-Builder Notation. A way of indicating the elements of a particular set. For example, the set A whose elements are 1, 2, 3, 4, 5 may be indicated using set-builder notation as $A = \{x : x \text{ is a natural number and } x < 6\}$. This is read "$A$ is the set of all elements, x, such that x is a natural number and x is less than 6." Note that the colon (:) in the set-builder notation is used to mean "such that."

Similarity. Let $ABC \cdots \longleftrightarrow A'B'C' \cdots$ be a one-to-one correspondence between the vertices of polygon $ABC \cdots$ and polygon $A'B'C' \cdots$. This correspondence is called a similarity between the polygons if corresponding angles are congruent and if the lengths of corresponding sides are proportional. If $ABC \cdots \longleftrightarrow A'B'C' \cdots$ is a similarity, we say that the polygons are *similar polygons* and that each of them is *similar* to the other. If $ABC \cdots \longleftrightarrow A'B'C' \cdots$ is a similarity with $AB = k \cdot A'B'$, $AC = k \cdot A'C'$, etc., then we call k the *constant of proportionality*, or the *proportionality constant*, for that similarity.

Slope.

of a line. The slope of a nonvertical line is the slope of any of its segments. The *slope of a nonvertical ray* is the slope of the line that contains the ray.

of a segment. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points and if $x_1 \neq x_2$, then the slope of $\overline{AB}$ is given by $y_2 - y_1/x_2 - x_1$.

Sphere. The set of all points in space that are equidistant from a fixed point in space.

center of. The point in the interior of a sphere from which all points of the sphere are equidistant.

chord of. A segment whose endpoints are on a sphere.

concentric spheres. Two or more spheres that have the same center.

congruent spheres. Two spheres are called congruent spheres if and only if their radii are equal.

diameter of. A chord of a sphere which contains the center of the sphere.

exterior of. The set of all points in space whose distance from the center of a sphere is greater than the radius of the sphere.

great circle of. The intersection of a sphere and a plane that contains the center of the sphere.

interior of. The set of all points in space whose distance from the center of a sphere is less than the radius of the sphere.

radius of. A segment whose endpoints are the center of a sphere and a point of the sphere. Also the number that is the length of all radii of the same sphere.

secant of. A line that intresects a sphere in two distinct points.

surface area of. The surface area S of a sphere with radius r is given by the formula $S = 4\pi r^2$.

tangent plane of. A plane is tangent to a sphere if it intersects the sphere in exactly one point.

volume of. The volume V of a sphere with radius r is given by the formula $V = \frac{4}{3}\pi r^3$.

Square. A rectangle with two adjacent sides congruent.

Theorem. A general statement of a mathematical principle that can be deduced, or proved, from other statements (postulates, theorems, definitions).

Trapezoid. A convex quadrilateral with at least two parallel sides.

Triangle. If A, B, C are three noncollinear points, then the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is a triangle. Each of the points A, B, C is called a *vertex* of $\triangle ABC$. Each of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is called a *side* of $\triangle ABC$. Each of the angles $\angle ABC$, $\angle BCA$, $\angle CAB$ is called an *angle* of $\triangle ABC$.

acute triangle. A triangle with three acute angles.

equiangular triangle. A triangle with three congruent angles.

equilateral triangle. A triangle with three congruent sides.

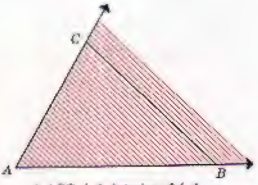
exterior angle of. An angle that forms a linear pair with an interior angle of a triangle is called an exterior angle of the triangle.

exterior of. The set of all points in the plane of the triangle that are neither points of the triangle, nor points of the interior of the triangle.

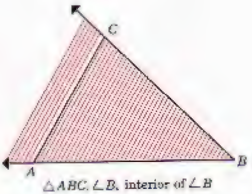
inscribed. A triangle is inscribed in a circle and is called an inscribed triangle of of the circle if all three of its vertices lie on the circle.

interior angle of. Each angle of a triangle is called an interior angle of the triangle.

interior of. The intersection of the interiors of the three angles of a triangle is the interior of the triangle.

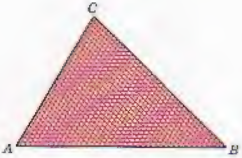


$\triangle ABC$, $\angle A$, interior of $\angle A$



$\triangle ABC$, $\angle B$, interior of $\angle B$

$\triangle ABC$, $\angle C$, interior of C



$\triangle ABC$, interior $\triangle ABC$

isosceles triangle. An isosceles triangle is a triangle with (at least) two congruent sides. If two sides are congruent, then the remaining side is called the *base*. The angle opposite the base is called the *vertex angle*. The two angles that are opposite the congruent sides are called the *base angles*.

median of a triangle. A segment whose endpoints are a vertex of the triangle and the midpoint of the side opposite that vertex.

obtuse triangle. A triangle with one obtuse angle.

right triangle. A triangle with one right angle. The *hypotenuse* of a right triangle is the side that is opposite the right angle. The other two sides of a right triangle are called *legs*.

xy-Coordinate System. Given an x-axis and a y-axis, the one-to-one correspondence between the set of all points in the xy-plane and the set of all ordered pairs of real numbers in which each point P in the plane corresponds to the ordered pair (a, b), where a, b are the x-, y-coordinates, respectively, of P, is the xy-coordinate system.

xyz-Coordinate System. Given an x-axis, a y-axis, and a z-axis, the one-to-one correspondence between the set of all points and the set of all ordered triples of real numbers in which each point P corresponds to the ordered triple (a, b, c), where a, b, c are the x-, y-, z-coordinates, respectively, of P, is the xyz-coordinate system.

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